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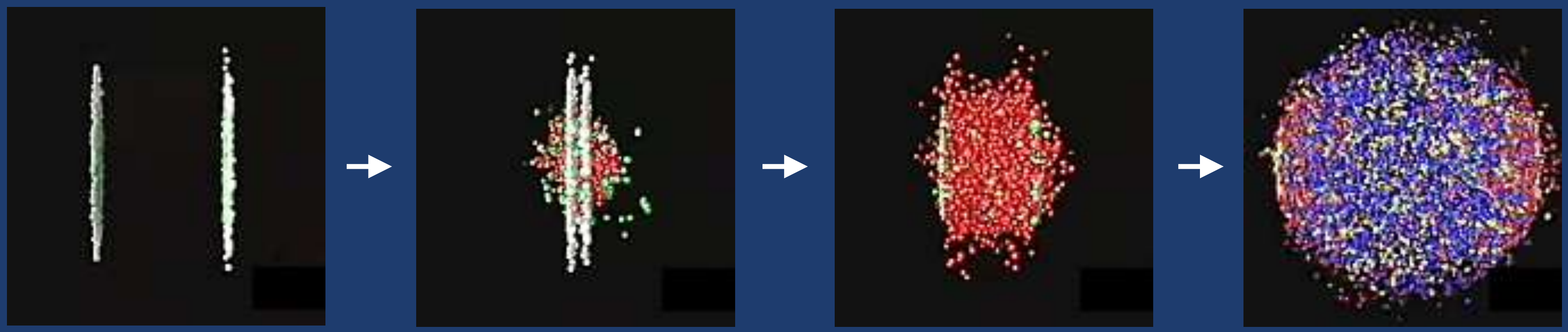
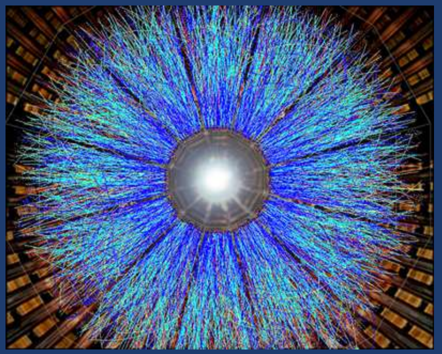
THE COMPLEX STRUCTURE OF CLASSICAL HYDRODYNAMICS: CONVERGENCE, QUANTUM CHAOS AND BOUNDS

ICTS, 7.10.2020

MOTIVATION





"hydrodynamics is an asymptotic series" (Bjorken flow, attractor, …)

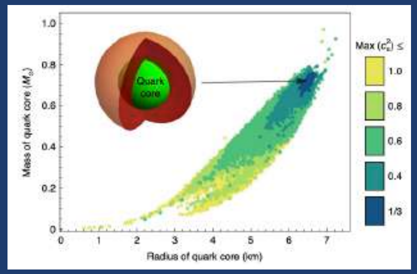
"hydrodynamics works unreasonably well"

holographic calculation of η/s works extremely well

is there a bound? $\frac{\eta}{s} \geq \mathcal{C}\,\frac{1}{4\pi}$

quantum chaotic mess leads to (collective) hydrodynamisation and thermalisation

is the speed of sound bounded?
[Annala, Gorda, Kurkela, Nättilä, Vuorinen (2020)]



OUTLINE

- introduction: hydrodynamics and holographic duality

I. complex spectral curves and convergence

II. quantum chaos through *pole-skipping*

III. bounds from univalence

- future directions

INTRODUCTION: HYDRODYNAMICS AND HOLOGRAPHIC DUALITY

HYDRODYNAMICS

- low-energy limit of QFTs – a Schwinger-Keldysh effective field theory
  [Grozdanov, Polonyi (2013); Crossley, Glorioso, Liu (2015); Haehl, Loganayagam, Rangamani (2015); …]

- expressed through conservation laws (equations of motion) of globally conserved operators

$$\nabla_\mu T^{\mu\nu} = 0 \qquad \nabla_\mu J^\mu = 0 \qquad \dots \nabla_\mu J^{\mu\nu_1\dots\nu_n} = 0$$

- tensor structures (symmetries and phenomenological gradient expansions) with transport coefficients (microscopic)

$$\partial u^\mu \sim \partial T \ll 1$$

$$T^{\mu\nu}\left(u^\lambda, T, \mu\right) = (\varepsilon + P)\, u^\mu u^\nu + P g^{\mu\nu} - \eta\sigma^{\mu\nu} - \zeta \nabla \cdot u \Delta^{\mu\nu} + \dots$$

- small ∂ = small frequency-momentum $u^\mu, T \sim e^{-i\omega t + i\mathbf{q}\cdot\mathbf{x}}$: $\omega/T \sim q/T \ll 1$

- dispersion relations:

shear diffusion: $\omega = -iDq^2$

sound: $\omega = \pm v_s q - i\Gamma q^2$

equilibrium temperature

$q = \sqrt{\mathbf{q}^2}$

HYDRODYNAMICS

- infinite, all-order hydrodynamic expansion

$$T^{\mu\nu} = \sum_{n=0}^{\infty} \left[\sum_i^N \lambda_i^{(n)} \mathcal{T}^{\mu\nu}_{(n)} \right] \xrightarrow[u^\mu \sim T \sim e^{-i\omega t + iqz}]{\nabla_\mu T^{\mu\nu} = 0} \boxed{\omega(q) = \sum_{n=1}^{\infty} \alpha_n q^n}$$

- non-analytic corrections due to statistical (quantum) corrections; long-time tails

- conformal symmetry constrains the series

- state of the art for relativistic neutral hydrodynamics

CFT:
Weyl covariance
$T^\mu_{\ \mu} = 0$

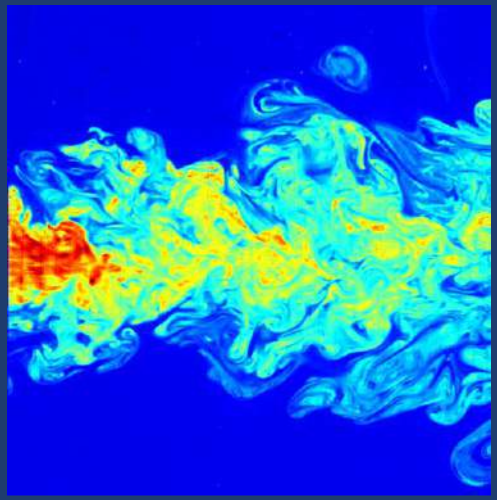
	max *N*	max *N* in CFT	
first order	*2*	*1*	Navier-Stokes (1821)
second order	*15*	*5*	BRSSS (2007)
third order	*68*	*20*	Grozdanov, Kaplis, PRD (2016)

[also Diles, Mamani, Miranda, Zanchin, JHEP (2020) and A. Jaiswal, PRC (2013) for kinetic theory]

HOLOGRAPHIC DUALITY

- duality: theory A = theory B
- holographic or gauge/gravity duality is a result of string theory, which is a quantum theory of gravity [Maldacena (1997)]

strongly coupled quantum theory	$=$	*weakly coupled gravity*
(extremely hard)	$=$	(much easier)



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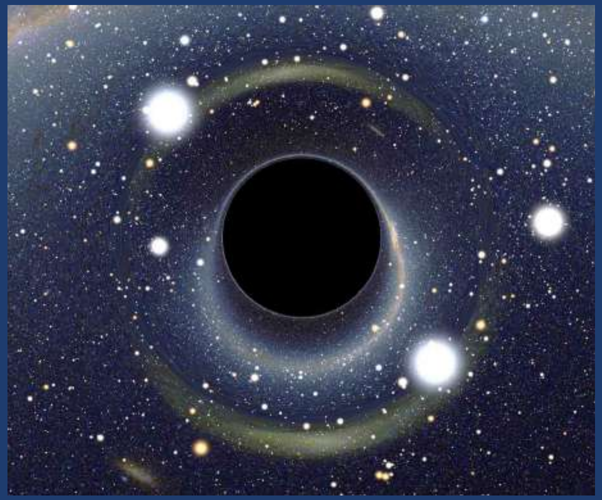


- weakly interacting gravity allows to analyse strongly coupled microscopic QFTs
- invaluable explicit (toy) models, e.g., the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory

HOLOGRAPHIC HYDRODYNAMICS

- holography is an extremely useful tool for studying the structure of thermal spectra
- example: the spectrum (physical excitations) of a free (zero coupling) massive relativistic theory plotted for complex frequency $\omega \in \mathbb{C}$:

energy-momentum
(frequency-wavevector) relation

$$E^2 = \left(mc^2\right)^2 + p^2c^2$$

$$E = \hbar\omega$$

$$p = \hbar q$$

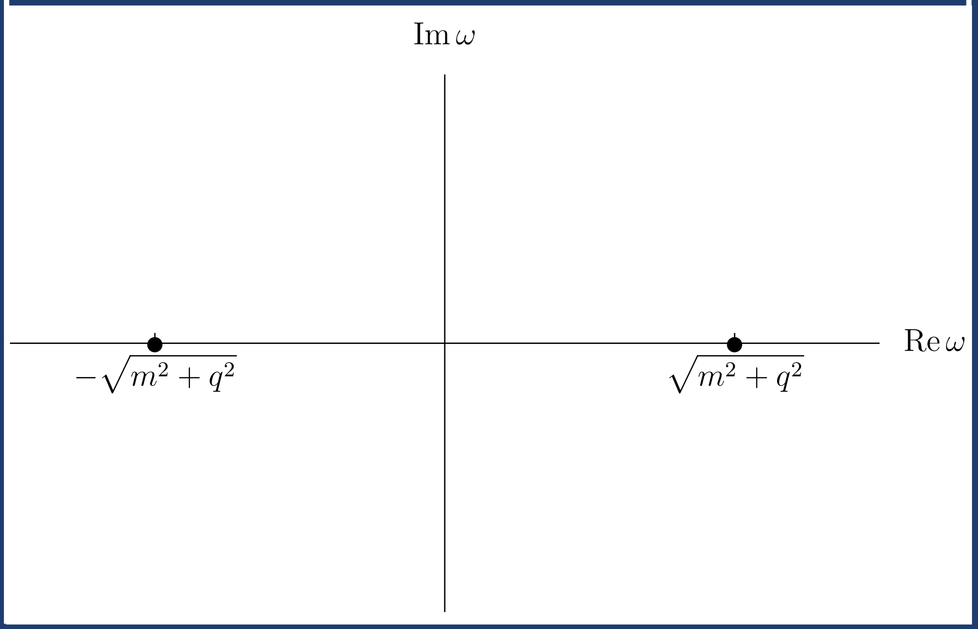
$$\downarrow c = \hbar = 1$$

$$\boxed{\omega = \pm\sqrt{m^2+q^2}}$$

dispersion relation
no damping (no imaginary part)

