Homogeneity results for invariant distributions on p-adic symmetric spaces

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Characters

$\mathrm{G}(F)$ p-adic reductive group: F local field, G/F connected reductive.

Character of $\pi : \mathrm{G}(F) \to \mathrm{GL}_{\mathbb{C}}(V)$, finite length smooth representation,

is the $\mathrm{Int}\,\mathrm{G}(F)$-invariant distribution:

$$\Theta_\pi : C_c^\infty(\mathrm{G}(F)) \to \mathbb{C},$$

$$f \mapsto \operatorname{tr}\left(v \mapsto \int_{\mathrm{G}(F)} f(g)\pi(g) \cdot v \, dg\right) \qquad \text{(in short, } f \mapsto \operatorname{tr}\pi(f)\text{)}.$$

Fact (Harish-Chandra, Raphaël Beuzart-Plessis):

$$\Theta_\pi(f) = \int_{\mathrm{G}(F)} \Theta_\pi(g) f(g)\, dg, \quad \Theta_\pi \in L^1_{\mathrm{loc}}(\mathrm{G}(F)), \quad \text{locally const. on } \mathrm{G}(F)_{\mathrm{reg}}.$$

Why characters?

- Have linear independence of characters.
- Crucially used in trace formula, endoscopy theory of L-packets, work on Gan-Gross-Prasad conjectures etc.
- Characters reflect properties: e.g., growth of Θ_π near $1 \in \mathsf{G}(F)$ tells us how "big" of a representation π is, existence of Whittaker models etc.
- Study characters near 1, by pulling them down to Ad $\mathsf{G}(F)$-invariant distributions $\mathfrak{g}(F) := \operatorname{Lie}\mathsf{G}(F)$.

$$\mathfrak{g}(F) \overset{\text{open}}{\supset} \underset{\substack{\Psi \\ 0}}{\mathcal{U}} \overset{\exp}{\to} \underset{\substack{\Psi \\ 1}}{\mathcal{V}} \overset{\text{open}}{\subset} \mathsf{G}(F).$$

$\Theta_\pi|_{\mathcal{V}} \rightsquigarrow \theta_\pi := \Theta_\pi|_{\mathcal{V}} \circ \exp$ on $\mathcal{U}$ or on $\mathfrak{g}(F)$.

- "Slogan:" Lie algebra analogue of characters = Fourier transforms of orbital integrals, $\hat{\mu}_{\mathcal{O}_X}$, $X \in \mathfrak{g}^*$ (defined in next two slides).

Fourier transforms

Fourier transform on a f.d.v.s $V/\mathbb{R}$:

$$\hat{f}(x) = \int_V f(y) \exp(i\langle x, y\rangle)\, dy,$$

$\langle\cdot,\cdot\rangle$ Euclidean inner product.

Similarly, $C_c^\infty(\mathfrak{g}(F)) \ni f \mapsto \hat{f} \in C_c^\infty(\mathfrak{g}(F))$:

$$\hat{f}(X) = \int_{\mathfrak{g}(F)} f(Y)\, \psi(B(X, Y))\, dY,$$

- $B : \mathfrak{g} \times \mathfrak{g} \to F$, symmetric nondegenerate Ad G-invariant bilinear form;
- $\psi : F \to \mathbb{C}^\times$ nontrivial additive character.

Better: $C_c^\infty(\mathfrak{g}(F)) \ni f \mapsto \hat{f} \in C_c^\infty(\mathfrak{g}^*(F)), \quad X \mapsto \int_{\mathfrak{g}(F)} f(Y)\, \psi(\langle X, Y\rangle)\, dY$.

Fourier transform for distributions: $\hat{T}(f) := T(\hat{f})$.

