On the sticky particle solutions to the pressureless Euler system in general dimension

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The pressureless Euler system

We consider the pressureless Euler system in $[0,1] \times \mathbb{R}^d$, $d \geq 2$

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho v) = 0 \\ \partial_t(\rho v) + \mathrm{div}(\rho v \otimes v) = 0 \\ \rho_{|_{t=0}} = \rho_0, \quad v_{|_{t=0}} = v_0 \end{cases} \tag{1}$$

where ρ is the distribution of particles and v is their velocity.

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where ρ is the distribution of particles and v is their velocity.

In particular we are interested in sticky particle solutions, namely solutions to (1) which satisfy the following adhesion principle: if two particles of fluid do not meet, they move freely keeping constant velocity, otherwise they join with velocity given by the balance of momentum.

Finite particle solutions

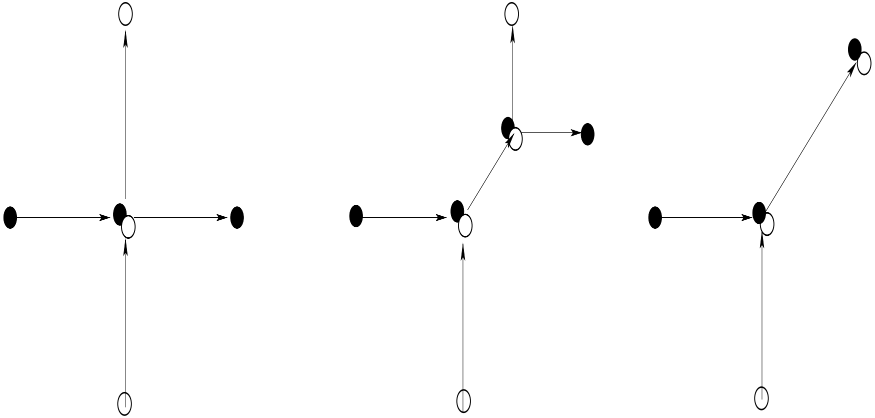


Figure: In the first and in the third figure the total kinetic energy is dissipated. In the second figure dissipation is violated. The third figure depicts a sticky particle solution. In particular, this is the maximally dissipating solution.

Finite sticky particle solutions

Given N particles

$$P_i(t) := (m_i, x_i(0), v_i(0)), \quad i = 1, \dots, N$$

one can always find a sticky particle solution in the following way. Particles evolve as $\dot{x}_i(t) = v_i(t)$, where $v_i(t) = v_i(0)$ up to the first time $\bar{t}$ such that there exists $j \neq i$ such that $x_j(\bar{t}) = x_i(\bar{t})$. Then, defining

$$J_i(t) = \{j : x_j(t) = x_i(t)\}$$

set

$$v_i(t) = \frac{\sum_{j \in J_i(t)} m_j v_j(\bar{t}-)}{\sum_{j \in J_i(t)} m_j}.$$

Then the particles $x_j(t)$ with $j \in J_i(t)$ join $x_j(t)$ after collision, the sets $J_i(t)$ are nondecreasing and their cardinality has a discontinuity in at most N times.

The one-dimensional case

In dimension $d = 1$, the collision of particles is a stable condition: slightly perturbing the initial position or velocity particles are going to collide.

- Grenier and indipendently E, Rykohv and Sinai proved existence of meaure valued solutions as limits of finite sticky particle solutions.
- Brenier and Grenier characterized the weak solutions ρ in terms of their cumulative distribution function $M_\rho(x) = \rho((-\infty, x])$ proving that $M(t, \cdot) = M_{\rho_t}(\cdot)$ is the unique monotone entropy solution of the scalar conservation law

$$\partial_t M + \partial_x A(M) = 0,$$

where A is a continuous flux function depending on ρ_0 and v_0. In particular one obtains the L^1-stability and convergence of numerical schemes and vanishing viscosity solutions.