4*d* QFT spectrum

$$\langle \phi(-\omega,-q), \phi(\omega,q) \rangle$$

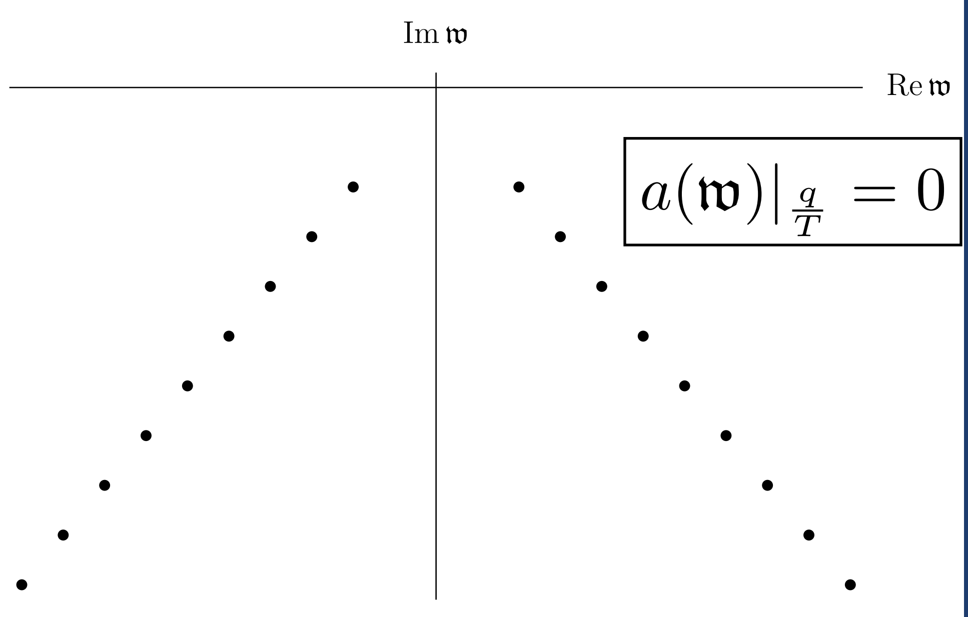


HOLOGRAPHIC HYDRODYNAMICS

- holography is an extremely useful tool for studying the structure of thermal spectra
- the spectrum of field theory correlators equals the quasinormal spectrum of frequencies of dual black branes, plotted for $\mathfrak{w} \equiv \frac{\omega}{2\pi T} \in \mathbb{C}$:
- analytically known dispersion relations in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory at infinite 't Hooft coupling and $N_c \to \infty$

  [Grozdanov, Kovtun, Starinets, Tadić, JHEP (2019)]

$$\langle T_{\mu\nu}(-\omega,-q), T_{\rho\sigma}(\omega,q)\rangle_R \sim \frac{b(\omega,q)}{a(\omega,q)}$$



- sound:

$$\omega = \pm\frac{1}{\sqrt{3}}q - \frac{i}{6\pi T}q^2 \pm \frac{3-2\ln 2}{24\sqrt{3}\pi^2T^2}q^3 - \frac{i\left(\pi^2 - 24 + 24\ln 2 - 12\ln^2 2\right)}{864\pi^3T^3}q^4 \pm \cdots$$

- shear:

$$\omega = -\frac{i}{4\pi T}q^2 - \frac{i(1-\ln 2)}{32\pi^3T^3}q^4 - \frac{i(24\ln^2 2 - \pi^2)}{96\,(2\pi T)^5}q^6 - \frac{i\left[2\pi^2(\ln 32 - 1) - 21\zeta(3) - 24\ln 2(1+\ln 2(\ln 32 - 3))\right]}{384\,(2\pi T)^7}q^8 + \cdots$$

THE REST OF THE TALK:

$$\omega(q) = \sum_{n=1}^{\infty} \alpha_n q^n$$

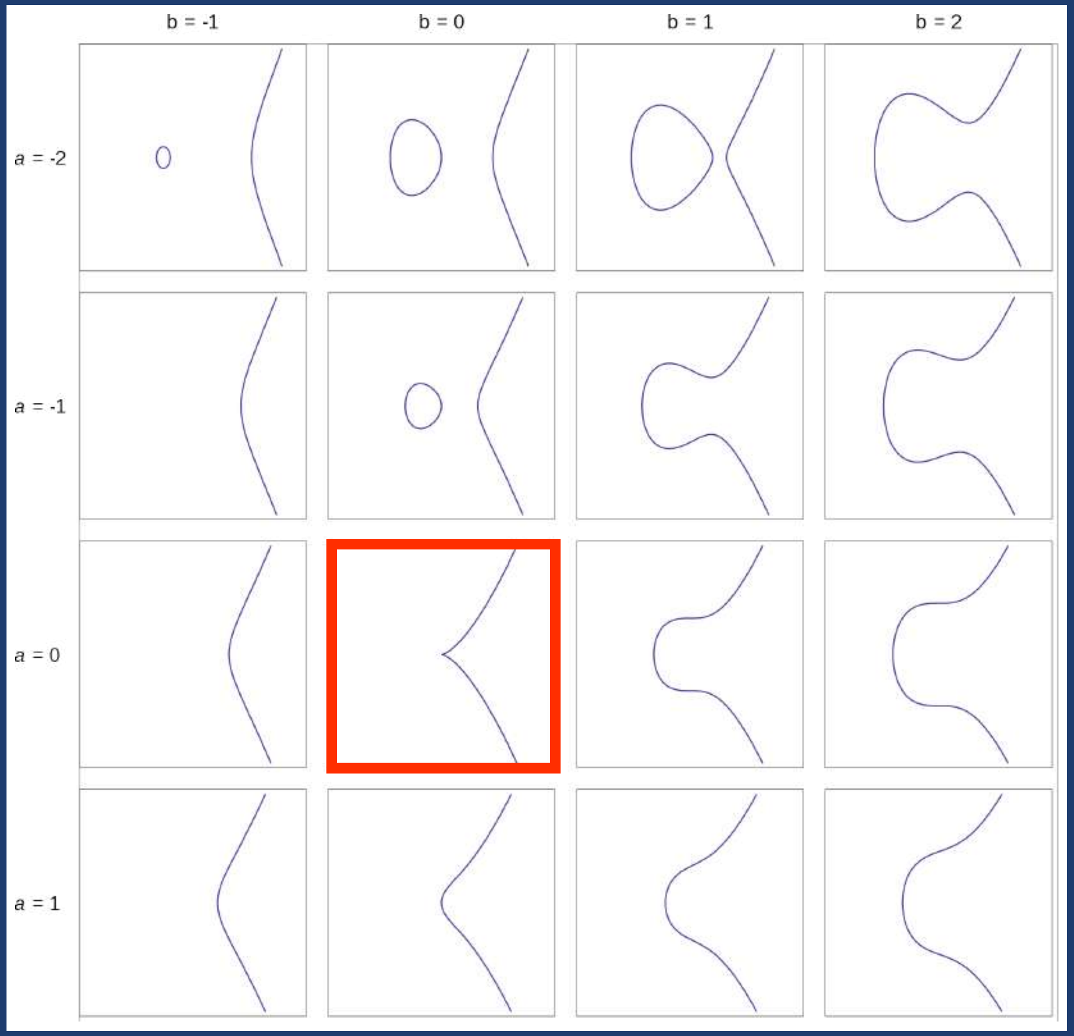
$$\omega, q \in \mathbb{C}$$

I. COMPLEX SPECTRAL CURVES AND CONVERGENCE

COMPLEX SPECTRAL CURVES

- algebraic curves are solutions to polynomial equations $P(x,y) = 0 \Rightarrow y(x)$

- e.g.: elliptic curves are non-singular solutions of $y^2 = x^3 + ax + b,\ x, y \in \mathbb{R}$



- we will be interested in critical points, such as cusps, self-intersections, …, of complex spectral curves (with *P(x,y)* not necessarily a polynomial)

$$P(x,y) = 0 \Rightarrow y(x),\ x, y \in \mathbb{C}$$

LOCAL ANALYSIS: PUISEUX SERIES

- **Taylor series** is a series in integer powers of the expansion parameter
- **Puiseux series** is a series in fractional powers of the expansion parameter
- consider a simple example of an algebraic curve for $x, y \in \mathbb{C}$

$$P(x,y) = x^2 + y^2 - 1 = 0$$

- we want to find series solutions for $y(x)$
- a regular point is defined by $P(x_r, y_r) = 0,\ \partial_y P(x_r, y_r) \neq 0$ at the regular point $(x_r, y_r) = (0,1)$, the solution gives a Taylor series

$$y = y^{(T)}(x) = 1 - \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

- a critical point (of order 2) is defined by $P(x_*, y_*) = 0,\ \partial_y P(x_*, y_*) = 0,\ \partial_y^2 P(x_*, y_*) \neq 0$ here, two such points, $(x_*, y_*) = (\pm 1, 0)$, each with two branches of Puiseux series, e.g.

at $(x_*, y_*) = (1,0)$:

$$y = y_1^{(P)}(x) = i\sqrt{2}(x-1)^{\frac{1}{2}} + i2^{-\frac{3}{2}}(x-1)^{\frac{3}{2}} + \cdots$$
$$y = y_2^{(P)}(x) = -i\sqrt{2}(x-1)^{\frac{1}{2}} - i2^{-\frac{3}{2}}(x-1)^{\frac{3}{2}} + \cdots$$

- **radius of convergence** is distance to the nearest critical point: $R_x^{(T)} = 1,\ R_x^{(P)} = 2$

CONVERGENCE OF HYDRODYNAMICS

- hydrodynamic modes as complex spectral (or infinite-order algebraic) curves [Grozdanov, Kovtun, Starinets, Tadić, PRL (2019) and JHEP (2019)]

hydro: $\det \mathcal{L}(\mathbf{q}^2, \omega) = 0$
QNM: $a(\mathbf{q}^2, \omega) = 0$ $\longrightarrow$ $\boxed{P(\mathbf{q}^2, \omega) = 0} \Longrightarrow \boxed{\omega_i(\mathbf{q}^2)}$ $\quad \mathfrak{w} = \frac{\omega}{2\pi T}, \; \mathfrak{q} = \frac{|\mathbf{q}|}{2\pi T} \in \mathbb{C}$

- e.g., first-order hydrodynamics: $P_1(\mathbf{q}^2, \omega) = \left(\omega + iD\mathbf{q}^2\right)^2 \left(\omega^2 + i\Gamma\omega\mathbf{q}^2 - v_s^2\mathbf{q}^2\right) = 0$
- analytic implicit function theorem (a regular point)

$$\boxed{P(\mathbf{q}_*^2, \omega_*) = 0, \; \partial_\omega P(\mathbf{q}_*^2, \omega_*) \neq 0}$$

- Puiseux theorem: there exists a convergent series around a critical point $(\mathbf{q}_*^2, \omega_*)$

$$\boxed{P(\mathbf{q}_*^2, \omega_*) = 0, \; \partial_\omega P(\mathbf{q}_*^2, \omega_*) = 0, \; \ldots, \; \partial_\omega^p P(\mathbf{q}_*^2, \omega_*) \neq 0}$$

$$\boxed{\begin{aligned} p_{\text{shear}} &= 1 \\ p_{\text{sound}} &= 2 \end{aligned}}$$