Orbital integrals

Orbital integral: if $\mathcal{O} := \operatorname{Ad} \mathsf{G}(F)(X) =$ adjoint orbit of $X \in \mathfrak{g}(F)$,

$$O_X(f) := \mu_{\mathcal{O}}(f) := \int_{\mathsf{G}(F)/\mathsf{G}^X(F)} f(\operatorname{Ad} g(X)) \frac{dg}{dg^X} = \int_{\mathcal{O} := \operatorname{Ad} \mathsf{G}(F)(X)} f(X) d\mu_{\mathcal{O}}.$$

This is well-defined if F has characteristic zero (or if $p \gg 0$):

- The centralizer $\mathsf{G}^X(F) = \{g \in \mathsf{G}(F) \mid \operatorname{Ad} g(X) = X\}$ is unimodular $\Rightarrow$ have a $\mathsf{G}(F)$-invariant measure $dg/dg^X = d\mu_{\mathcal{O}}$ on $\mathcal{O}$;
- $\forall f \in C_c^\infty(\mathfrak{g}(F))$, the integral $\mu_{\mathcal{O}}(f)$ conv. (Ranga Rao if $\mathrm{char}\, F = 0$).

Of partic. interest: when $\mathcal{O} \in \mathcal{O}(0) := \{$nilpotent $\operatorname{Ad} \mathsf{G}(F)$-orbits in $\mathfrak{g}(F)\}$,
i.e., when $\mathcal{O} \subset \mathcal{N}(\mathfrak{g}) :=$ the set of nilpotent elements of $\mathfrak{g}(F)$,
i.e., when $0 \in \bar{\mathcal{O}}^{\mathrm{Hausdorff}} = \overline{\operatorname{Ad} \mathsf{G}(F) \cdot X}^{\mathrm{Hausdorff}}$.

Fact: $J(\mathcal{N}(\mathfrak{g})) :=$ the set of $\operatorname{Ad} \mathsf{G}(F)$- invariant distribns supported on $\mathcal{N}(\mathfrak{g})$ $= \bigoplus_{\mathcal{O} \in \mathcal{O}(0)} \mathbb{C} \cdot \mu_{\mathcal{O}} =$ sum of nilpotent orbital integrals.

Similarly, have nilp. orb. int. on $\mathfrak{g}^*(F) := \operatorname{Hom}(\mathfrak{g}(F), F)$ with "Ad*"-action.

Fourier transforms of orbital integrals

Set $\hat{O}_X := \hat{\mu}_{\mathcal{O}} :=$ the Fourier transform of $\mu_{\mathcal{O}}$.

$$\mathfrak{g}(F) \overset{\text{open}}{\supset} \underset{\substack{\Downarrow \\ 0}}{\mathcal{U}} \overset{\exp}{\to} \underset{\substack{\Downarrow \\ 1}}{\mathcal{V}} \overset{\text{open}}{\subset} \mathsf{G}(F)$$

$\Rightarrow (\Theta_\pi : C_c^\infty(\mathsf{G}(F)) \to \mathbb{C}) \rightsquigarrow (\Theta_\pi|_{\mathcal{V}} : C_c^\infty(\mathcal{V}) \to \mathbb{C}) \overset{\exp}{\rightsquigarrow} (\theta_\pi : C_c^\infty(\mathcal{U}) \to \mathbb{C})$,

or $\theta_\pi : C_c^\infty(\mathfrak{g}(F)) \to \mathbb{C}$, supported on $\mathcal{U}$. Informally: $\theta_\pi = \Theta_\pi \circ \exp$.

Howe-Harish-Chandra character expn: $\mathcal{U}$ sufficiently small $\Rightarrow \forall f \in C_c^\infty(\mathcal{U})$:

$$\theta_\pi(f) := \Theta_\pi(f \circ \log) = \sum_{\mathcal{O} \in \mathcal{O}(0)^*} \underbrace{c_{\mathcal{O}}}_{\in \mathbb{C}} \cdot \hat{\mu}_{\mathcal{O}}(f),$$

if we assume that F has characteristic zero.