A variational viewpoint

- Let X_ρ be the pseudo-inverse of the distribution function M_ρ. X_ρ is a monotone function.
  Natile and Savaré proved that the solution X_{ρ_t} (found by approximation and convergence through Lipschitz estimates in the Wasserstein space) admits the following characterization

$$X_{\rho_t} = P_K(X_{\rho_0} + tV_0), \quad V_0 = v_0 \circ X_{\rho_0},$$

$$\frac{d}{dt}X_{\rho_t} + \partial I_K(X_{\rho_t}) \ni V_0, \quad t\frac{d}{dt}X_{\rho_t} + \partial I_K(X_{\rho_t}) \ni X_{\rho_t} - X_{\rho_0},$$

  I_K and P_K denoting respectively the indicator function and the projection operator on the cone of monotone maps K. The velocity v_t is uniquely determined by the relation $\frac{d}{dt}X_{\rho_t} = v_t \circ X_{\rho_t}$.
- Cavalletti, Sedjro and Westdickenberg gave an alternative proof of the existence of sticky particle solutions using directly the above characterization.

The multi-dimensional case

The main problem in a stability result that would allow to find sticky particle solutions from general initial data starting from finite particle solutions is that the collision condition is not stable if $d \geq 2$.

Counterexamples to existence in dimension $d \geq 2$

Bressan and Nguyen produced the following counterexample to existence of sticky particle solutions in dimension $d = 2$. First, construct the following one-dimensional sticky solution.

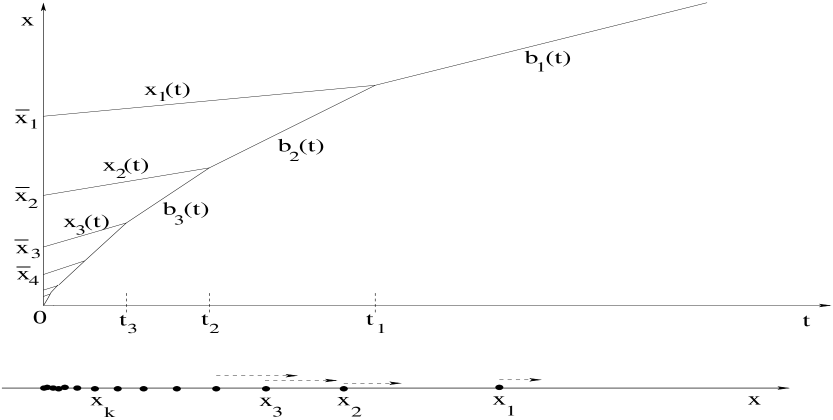


Figure: A sticky solution containing infinitely many particles moving on the x-axis

One gets the following two-dimensional counterexample.

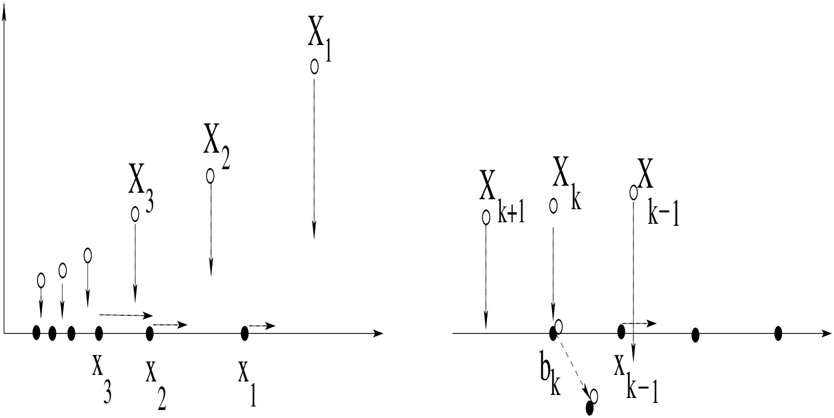
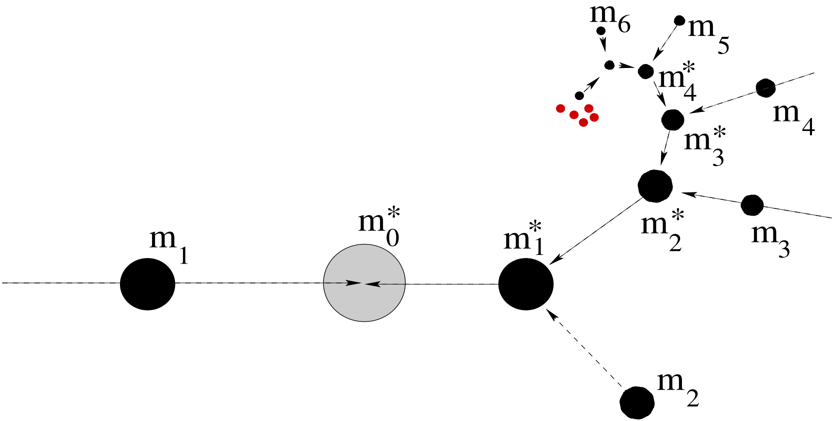