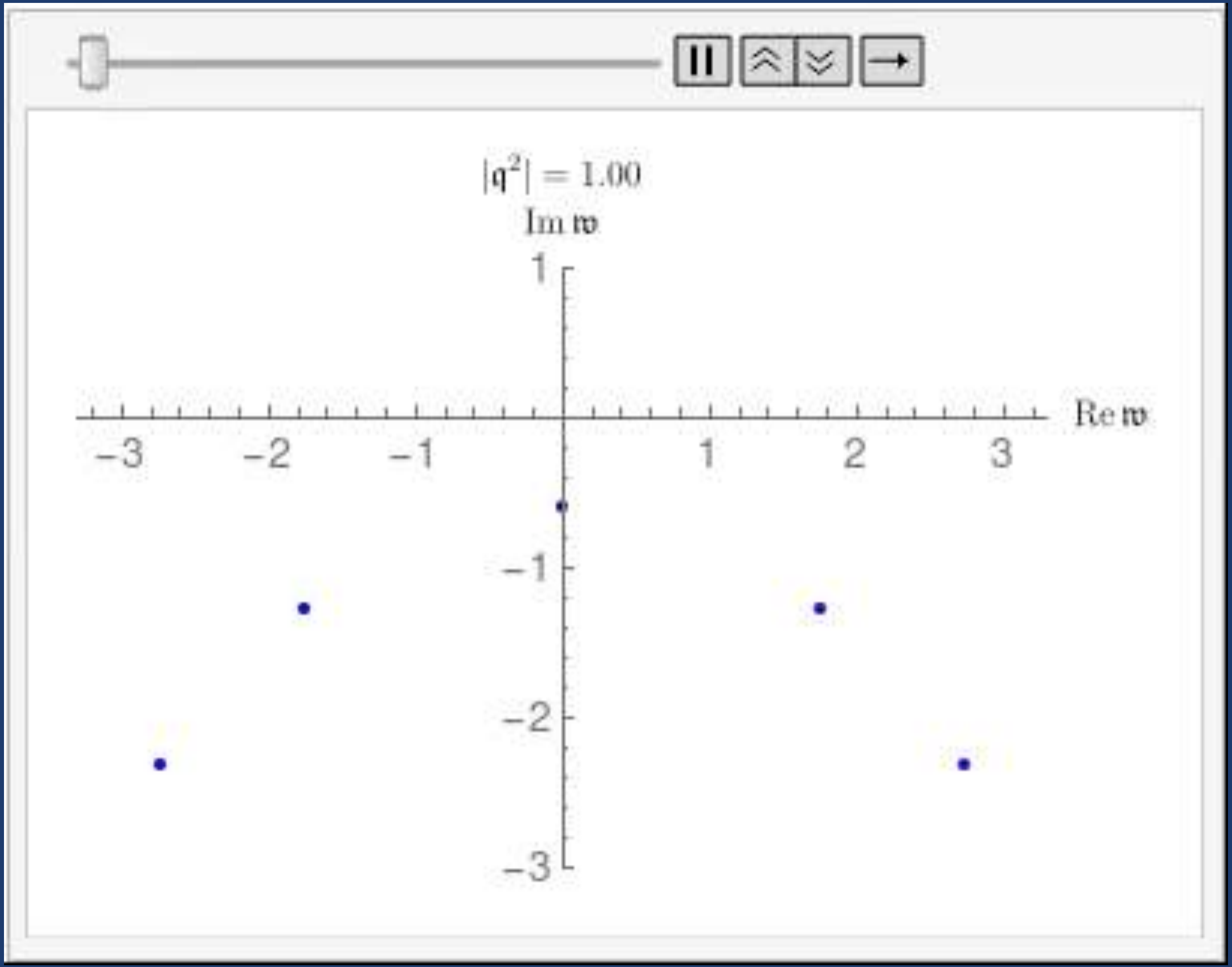
- hydrodynamic series are Puiseux series around $(\mathbf{q}^2, \omega)_{\text{shear}}^{(\text{regular})} = (\mathbf{q}_*^2, \omega_*)_{\text{sound}}^{(\text{critical})} = (0, 0)$

$$\boxed{\mathfrak{w}_{\text{shear}} = -i \sum_{n=1}^{\infty} c_n \left(\mathfrak{q}^2\right)^n = -i\mathfrak{D}\mathfrak{q}^2 + \ldots}$$

$$\boxed{\mathfrak{w}_{\text{sound}} = -i \sum_{n=1}^{\infty} a_n e^{\pm \frac{i\pi n}{2}} \left(\mathfrak{q}^2\right)^{n/2} = \pm v_s \mathfrak{q} - \frac{i}{2}\mathfrak{G}\mathfrak{q}^2 + \ldots}$$

CONVERGENCE OF HYDRODYNAMICS

- complexify $\mathfrak{q}^2 = \left|\mathfrak{q}^2\right| e^{i\theta}$, fix the absolute value and vary the argument in the QNM spectrum of the shear (diffusive) channel in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory



[animation by P. Tadić]

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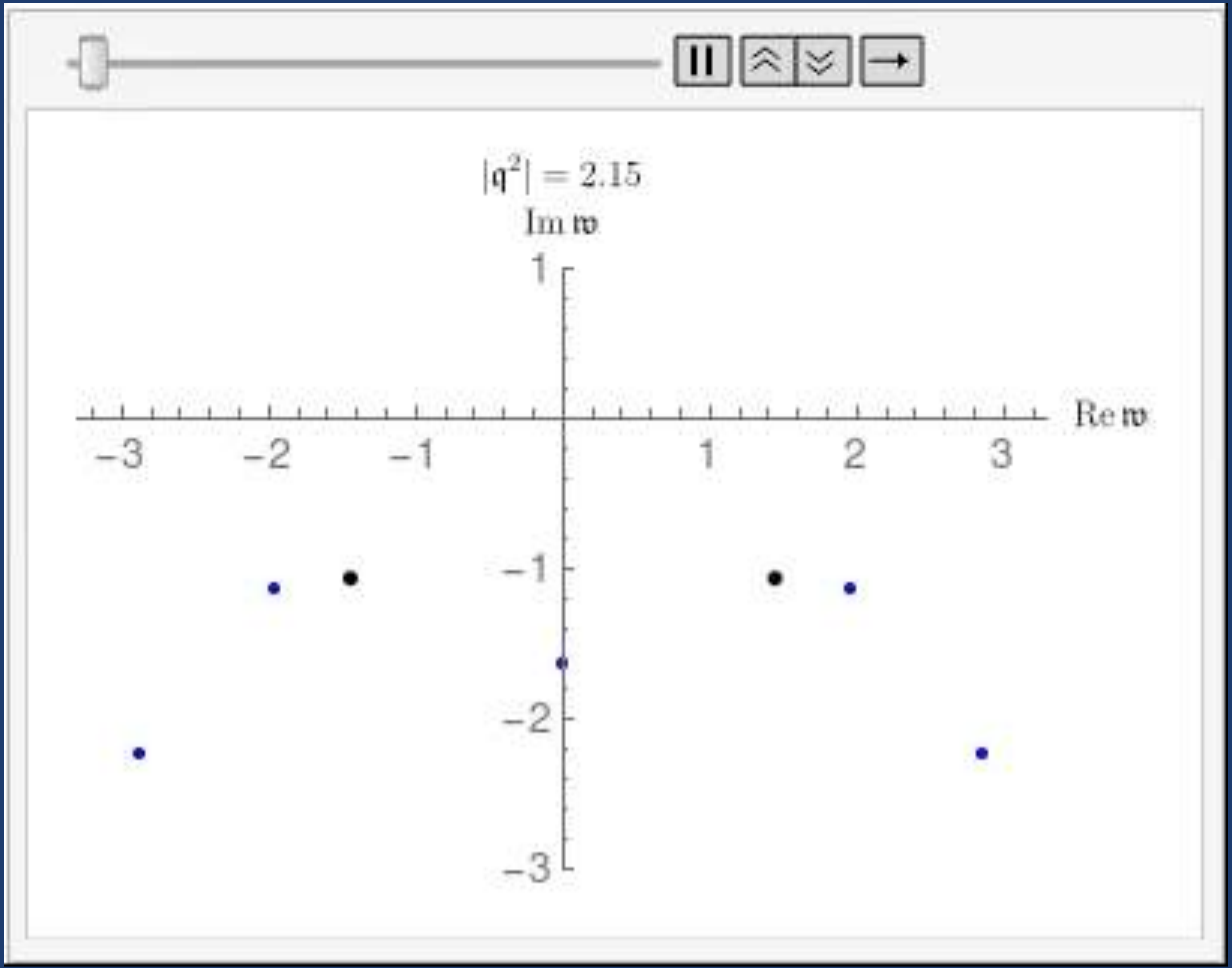

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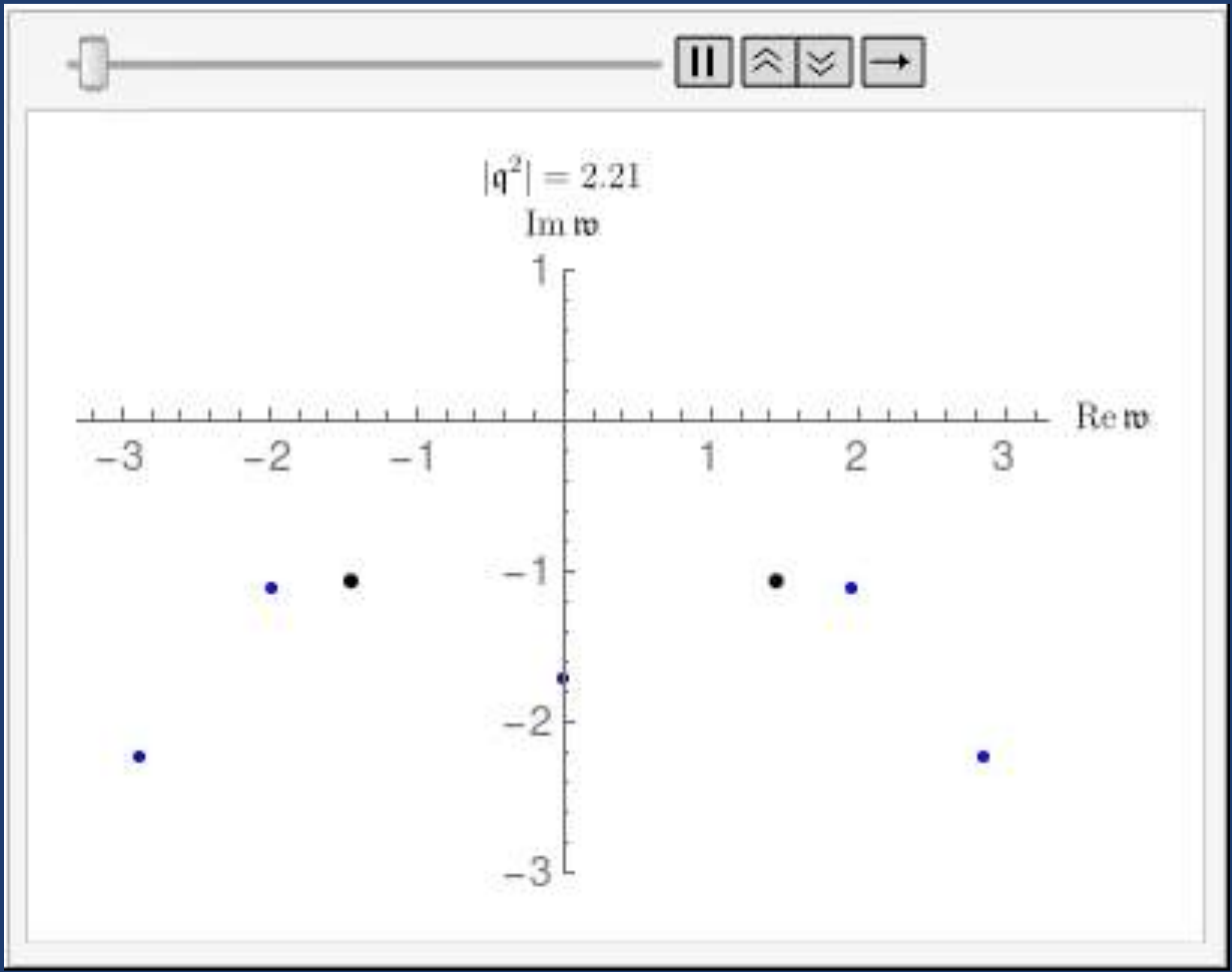