In other words: $\theta_\pi|_{\mathcal{U}} \in J(\widehat{\mathcal{N}(\mathfrak{g}^*)})|_{\mathcal{U}}$.

Question: How big can $\mathcal{U}$ be so this holds ("range of validity of the character expansion")?

Crude comments on the Howe-Harish-Chandra argument

H.-C. proves the expansion, more generally, for "admissible distributions" Θ.

If $K_0 \subset \mathrm{G}(F)$ compact open subgroup such that $\pi^{K_0} \neq 0$ (say π irreducible):

$$\pi|_{K_0} = \bigoplus_{\xi \in \mathrm{Irr}(K_0)} n_\xi \, \xi,$$

where $n_\xi = 0$ unless $\exists\, g \in \mathrm{G}(F)$ s.t. ξ has a nonz. $gK_0g^{-1} \cap K_0$-fixd vector.

Rephrase: $\Theta_\pi * \xi = 0$ unless $\exists\, g$ s.t. ξ has a nonz. $gK_0g^{-1} \cap K_0$-fixd vector.

Very crude idea, if Θ satisfies this: This forces the Fourier transform of $\theta = \Theta \circ \exp$, near 0, to have support near nilpotent cone $\mathcal{N}(\mathfrak{g}^*)$.

- Given π with a K_0-fixed vector, the above gives information on $\pi|_{K_0}$.
- Rep. th. of $K_0 = \exp(L_0) \leftrightarrow K_0$-orbits on $\mathfrak{g}^*/L_0^\perp$ (Howe's Kirillov th.).
- Information about Θ_π near 1, or θ_π near 0.

Recall the question: How big can $\mathcal{U}$ be so the character expansion holds?

Moy-Prasad filtrations

If $x \in \mathcal{B}(\mathrm{G})$ = (enlarged) Bruhat-Tits building of G, have Moy-Prasad filtration subgroups (for $r \geqslant 0$) and sublattices (for $r \in \mathbb{R}$):

$$\mathrm{G}(F)_{x,r} \subset \mathrm{G}(F), \qquad \mathfrak{g}(F)_{x,r} \subset \mathfrak{g}(F), \qquad \mathfrak{g}^*(F)_{x,r} \subset \mathfrak{g}^*(F).$$

Exm: If $\mathrm{G} = \mathrm{GL}(V)/F$, x is given by a $\downarrow$ filtration $\{L_s\}_{s\in\mathbb{R}}$ of lattices in V:

$$\mathfrak{g}(F)_{x,r} = \{X \in \mathrm{End}(V) \mid X(L_s) \subset L_{s+r} \,\forall\, s\}.$$

Have Moy-Prasad G-domains:

$$\mathrm{G}(F)_r := \bigcup_{x\in\mathcal{B}(\mathrm{G})} \mathrm{G}(F)_{x,r} \overset{\text{open}}{\subset} \mathrm{G}(F), \qquad \mathfrak{g}(F)_r := \bigcup_{x\in\mathcal{B}(\mathrm{G})} \mathfrak{g}(F)_{x,r} \overset{\text{open}}{\subset} \mathfrak{g}(F),$$

similarly $\mathfrak{g}^*(F)_r$. Open, closed, $\mathrm{G}(F)$-invariant, and have

$$\mathfrak{g}(F)_r = \bigcap_{x\in\mathcal{B}(\mathrm{G})} \left(\mathfrak{g}(F)_{x,r} + \mathcal{N}(\mathfrak{g})\right)$$

(Part of Moy-Prasad theory, take by Adler and DeBacker).

$$\exp : \mathfrak{g}(F)_{x,r} \to \mathrm{G}(F)_{x,r}, \quad \mathfrak{g}(F)_r \to \mathrm{G}(F)_r, \qquad \text{if } p \gg 0.$$

$$\mathfrak{g}(F)_{x,r+} := \bigcup_{s>r} \mathfrak{g}(F)_{x,s}, \qquad \mathfrak{g}(F)_{r+} := \bigcup_{x\in\mathcal{B}(\mathrm{G})} \mathfrak{g}(F)_{x,r+}, \quad \text{etc.}$$