Figure: If the white particle X_k hits the black particle x_k at time τ_k and sticks to it removing it from the x-axis, then it means than no particle x_j with $j > k$ has been hit and by white particles and removed from the x-axis. On the other hand, if no white particle X_j with $j > k$ hits a target particle x_j and removes it from the x-axis, then the white particle X_{k+1} hits its target.

Counterexamples to uniqueness in dimension $d \geq 2$

Bressan and Nguyen produced the following counterexample to uniqueness of sticky particle solutions in dimension $d = 2$.



A variational scheme for measure-valued solutions

Cavalletti, Sedjro and Westdickenberg introduced a variational in time discretization in the spirit of minimizing movements scheme and show, in dimension $d \geq 2$, the existence of Young-measure solutions to the pressureless Euler system and to the full compressible Euler system.

For the pressureless Euler system the approximating discrete in time sequence is sticky, but no sticky particle property for the limit trajectories is provided.

Main questions

- Does there exist a “large” set of initial data for which the Cauchy problem for the pressureless Euler system in dimension $d \geq 2$ in the class of sticky particle solutions is well-posed?

- How large is the dissipation of such sticky particle solutions? Do they fulfil a maximal dissipation criterion?

Dissipative solutions

In order to answer the previous questions, we will relax the notion of sticky particle solution allowing that particles meet without interacting, namely without changing their speed.

However, if particles interact at some time they will start to move with velocity given by the balance of momentum and they will stick together after that time.

The relaxing is then due to the fact that the condition of passing trough the same point (t, x) does not mean that the particles are interacting.

We will call such solutions dissipative solutions.

A Lagrangian notion of solution

In order to encode the sticky/dissipative property for general solutions we use a lagrangian notion of solution.
Define the space of curves with finite energy

$$\Gamma := \left\{ \gamma \in W^{1,2}\big((-1,1), \mathbb{R}^d\big) : \dot{\gamma}\llcorner_{(-1,0)} \text{ constant} \right\}.$$

with a metric d_Γ which metrizes the strong L^2 topology on $W^{1,2}$-bounded sets. To every $\gamma \in \Gamma$ we associate the initial velocity field

$$V_0(\gamma) = \gamma(0) - \gamma(-1).$$

The solutions to the pressureless Euler equations are a subset of $\mathcal{M}(\Gamma)$, where

$$\mathcal{M}(\Gamma) = \left\{ \eta \in \mathcal{P}(\Gamma) : \int |\gamma(0)|^2 \eta(d\gamma) \leq 1, \int \|\dot{\gamma}\|^2_{L^2(-1,1)} \eta(d\gamma) \leq 1 \right\}.$$

$\mathcal{M}(\Gamma)$ is compact w.r.t. the weak topology.

For any $t \in (0,1)$, define

$$\Gamma_t := W^{1,2}\big((t,1),\mathbb{R}^d\big).$$

Let $R_t : \Gamma \to \Gamma_t$ be the restriction map

$$R_t(\gamma) = \gamma \llcorner_{(t,1)}.$$

The map R_t induces and equivalence relation on Γ and every $\eta \in \mathcal{M}(\Gamma)$ has a unique disintegration

$$\eta = \int \omega^t_{\gamma'} \eta_t(d\gamma'), \qquad \eta_t = (R_t)_\sharp \eta,$$

where $\omega^t_{\gamma'} \in \mathcal{M}(\Gamma)$ satisfies $\omega^t_{\gamma'}(R_t^{-1}(\gamma')) = 1$.