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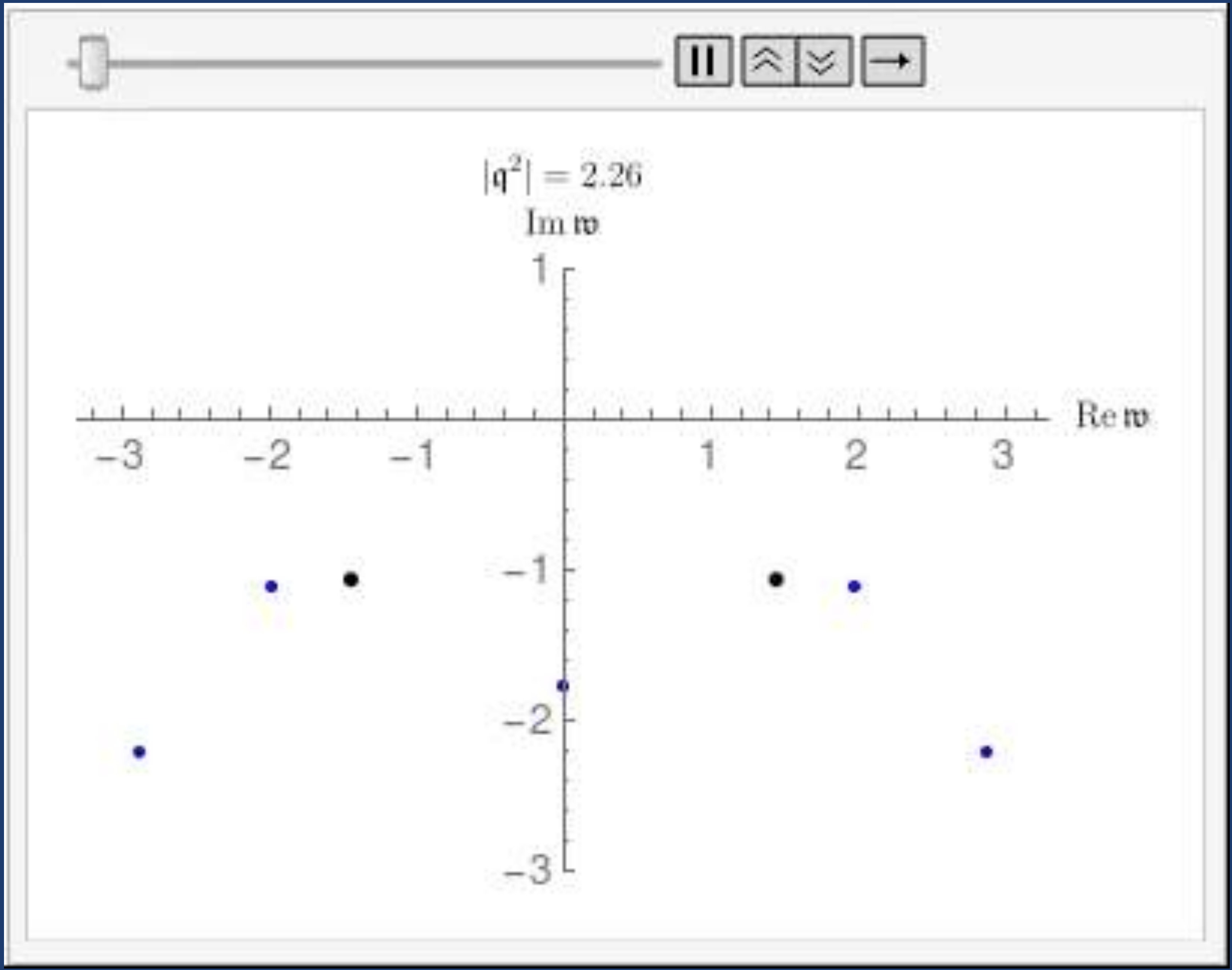
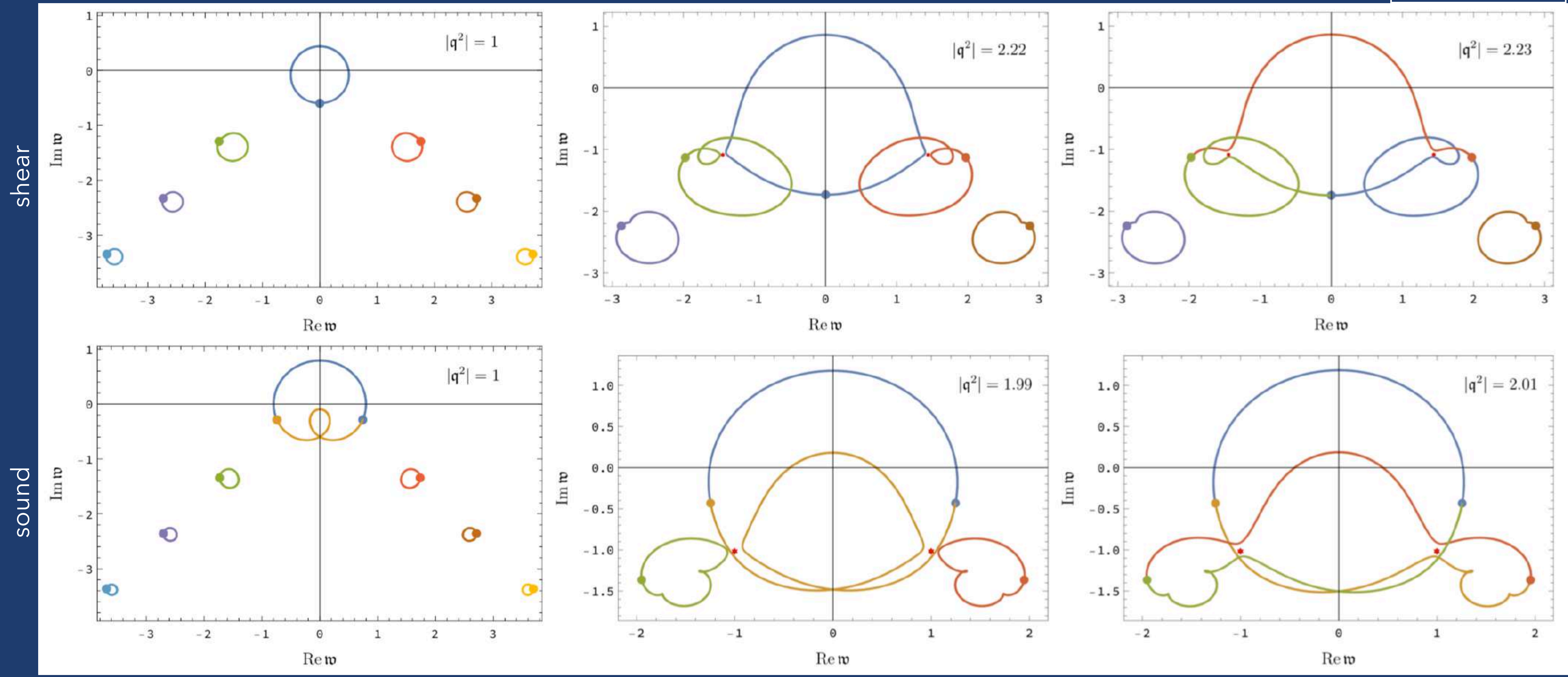


[animation by P. Tadić]

CONVERGENCE OF HYDRODYNAMICS

- radius of convergence of $\mathfrak{w}(\mathfrak{q}) = \sum_{i=1}^{\infty} c_n \mathfrak{q}^n$, i.e. $|\mathfrak{q}| < \mathfrak{q}_*$, is set by the lowest momentum at which the hydro pole collides (level-crossing): $\mathfrak{q}_* = \min\left[|\mathfrak{q}_{\text{collision}}|\right]$

$\mathfrak{q}^2 = |\mathfrak{q}^2| e^{i\theta}$

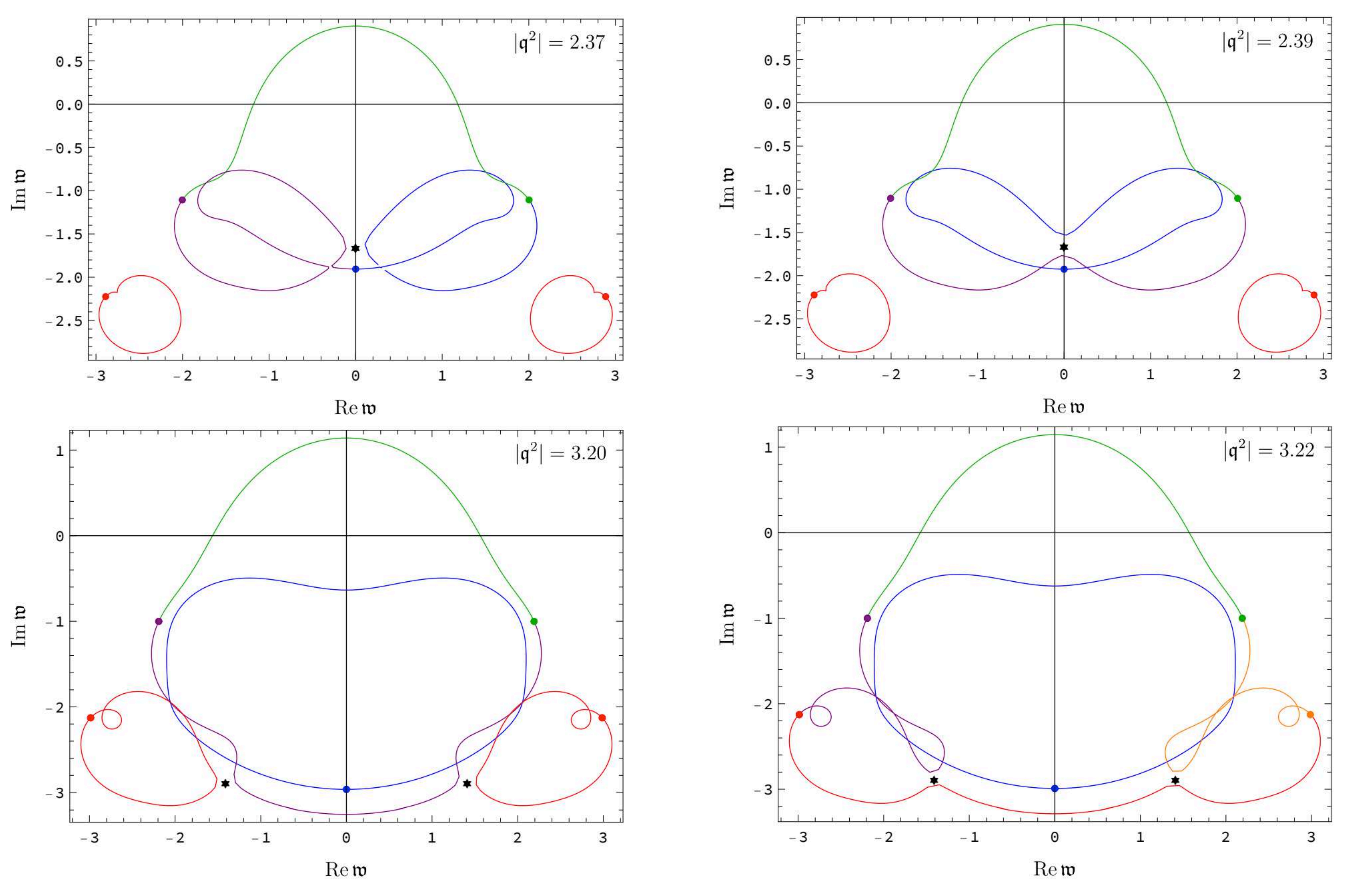


shear:	$\mathfrak{q}_* \approx 1.49131$ $\mathfrak{w}(\mathfrak{q}_*) \approx \pm 1.4436414 - 1.0692250i$

$\mathcal{N} = 4$ SYM

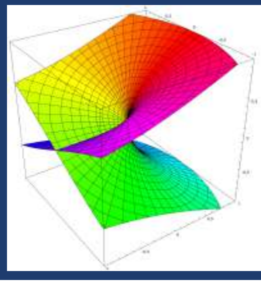
sound:	$\mathfrak{q}_* = \sqrt{2} \approx 1.41421$ $\mathfrak{w}(\mathfrak{q}_*) = \pm 1 - i$