Moy-Prasad Depth, consequences for $\hat{\theta}_\pi$

Definition (Moy and Prasad)

The depth of π is the smallest $\rho(\pi) \geqslant 0$ such that π has a nonzero $\mathsf{G}(F)_{x,\rho(\pi)+}$-fixed vector, for some $x \in \mathcal{B}(\mathsf{G})$.

Conjecture (Hales-Moy-Prasad)

For "good" p, the character expn. for $\theta_\pi = \Theta_\pi \circ \exp$ holds on $\mathfrak{g}(F)_{\rho(\pi)+}$, i.e., the "range of validity" of character expansion contains $\mathfrak{g}(F)_{\rho(\pi)+}$.

DeBacker: True if $p \gg 0$. (Applies more gen. to "r-admissible distribns").

$$(\mathsf{G}(F)_{x,r}/\underset{\substack{\uplus\\ \xi}}{\mathsf{G}}(F)_{x,r+})^\wedge \overset{\log}{=} (\mathfrak{g}(F)_{x,r}/\mathfrak{g}(F)_{x,r+})^\wedge \cong \mathfrak{g}^*(F)_{x,-r}/\underset{\substack{\uplus\\ \hat{\xi}}}{\mathfrak{g}^*}(F)_{x,(-r)+}.$$

Moy-Prasad $\Rightarrow$ For $r > \rho(\pi)$, if ξ occurs in π, then $\hat{\xi} \cap \mathcal{N}(\mathfrak{g}^*) \neq 0$.
Thus, if $\hat{\theta}_\pi(\mathbb{1}_{\hat{\xi}}) \neq 0$, then $\hat{\xi} \cap \mathcal{N}(\mathfrak{g}^*) \neq 0$.

Gener.: $f \in C_c(\mathfrak{g}^*(F)_{x,-s}/\mathfrak{g}^*(F)_{x,(-r)+})$ ($\hat{f} \in C_c(\mathfrak{g}(F)_{x,r}/\mathfrak{g}(F)_{x,s+})$), $s \geqslant r$, if $\hat{\theta}_\pi(f) \neq 0$, then $\mathrm{supp}\ f$ intersects $\mathfrak{g}^*(F)_{x,(-s)+} + \mathcal{N}(\mathfrak{g}^*)$ (uses exp).

Steps

Rename $r \rightsquigarrow -r$, $s \rightsquigarrow -s$. Have $T = \hat{\theta}_\pi$ distribution on

$$\mathrm{FT}(C_c^\infty(\mathfrak{g}(F)_{-r})) = \sum_{x \in \mathcal{B}(\mathsf{G})} \sum_{s \leqslant r} C_c(\mathfrak{g}^*(F)_{x,s}/\mathfrak{g}^*(F)_{x,r+}) =: \mathcal{D}_r,$$

some nice behavior, for all $s \leqslant r$, on $C_c(\mathfrak{g}^*(F)_{x,s}/\mathfrak{g}^*(F)_{x,r+})$.

Let $\tilde{J}(r) :=$ the space of such distributions, "Waldspurger-DeBacker space".

To show: it is a linear combination of nilpotent orbital integrals, i.e.,

Show: $\tilde{J}(r)|_{\mathcal{D}_r} = J(\mathcal{N}(\mathfrak{g}^*))|_{\mathcal{D}_r}$.