Dissipative and sticky particle solutions

Define, for all $t \in [0, 1]$

$$V_t(\gamma) := \int V_0(\gamma')\omega^t_{R_t(\gamma)}(d\gamma').$$

Definition

We say that $\eta \in \mathcal{M}(\Gamma)$ is a dissipative solution of the pressureless Euler equations if it holds

$$\dot{\gamma}(t) = V_t(\gamma) \qquad (\mathcal{L}^1 \times \eta)\text{-a.e. on } (-1, 1) \times \Gamma.$$

Definition

We say that $\eta \in \mathcal{M}(\Gamma)$ is a sticky particle solution of the pressureless Euler equations if it is dissipative and moreover it is concentrated on a subset of Γ on which, for all $t \in [0, 1]$, the maps R_t and e_t induce the same equivalence relation, being e_t the evaluation map $e_t : \Gamma \to \mathbb{R}^d$, $e_t(\gamma) = \gamma(t)$.

Energy dissipation

Proposition

Let $\Psi : \mathbb{R}^d \to \mathbb{R}$ be a convex function and let $\eta \in \mathcal{M}(\Gamma)$ be a dissipative solution. Then, the map

$$t \mapsto \int \Psi(V_t(\gamma))\eta(d\gamma)$$

is nonincreasing. In particular, taking $\Psi = |\cdot|^2$ one has that $\forall\, s \le t, s, t \in [0,1]$

$$\int |V_s(\gamma) - V_t(\gamma)|^2 \eta(d\gamma) = \int |V_s(\gamma)|^2 \eta(d\gamma) - \int |V_t(\gamma)|^2 \eta(d\gamma).$$

Moreover, the map $t \mapsto V_t$ belongs to $BV^{1/2}([0,1]; L^2_\eta(\Gamma; \mathbb{R}^d))$ and it is right continuous.

Main results

Definition (Initial data for sticky particle solutions)

Let $V_0 : \Gamma \to \mathbb{R}^d$ be the continuous map defined by $V_0(\gamma) = \gamma(0) - \gamma(-1)$. We say that a vector valued measure ν_0 on $\mathbb{R}^d \times \mathbb{R}^d$ is an initial data of a dissipative solution η if $\nu_0 = (e_0, V_0)_{\#}\eta$.

Theorem (Bianchini, D. 2020)

For every initial data as above there exists at least one dissipative solution.
The set of dissipative solutions is a compact closed set w.r.t. the weak L^2 topology.
The family of finite sticky particle solutions is dense in the set of dissipative solutions.

Main results

Definition (Free flow)

We say that a dissipative solution $\eta \in \mathcal{M}(\Gamma)$ is a free flow if η is concentrated on a set of straight lines with empty mutual intersection.

Theorem (Bianchini, D. 2020)

There is a set $D_0 \subset \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ such that, for any $\nu_0 \in D_0$ there exists a unique dissipative solution η with initial data ν_0 and it is given by a free flow. Such a set is a G_δ and dense w.r.t. the weak topology on $\mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$. Moreover, the corresponding solutions are measure solutions to the pressureless Euler system.

A G_δ set of zero dissipation data

- Since $d \geq 2$, given a finite sticky particle solution it is easy to perturb the initial datum $(e_0, V_0)_\# \eta \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ slightly in order to have data generating uniquely a free flow whose trajectories have mutual distance $0 < \delta \ll 1$.

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- Define the total dissipation of a measure $\eta \in \mathcal{M}(\Gamma)$ as

$$D(\eta) = \|V_0\|^2_{L^2_\eta} - \int \int_0^1 |V_t(\gamma)|^2 dt \eta(d\gamma). \tag{2}$$

Such data generating uniquely a free flow belong to the set

$$D_0 = \left\{\mu \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d) : \ D(\eta) = 0 \, \forall \, \eta \in \mathcal{M}(\Gamma) \text{ dissipative s.t. } V_{0\#}\eta = \mu\right\}.$$

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- By compactness of dissipative solutions and upper semicontinuity of the total dissipation $D(\eta)$, the sets

$$D_{1/k} = \Big\{\mu \in \mathcal{P}_2(\mathbb{R}^d) : \ D(\eta) < 1/k \, \forall \, \eta \in \mathcal{M}(\Gamma) \text{ dissipative s.t. } V_{0\#}\eta = \mu\Big\}.$$

  are open and $D_0 = \cap_{k \in \mathbb{N}} D_{1/k}$.