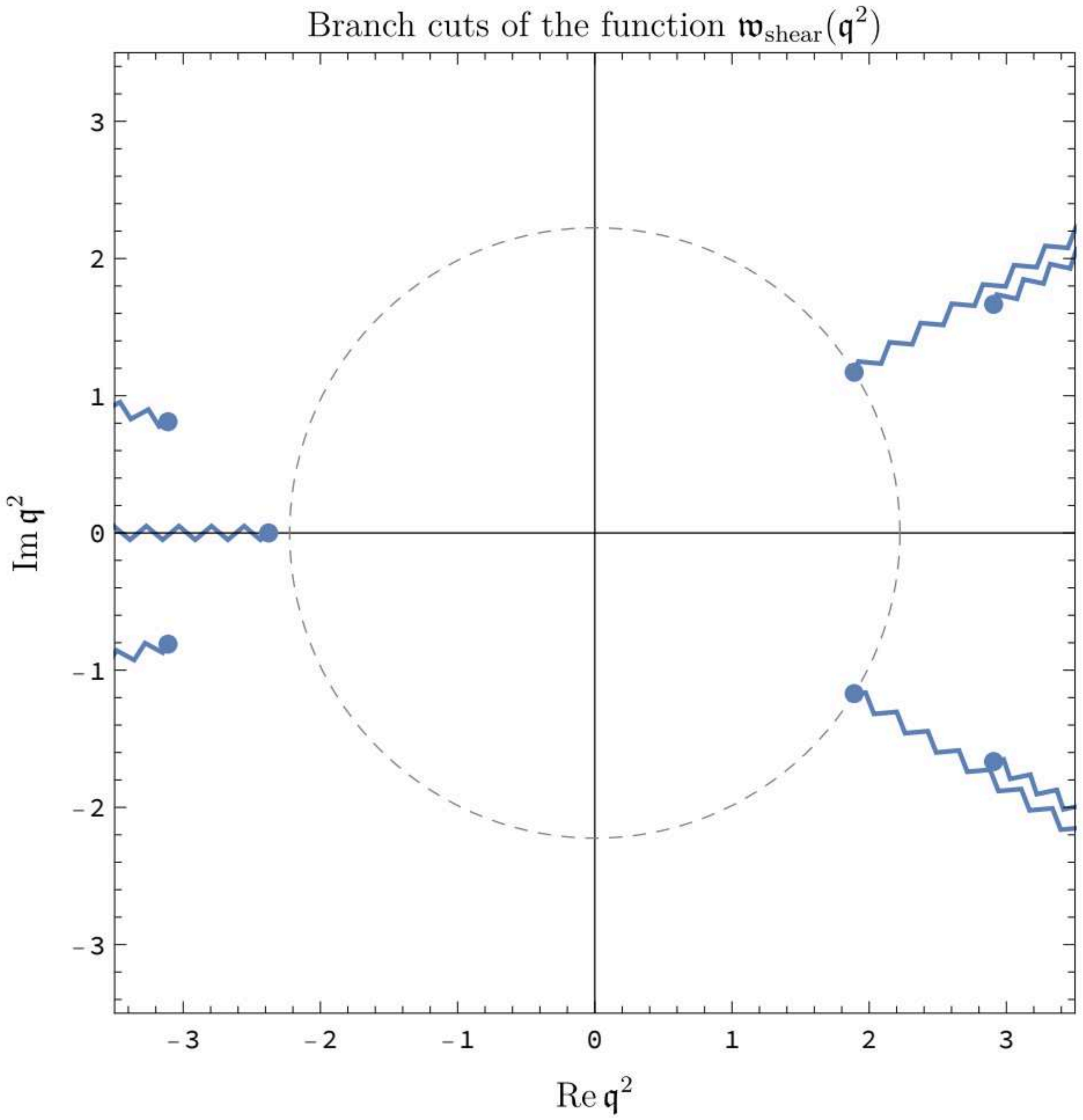
HIGHER CRITICAL POINTS (E.G. SHEAR/DIFFUSIVE CHANNEL)

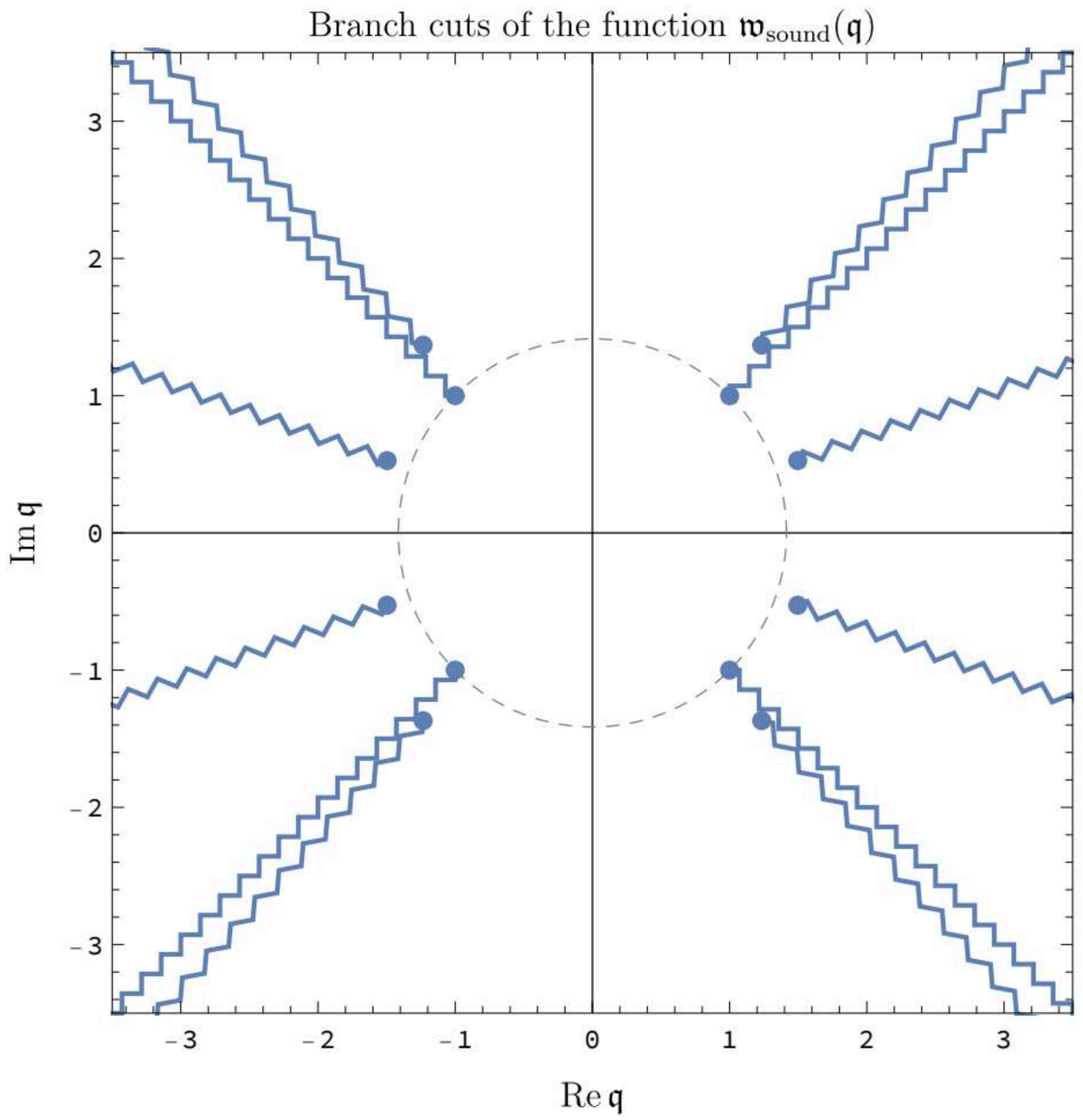


ANALYTIC STURCTURE

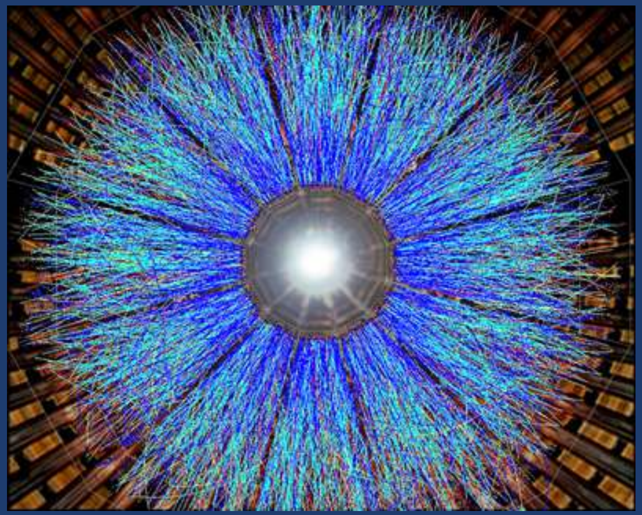
- analytic structure of dispersion relations in complexified momentum space
- dispersion relations are complicated, multi-sheeted Riemann surfaces "connecting various modes in the spectrum into one entity"







UNREASONABLE EFFECTIVENESS



"**unreasonable**": hydro works for large derivatives

$$\omega(q) = \sum_{n=1}^{\infty} \alpha_n q^n$$

radius of convergence in *N*=4 SYM at infinite coupling →

$$q/T \sim O(10)$$

↑

microscopic input from holography

orders of magnitude larger radius of convergence than naive $q/T \ll 1$ – if this is true in general, it may explain the **"unreasonable effectiveness of hydrodynamics"**

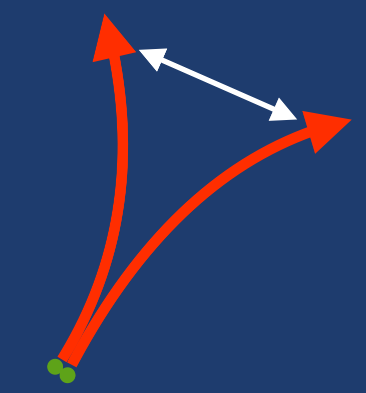
hydrodynamics is neither convergent nor asymptotic; it depends on the observable!

II. QUANTUM CHAOS AND *POLE-SKIPPING*

CHAOS

Lyapunov exponent — butterfly velocity

$$|\Delta Z(t,\mathbf{x})| \approx |\Delta Z(t_i,\mathbf{x}_i)|\, e^{\lambda_L (t-|\mathbf{x}|/v_B)}$$



$t = t_i$

- classical chaos means extreme sensitivity to initial conditions
- exponential Lyapunov divergence of trajectories and the butterfly effect
- in quantum systems, molecules collide chaotically
- “out-of-time-ordered” correlation functions [Larkin, Ovchinnikov; Kitaev]

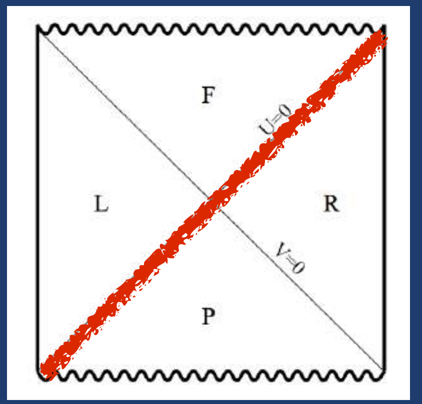
$$C(t,\mathbf{x}) = \left\langle [W(t,\mathbf{x}),V(0,\mathbf{0})]^\dagger [W(t,\mathbf{x}),V(0,\mathbf{0})] \right\rangle_T \sim \epsilon\, e^{\lambda_L (t-|\mathbf{x}|/v_B)}$$

butterfly velocity

scrambling time $\boxed{t_* = \dfrac{1}{\lambda_L} \ln N}$ ⟵ typically, $\epsilon = 1/N_c^2 \ll 1$ — Lyapunov exponent

- its “build-up” describes the quantum butterfly effect
- standard lore: *“microscopic quantum information is smeared out at large distances”*