Strategy (Waldspurger-DeBacker)

Show: $J(\mathcal{N})|_{\mathcal{D}_r} = \tilde{J}(r)|_{\mathcal{D}_r}$, $\quad \mathcal{D}_r := \sum_{x \in \mathcal{B}(\mathsf{G})} \sum_{s \leqslant r} C_c(\mathfrak{g}^*(F)_{x,s}/\mathfrak{g}^*(F)_{x,r+})$

(set $\mathcal{N} = \mathcal{N}(\mathfrak{g}^*)$, may informally think $\mathfrak{g}(F)_{x,r} = \mathfrak{g}^*(F)_{x,r}$ etc.).

- Easier steps:
  - Easy inclusion: $J(\mathcal{N}) \subset \tilde{J}(r)$, so also when restricted to $\mathcal{D}_r$.
  - For good p, $\dim_{\mathbb{C}} J(\mathcal{N}) = \#\mathcal{O}(0)^* :=$ number of nilpotent orbits in $\mathfrak{g}^*(F)$, basis = nilpotent orbital integrals (also when restricted to $\mathcal{D}_r$).
  - Therefore, enough to show: $\dim_{\mathbb{C}} \tilde{J}(r)|_{\mathcal{D}_r} \leqslant \#\mathcal{O}(0)^*$.
- "Descent and recovery": Can replace $\mathcal{D}_r \rightsquigarrow$ the smaller $\mathcal{D}_r^{r+} := \sum C_c(\mathfrak{g}^*(F)_{x,r}/\mathfrak{g}^*(F)_{x,r+})$: $\tilde{J}(r)|_{\mathcal{D}_r} \to \tilde{J}(r)|_{\mathcal{D}_r^{r+}}$ is an isomorphism.
- Every element of $\tilde{J}(r)|_{\mathcal{D}_r^{r+}}$ is uniquely determined by its values on a set of $\{X_i + \mathfrak{g}^*(F)_{x_i,r+}\}_i$, where $\{X_i\}$ runs over a set of representatives for nilpotent orbits in $\mathfrak{g}^*(F)$ $\quad \Rightarrow \dim_{\mathbb{C}} \tilde{J}(r)|_{\mathcal{D}_r} \leqslant \#\mathcal{O}(0)^*$.

The last step uses DeBacker's classification of nilp. orbits in terms of

$$\{X + \mathfrak{g}^*(F)_{x,r+} \mid x \in \mathcal{B}(\mathsf{G}), X \in \mathfrak{g}^*(F)_{x,r}\}$$

(identify $\mathfrak{g} = \mathfrak{g}^*$; sort of an affine version of the Bala-Carter classification).

Spherical characters

Abstract setting. Let $\mathcal{G}$ finite group, $\mathcal{H} \subset \mathcal{G}$ subgroup. Consider "relative harmonic analysis", for the $\mathcal{G}$-space $\mathcal{G}/\mathcal{H}$, in terms of representations (π, V) of $\mathcal{G}$ such that $V^{\mathcal{H}} \neq 0$.

A spherical character is associated to $(\pi, V, \ell, \ell^\vee)$, where (π, V) rep of $\mathcal{G}$, $\ell^\vee \in V^{\mathcal{H}}, \ell \in (V^\vee)^{\mathcal{H}}$ (equiv., $\ell : V \to \mathbb{C}, \ell^\vee : V^\vee \to \mathbb{C}$ $\mathcal{H}$-invariant):

$$g \mapsto \langle \ell, \pi(g)\ell^\vee \rangle =: \Theta_{\pi,\ell,\ell^\vee},$$

a function on $\mathcal{H}\backslash\mathcal{G}/\mathcal{H}$, i.e., an $\mathcal{H}$-invariant function on $\mathcal{G}/\mathcal{H}$.

Example: ("Group case") $\mathcal{G} = \mathcal{H} \times \mathcal{H} \supset \mathcal{H}$ 'diagonally',

$$\begin{array}{ccccc} \mathcal{H} \circlearrowright & \mathcal{G}/\mathcal{H} & \cong & \mathcal{H} & \circlearrowleft \mathcal{H} \text{ (conjugation).} \\ & \Cup & & \Cup & \\ & (h_1, h_2)\mathcal{H} & \mapsto & h_1 h_2^{-1} & \end{array}$$

$\pi = \sigma \otimes \sigma^\vee, \pi^\vee = \sigma^\vee \otimes \sigma$ (of $\mathcal{G} = \mathcal{H} \times \mathcal{H}$), $\ell, \ell^\vee =$ "obvious pairings".