Further steps

1. Compactness of dissipative solutions,
2. Density of finite sticky particle solutions.

The second step is proved through a series of approximation procedures:

i) Discrete in time approximation,
ii) Countable sticky particle solutions,
iii) Finite sticky particle solutions.

Compactness of dissipative solutions

Issue: To pass to the limit in the relation

$$\dot{\gamma}(t) = V_t(\gamma) = \int V_0(\gamma')\omega^{n,t}_{R_t(\gamma)}(d\gamma') \qquad (\mathcal{L}^1 \times \eta^n)\text{-a.e. on } (-1,1) \times \Gamma.$$

as η^n converge weakly to η.

Discrete in time approximation

Definition (Discrete in time dissipative solutions)

A dissipative solution $\eta \in \mathcal{M}(\Gamma)$ is discrete in time if there exists a finite partition $0 = t_0 < t_1 < \cdots < t_N = 1$ such that $V_t = V_{t_i}$ for all $t \in [t_i, t_{i+1})$. In particular, the trajectories followed by the particles are piecewise affine with speeds possibly changing at times $t_0, \ldots t_N$.

Proposition

Let $\eta \in \mathcal{M}(\Gamma)$ be a dissipative solution and $\epsilon > 0$. Then there exists $\tilde{\eta}^\epsilon = \tilde{F}^\epsilon_\# \eta \in \mathcal{M}(\Gamma)$ discrete in time dissipative solution such that

$$V_0(\tilde{F}^\epsilon(\gamma)) = V_0(\gamma), \quad \|\tilde{V}^\epsilon_t \circ \tilde{F}^\epsilon - V_t\|^2_{L^2_\eta} \leq \epsilon^2, \quad \int |\tilde{F}^\epsilon(\gamma)(t) - \gamma(t)|^2 \eta(d\gamma) \leq \epsilon.$$

In particular, as $\epsilon \to 0$, the measures η^ϵ converge in $\mathcal{M}(\Gamma)$ to η.

Since $t \mapsto V_t$ belongs to $BV^{1/2}([0,1]; L^2_\eta(\Gamma; \mathbb{R}^d))$ and it is right continuous. Therefore, given $\epsilon > 0$, there exists a finite partition $0 = t_0 < t_1 < \cdots < t_N = 1$ such that either

$$\|V_{t_i}\|^2_{L^2_\eta} - \|V_{t_{i+1}}\|^2_{L^2_\eta} < \epsilon^2 \tag{3}$$

or

$$\|V_{t_i}\|^2_{L^2_\eta} - \lim_{s \nearrow t_{i+1}} \|V_s\|^2_{L^2_\eta} < \epsilon^2. \tag{4}$$

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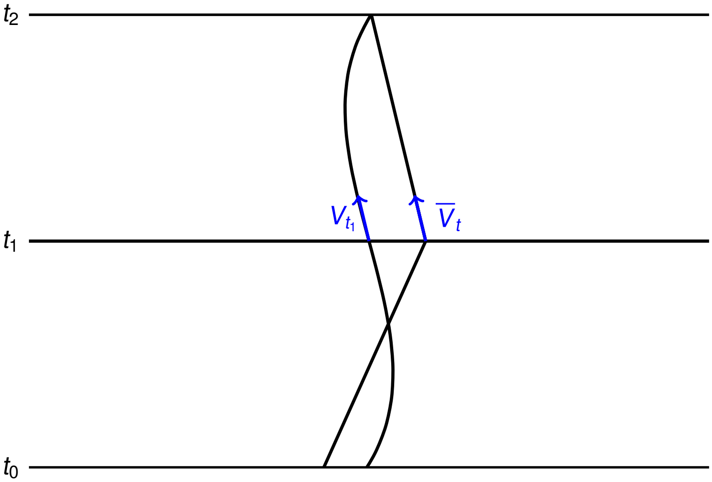
Define $\bar{V}_t = V_{t_i}$ if $t \in [t_i, t_{i+1})$. In particular, $\|\bar{V}_t - V_t\|_{L^2_\eta} \leq \epsilon$. Define

$$\tilde{F}^\epsilon : \Gamma \to \Gamma, \quad \tilde{F}^\epsilon(\gamma(t)) = \gamma(1) - \int_t^1 \bar{V}_s(\gamma) ds.$$