QUANTUM CHAOS



- **Exponential Lyapunov Chaos** – Lyapunov exponent and butterfly velocity in holographic theories can be computed from holography
- Lyapunov exponent saturates the Maldacena-Shenker-Stanford bound

OTOC of $\mathcal{O}(t,x)$

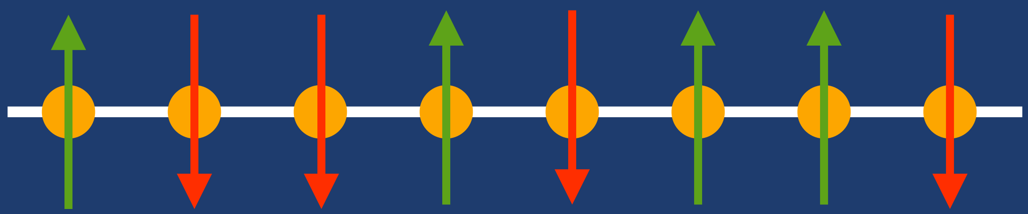
$$C(t,x) \sim \epsilon\, e^{\lambda_L (t - x/v_B)}$$

$$\lambda_L \leq 2\pi T$$

- in lattice spin systems, quantum chaos is not exponential (Lyapunov), but spreads polynomially with a bounded rate of maximal power-law growth – **Weak Quantum Chaos** [Kukuljan, Grozdanov, Prosen, PRB (2017)]

OTOC of $\int d^d x\, \mathcal{O}(t,x)$

$$c(t) \leq A t^{3d}$$



- all these theories exhibit macroscopic collective transport: hydrodynamics
- what is the precise connection between hydrodynamics and chaos [Hartnoll; Blake]?

$$D \sim v_B^2/\lambda_L \geq \text{“?”}$$

POLE-SKIPPING

- the phenomenon of pole-skipping makes precise the analytic connection between hydrodynamics and exponential chaos – true in "all" holographic theories [Grozdanov, Schalm, Scopelliti, PRL (2017); Blake, Lee, Liu, JHEP (2018); Blake, Davison, Grozdanov, Liu, JHEP (2018); Grozdanov, JHEP (2019)]

- resumed all-order hydrodynamic series (e.g. the sound channel)

$$\omega(q) = \sum_{n=1}^{\infty} \alpha_n q^n$$

passes through a special "chaos point" at imaginary momentum

$$\mathcal{P}_c : \qquad \omega(q = iq_0) = i\lambda_L, \qquad \lambda_L = 2\pi T, \qquad q_0 = \lambda_L / v_B$$

defined through the fact that the associated two point function has both a pole and a zero at this point (e.g. in the sound channel, this is the energy density correlator)

$$G^R_{T^{00}T^{00}}(\omega, q) = \frac{b(\omega, q)}{a(\omega, q)}, \qquad \lim_{(\omega,q)\to\mathcal{P}_c} a(\omega, q) = \lim_{(\omega,q)\to\mathcal{P}_c} b(\omega, q) = 0$$

POLE-SKIPPING

- simple example: the Sachdev-Ye-Kitaev chain [Gu, Qi, Stanford (2017)]

$$G^R_{T^{00}T^{00}}(\omega, q) = C\frac{i\omega\left(\frac{\omega^2}{\lambda_L^2}+1\right)}{-i\omega + D_E q^2}$$

pole (diffusion): $\omega = -iD_E q^2$

zero : $\omega = \pm i\lambda_L$

$$|q_0| = \frac{\lambda_L}{v_B} = \frac{\lambda_L}{\sqrt{\lambda_L D_E}}$$

- in $\mathcal{N}=4$ SYM theory at infinite N_c:

$$q_0 = \sqrt{6}\pi T$$
$$v_B = \lambda_L/q_0 = \sqrt{2/3}$$

point of chaos is inside the radius of convergence of

$$\omega(q) = \sum_{n=1}^{\infty} \alpha_n q^n$$

- the reason for pole-skipping in holography is a special, new property of Einstein's equations at the horizon [Blake, Davison, Grozdanov, Liu, JHEP (2018)]

POLE-SKIPPING

- in $\mathcal{N}=4$ SYM theory at infinite N_c and infinite coupling

$$\text{diffusion}: \ \omega_c = \omega(q_c = q_0) = -i\lambda_L$$
$$\text{sound}: \ \omega_c = \omega(q_c = i q_0) = i\lambda_L$$

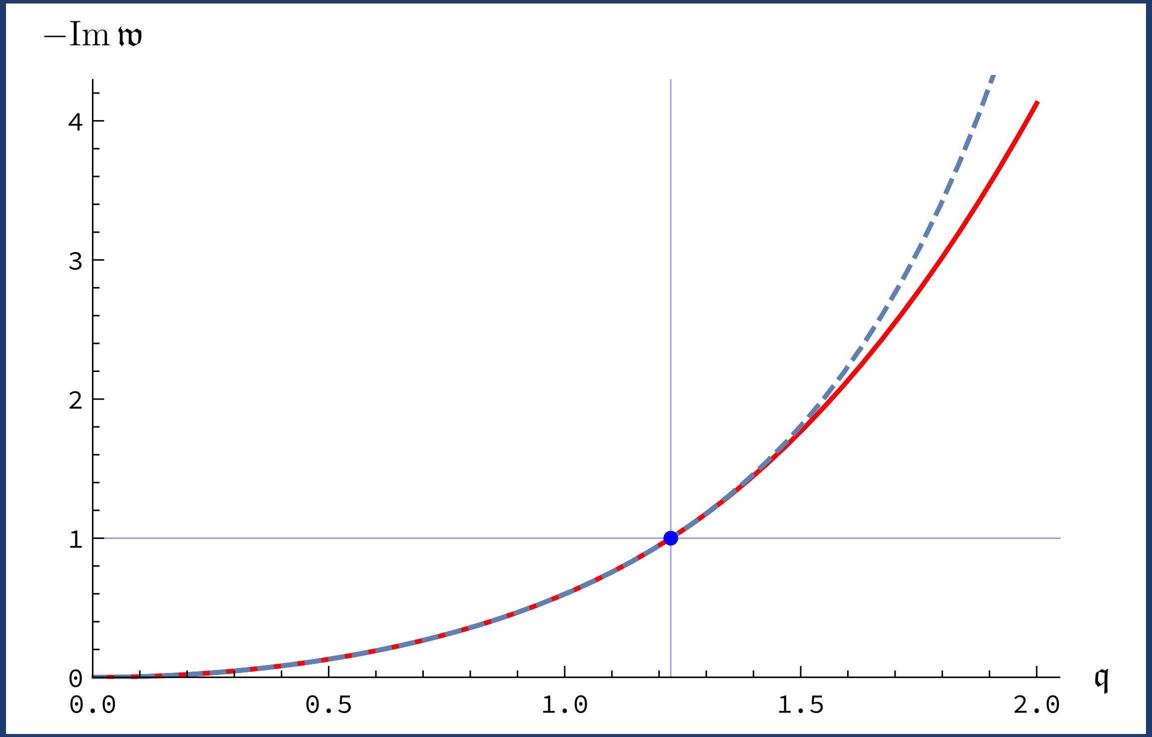
$q_0 \in \mathbb{R}$

[Grozdanov, Kovtun, Starinets, Tadić, JHEP (2019)]

$$\text{Lyapunov exponent}: \ \lambda_L = |\omega_c| = 2\pi T$$
$$\text{butterfly velocity}: \ v_B = |\omega_c / q_c|$$

shear (diffusion):

$$\langle T_{xz}(-\omega, -q_z), T_{xz}(\omega, q_z)\rangle$$



$$\omega = -\frac{i}{4\pi T}q^2 - \frac{i(1-\ln 2)}{32\pi^3 T^3}q^4 - \frac{i(24\ln^2 2 - \pi^2)}{96\,(2\pi T)^5}q^6 - \frac{i\left[2\pi^2(\ln 32 - 1) - 21\zeta(3) - 24\ln 2(1+\ln 2(\ln 32 - 3))\right]}{384\,(2\pi T)^7}q^8 + \cdots$$

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- in $\mathcal{N}=4$ SYM theory at infinite N_c and infinite coupling

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sound:

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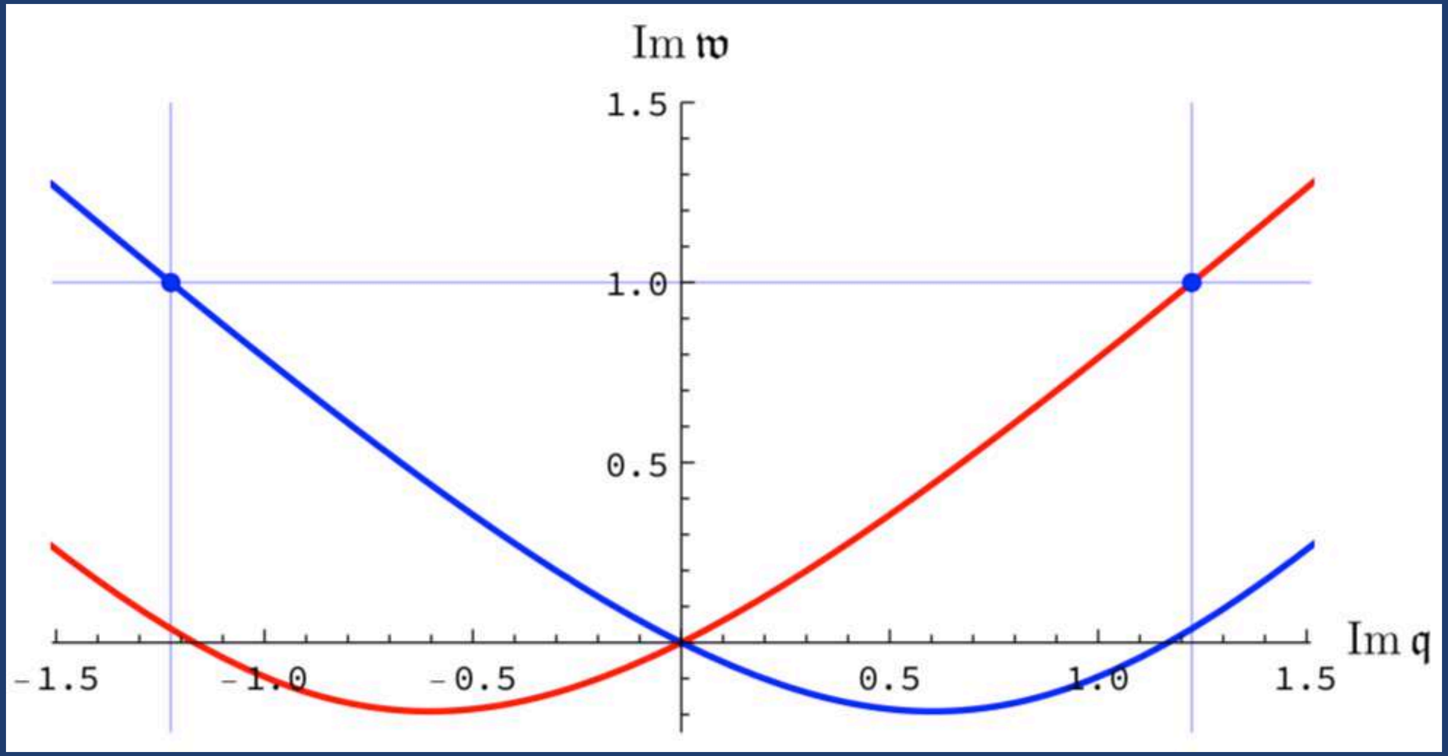


$$\omega = \pm \frac{1}{\sqrt{3}} q - \frac{i}{6\pi T} q^2 \pm \frac{3 - 2\ln 2}{24\sqrt{3}\pi^2 T^2} q^3 - \frac{i\left(\pi^2 - 24 + 24\ln 2 - 12\ln^2 2\right)}{864\pi^3 T^3} q^4 \pm \cdots$$

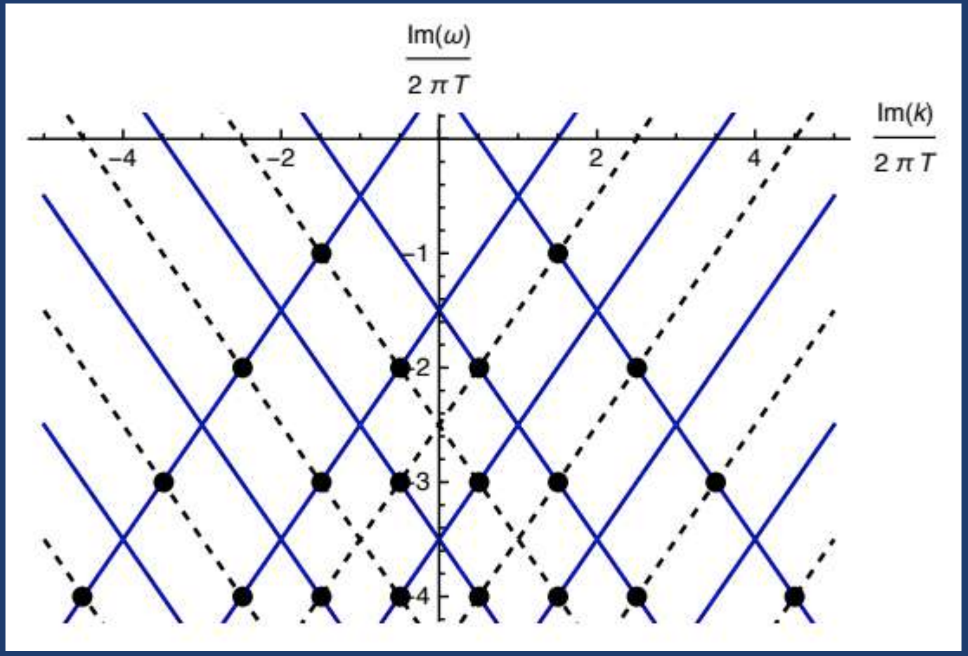
POLE-SKIPPING

- leading coupling corrections preserve pole-skipping [Grozdanov, JHEP (2019)]
- in general, multiple Lyapunov exponents, … ?
- at weak coupling (kinetic theory), existence of pole-skipping is unknown [Grozdanov, Schalm, Scopelliti, PRE (2018)]
- (holographic) CFTs exhibit an infinite tower of pole-skipping points (various operators) [Grozdanov, Kovtun, Starinets, Tadić, JHEP (2019); Blake, Davison, Vegh, JHEP (2019); …]
- frequencies are multiples of Matsubara frequencies
- pole-skipping imposes infinite constraints on the structures of field theory correlators
- example: 2*d* CFT dual to a 3*d* black hole

in $\mathcal{N}=4$ SYM:

$$\lambda_L = 2\pi T$$

$$v_B = \sqrt{\frac{2}{3}}\left(1+\frac{23\zeta(3)}{16}\frac{1}{\lambda^{3/2}}+\cdots\right)$$



[plot from Blake, Davison, Vegh, JHEP (2019)]

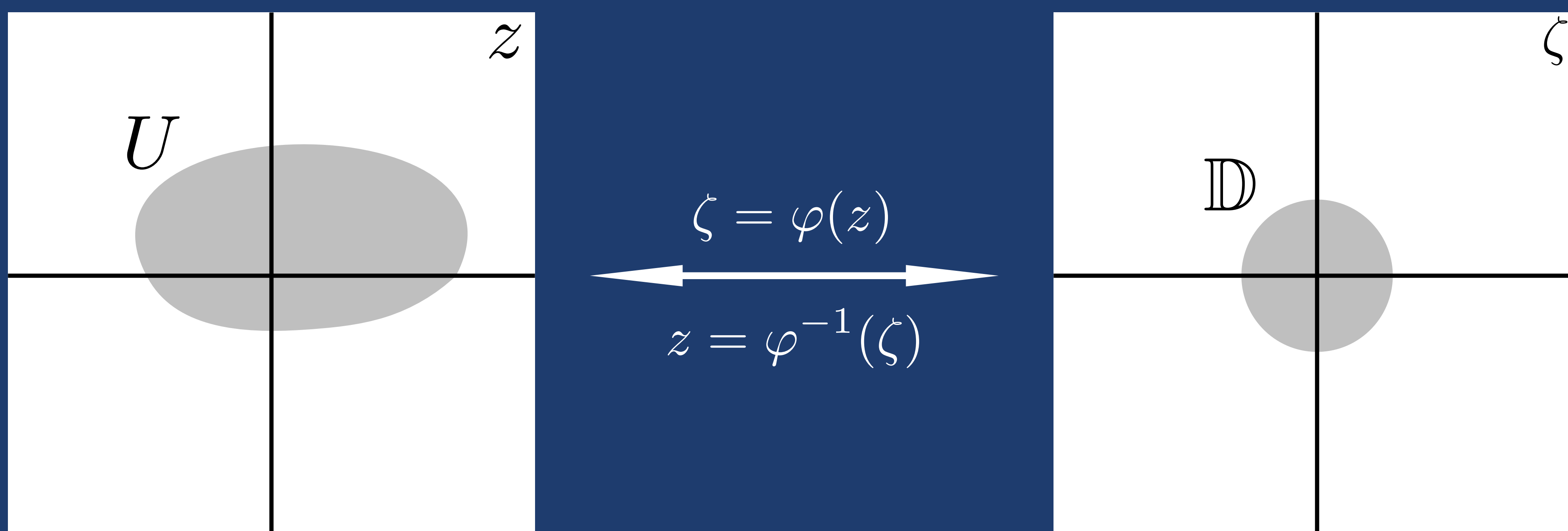
III. BOUNDS FROM UNIVALENCE

[Grozdanov, arXiv:2008.00888]

$$\lambda_L \leq 2\pi T \qquad (?) \implies (?) \qquad \frac{\eta}{s} \geq \mathcal{C}_? \frac{1}{4\pi}, \quad \mathcal{C}_? \frac{v_B^2}{\lambda_L} \leq D \leq \mathcal{C}_? v^2 \tau_{eq}, \quad v_s \leq \mathcal{C}_? \sqrt{\frac{1}{d-1}}$$

UNIVALENT FUNCTIONS

- univalent function $f(z),\ z \in \mathbb{C}$ is a complex, holomorphic and injective function
- injectivity: $f(z_1) \neq f(z_2)$ for all $z_1 \neq z_2$
- assume that $f(z)$ has a finite (open) region of univalence U
- the Riemann mapping theorem:



- function is now univalent on the open unit disk: $\mathbb{D} = \{\zeta \,|\, |\zeta| < 1\}$

$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n$$

UNIVALENT FUNCTIONS

recall:
$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} b_n \zeta^n$$

- holomorphic functions are "stiff", univalent function even more so…
- the growth theorem:

$$\frac{|\zeta|}{(1+|\zeta|)^2} \leq |f(\zeta)| \leq \frac{|\zeta|}{(1-|\zeta|)^2}$$

- the famous Bieberbach conjecture (1916), now de Branges's theorem (1985):

$$|b_n| \leq n, \quad \text{for all } n \geq 2$$

- when is $f(z)$ univalent and what is U?
- local univalence:

$$f'(z) \neq 0$$

- global univalence is tricky

$$\mathrm{Re}\, f'(z) > 0 \text{ in any convex } z \in U \subset \mathbb{C}$$

$f(z)$ is univalent if
$$|\{f(z), z\}| \leq \frac{2}{(1-|z|^2)^2}, \quad |z| < 1$$

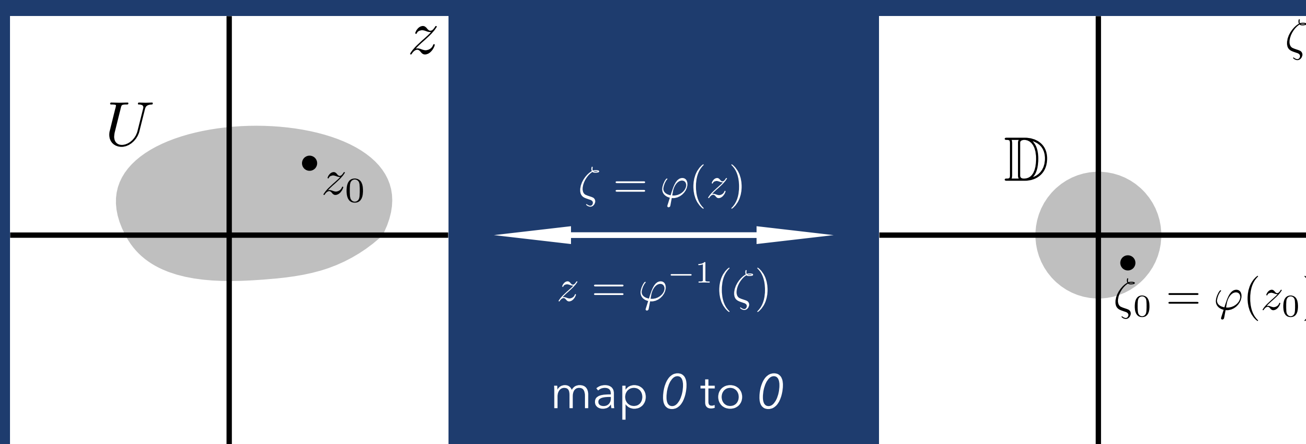
if $f(z)$ is univalent, then
$$|\{f(z), z\}| \leq \frac{6}{(1-|z|^2)^2}, \quad |z| < 1$$

- if $\mathrm{Re}\, f'(\zeta) > 0, \ \zeta \in \mathbb{D}$, then

$$-|\zeta| + 2\ln(1+|\zeta|) \leq |f(\zeta)| \leq -|\zeta| - 2\ln(1-|\zeta|)$$
$$|b_n| \leq 2/n, \quad \text{for all } n \geq 2$$

DIFFUSIVE DISPERSION RELATION

- diffusive Puiseux (Taylor) series $\omega_{\text{diff}}(z \equiv \mathbf{q}^2) = -i \sum_{n=1}^{\infty} c_n z^n, \quad c_1 = D$
- assume we know some $U \ \Rightarrow \ \varphi(z)$, and one value $\omega_0 \equiv \omega_{\text{diff}}(z_0), \ z_0 \in U$
- define: $f_{\text{diff}}(z) = i\omega_{\text{diff}}(z) \ \longleftrightarrow$ $f_{\text{diff}}(\zeta) \equiv \dfrac{i\omega_{\text{diff}}(\varphi^{-1}(\zeta))}{D\partial_\zeta \varphi^{-1}(0)} = \zeta + \sum_{n=2}^{\infty} b_n^{\text{diff}} \zeta^n$



- exact bounds immediately follow:

$$\frac{|\omega_0|\,(1-|\zeta_0|)^2}{|\zeta_0|\,|\partial_\zeta \varphi^{-1}(0)|} \le D \le \frac{|\omega_0|\,(1+|\zeta_0|)^2}{|\zeta_0|\,|\partial_\zeta \varphi^{-1}(0)|}$$

$$|b_n| \le n \ \Rightarrow \ \left| c_2 + \frac{D}{2} \frac{\partial_\zeta^2 \varphi^{-1}(0)}{[\partial_\zeta \varphi^{-1}(0)]^2} \right| \le \frac{2D}{|\partial_\zeta \varphi^{-1}(0)|}, \ \ldots$$

- stronger bounds (logs, ...) exist if $\text{Re}\, f'(\zeta) > 0, \ |\zeta| < 1$