Then, $\tilde{F}^\epsilon_\# \eta$ is a dissipative solution satisfying the required estimates.


t_2
t_1
t_0
V_{t_1}
$\overline{V}_t$

Countable and finite particle approximation

Proposition

Let η be a discrete in time dissipative solution and $\delta > 0$. Then there exists $\hat{\eta}^\delta = \hat{F}^\delta_\# \eta$ discrete in time dissipative solution such that

$$|V_t(\hat{F}^\delta(\gamma)) - V_t(\gamma)| \leq \delta, \quad |\hat{F}^\delta(\gamma)(t) - \gamma(t)| \leq \delta. \tag{5}$$

Moreover, $\hat{\eta}^\delta$ is a dissipative countable particle solution, namely it is concentrated on a countable number of trajectories.

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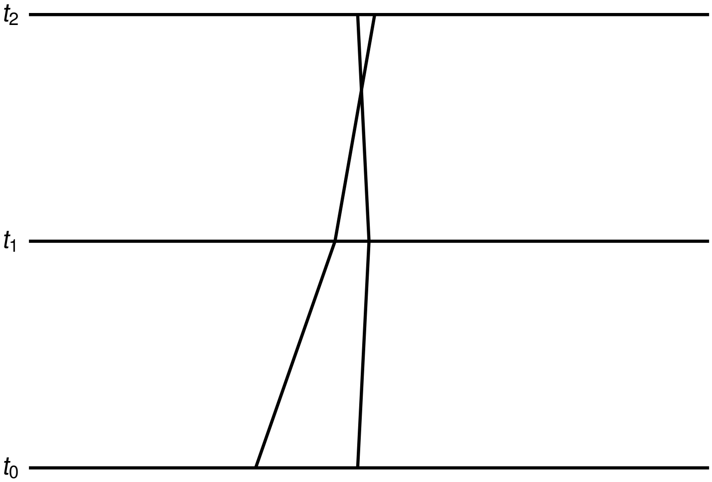
Proposition

Let $\eta \in \mathcal{M}(\Gamma)$ be a dissipative countable particle solution. Then, for every $\sigma > 0$, there exists η^σ finite sticky particle solution with the property that $\eta^\sigma \rightharpoonup \eta$ as $\sigma \to 0$.


t_2
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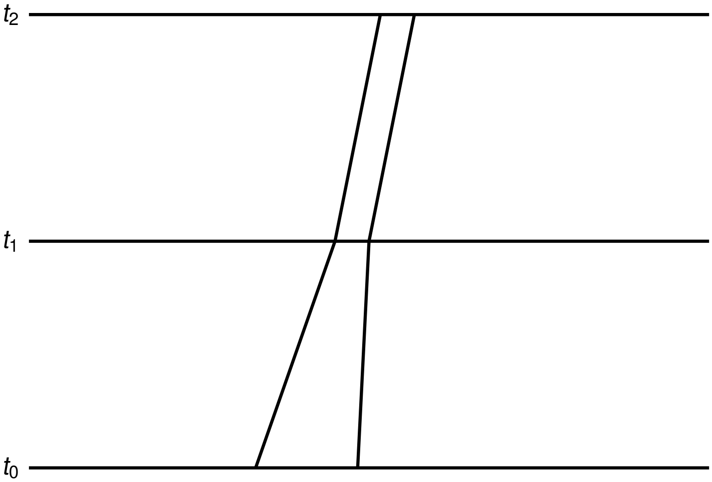

t_2
t_1
t_0

Conclusions

- We prove that for a comeager set of initial data in the weak topology the pressureless Euler system admits a unique sticky particle solution given by a free flow where trajectories are disjoint straight lines.
- Such an existence and uniqueness result holds for a broader class of solutions decreasing their kinetic energy, which we call dissipative solutions, and which turns out to be the compact weak closure of the classical sticky particle solutions.
  Therefore any scheme for which the energy is l.s.c. and is dissipated will converge, for a comeager set of data, to the solution given by the free flow.

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