DIFFUSIVE DISPERSION RELATION

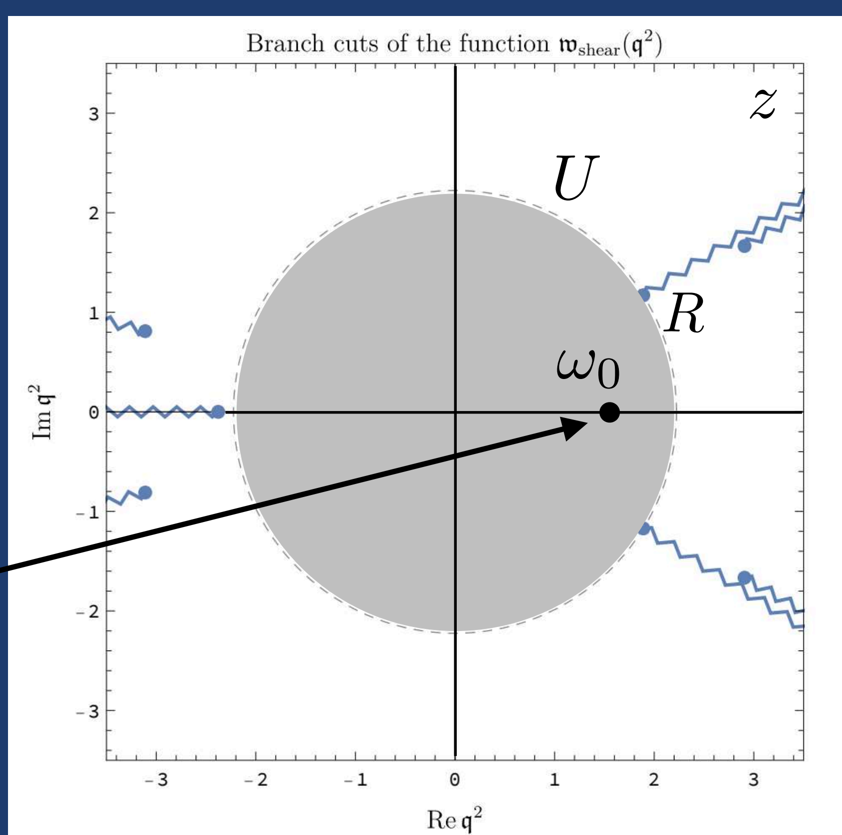
- are diffusive dispersion relations really univalent?

- function is holomorphic and invertible at $z = 0$ (Puiseux), hence locally univalent; a finite $U = \{z \,|\, |z| < \min[|z_g|, R]\}$ with group velocity $v_g = \partial\omega/\partial q$ and

$$\mathrm{Re}\, f'(\zeta) > 0 \implies z_g = q_g^2 \equiv \min q^2 \,|\, \mathrm{Re}\, v_g \,\mathrm{Im}\, q = \mathrm{Im}\, v_g \,\mathrm{Re}\, q$$

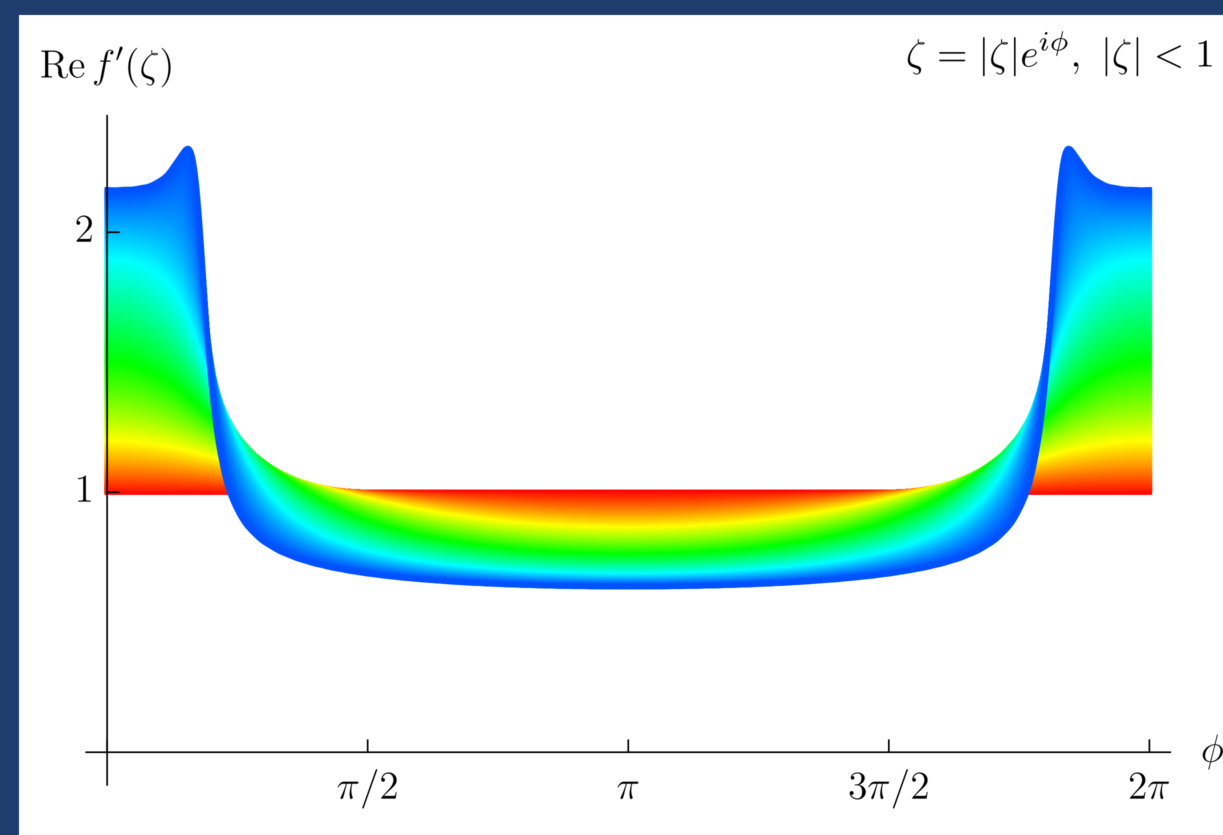
- example: $\mathcal{N} = 4$ SYM theory

$$0.046/T \leq D = 1/4\pi T \approx 0.080/T \leq 0.201/T$$



pole-skipping

Möbius transformation $\varphi : U \to \mathbb{D}$



SIMPLEST BOUND ON DIFFUSION

- assume a dispersion relation that is univalent everywhere except at a single branch cut
- example: self-dual axion model [Andrade, Withers (2013); Davison, Gouteraux (2014)]

energy diffusion: $$\omega(z=\mathbf{q}^2) = -i\pi T\left(1-\sqrt{1-\frac{z}{\pi^2T^2}}\right)$$

- optimal bounds (similar to Blake's proposal $D \gtrsim v_B^2/\lambda_L$):

energy diff:	$z_0 = -\frac{\lambda_L^2}{v_B^2} < 0:$	$\frac{v_B^2}{\lambda_L} \leq D \leq \frac{v_B^2}{\lambda_L} + \frac{\lambda_L}{R}$
momentum diff:	$0 < z_0 = \frac{\lambda_L^2}{v_B^2} < R:$	$\frac{v_B^2}{\lambda_L} - \frac{\lambda_L}{R} \leq D \leq \frac{v_B^2}{\lambda_L}$
higher orders:	$0 \leq c_2 \leq \frac{D}{R}, \quad \cdots$	

- infinite radius of convergence:

$$\omega_{\text{diff}}(\mathbf{q}^2) = -iD\mathbf{q}^2 = -i\frac{v_B^2}{\lambda_L}\mathbf{q}^2$$

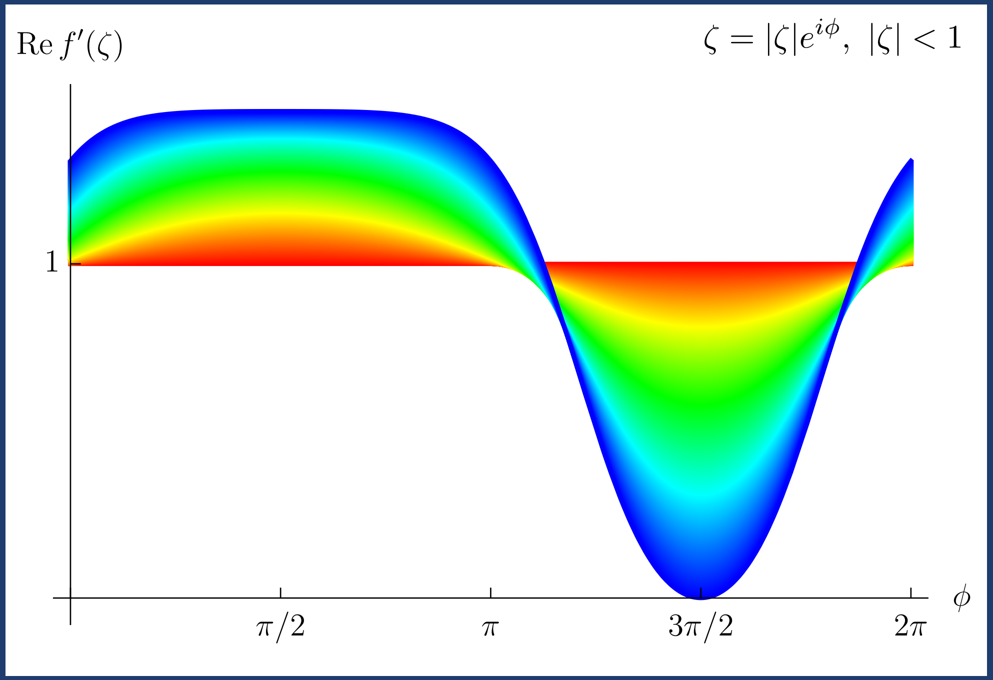
SOUND

- the story is analogous with univalence breakdown set by the local condition

$$f'(z) = 0 \implies v_g = 0$$

- example: $\mathcal{N}=4$ SYM theory

$$z_g = \sqrt{\mathbf{q}_g^2} \approx -3.791\, iT$$
$$z_g = \sqrt{\mathbf{q}_g^2} \approx -3iv_s/4D = -5.441\, iT$$



- construct a sufficient analyticity (univalence) condition for the conformal bound on the speed of sound [Cherman, Cohen, Nellore (2009); Hohler, Stephanov (2009)]

$$|\partial_\zeta \varphi^{-1}(0)| = 4\sqrt{3}|\omega_0(z_0)| \ \wedge \ |\zeta_0| = |\varphi(z_0)| \to 1 \quad \implies \quad 0 \le v_s \le \sqrt{\frac{1}{3}}$$

CONCLUSIONS AND FUTURE DIRECTIONS

CONCLUSIONS AND FUTURE DIRECTIONS

- complex analytic structures of transport can reveal new physical properties

- dispersion relations converge in momentum space, in x-space they sometimes converge and sometimes diverge [see Heller, Serantes, Spaliński, Svensson, Withers (2020)]

- pole-skipping imposes strong chaos constraints on transport and Green's functions
- what is the physical meaning of pole-skipping or "*0/0*" in QFT correlators?
- pole-skipping or its generalisation in weakly coupled QFTs and kinetic theory?
- chaos in heavy ion collisions?

- new methods that allow for rigorous derivations of lower and upper bounds on *all* coefficients of hydrodynamic dispersion relations
- we can find precise analytic conditions of a theory that lead to certain bounds
- further physical implications? equations of state vs. the conformal bound, …? [QCD, neturon starts, etc., see e.g. Annala, Gorda, Kurkela, Nättilä, Vuorinen (2020)]

- fluctuations, quantum corrections, long-time tails, "*1/N*" … what remains of this story?

THANK YOU!