Then $\Theta_{\pi,\ell,\ell^\vee}$ on $\mathcal{G}/\mathcal{H}$ = the character Θ_σ of the representation σ of $\mathcal{H}$.

Our setting (p-adic symmetric spaces)

$\theta \in \mathrm{Aut}(\mathrm{G})$, $\theta^2 = 1, (\mathrm{G}^\theta)^0 \subset \mathrm{H} \subset \mathrm{G}^\theta$. $\quad \mathfrak{g} = \mathfrak{g}^{\theta=1} \oplus \mathfrak{g}^{\theta=-1} =: \mathfrak{h} \oplus \mathfrak{p} \underset{\mathrm{Ad}}{\circlearrowleft} \mathrm{H}$.

(π, V) smooth fin. length representation of $\mathrm{G}(F)$, $\mathrm{H}(F)$-invariant functionals

$$\ell : V \to \mathbb{C}, \qquad \ell^\vee : V^\vee \to \mathbb{C}.$$

The spherical character of $(\pi, V, \ell, \ell^\vee)$ = the distribution

$$\Theta_{\pi,\ell,\ell^\vee} : C_c^\infty(\mathrm{G}(F)) \to \mathbb{C}, \quad \varphi \mapsto \langle \underbrace{\ell}_{V\to\mathbb{C}}, \underbrace{\pi(\varphi)\ell^\vee}_{\in V} \rangle.$$

It is an $\mathrm{H}(F)$-bi-invariant distribution on $\mathrm{G}(F)$, i.e., an $\mathrm{H}(F)$-invariant distribution on $\mathrm{G}(F)/\mathrm{H}(F)$.

Rader and Rallis, 1996: In characteristic zero, local character expansion at $0 \in T_e(\mathrm{G}(F)/\mathrm{H}(F)) = \mathfrak{g}(F)/\mathfrak{h}(F) = \mathfrak{p}(F)$ (though not generally loc. int.):

$$\theta_{\pi,\ell,\ell^\vee}(f) = \sum_{\mathcal{O} \in \mathcal{N}(\mathfrak{p}^*)/\mathrm{Ad}\,\mathrm{H}(F) =: \mathcal{O}^{\mathfrak{p}^*}(0)} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}(f) \ (f \in C_c^\infty(\mathfrak{p}(F)), \text{ supp. near } 0).$$

Moy-Prasad domains, bringing down to the Lie algebra

Assume $p \neq 2$, recall $\mathfrak{g}(F) = \mathfrak{h}(F) \oplus \mathfrak{p}(F)$. For $x \in \mathcal{B}(\mathsf{H}) \overset{\text{Prasad}-\text{Yu}}{=} \mathcal{B}(\mathsf{G})^{\theta}$,

$$\mathfrak{g}(F)_{x,r} = (\mathfrak{h}(F) \cap \mathfrak{g}(F)_{x,r}) \oplus (\mathfrak{p}(F) \cap \mathfrak{g}(F)_{x,r}) = \underbrace{\mathfrak{h}(F)_{x,r}}_{\text{fact}} \oplus \underbrace{\mathfrak{p}(F)_{x,r}}_{\text{defn}}.$$

$$\mathfrak{p}(F)_r := \bigcup_{x \in \mathcal{B}(\mathsf{H})} \mathfrak{p}(F)_{x,r} \overset{\exp}{\to} \bigcup_{x \in \mathcal{B}(\mathsf{H})} \mathsf{H}(F)_{x,r} \backslash \mathsf{G}(F)_{x,r} \overset{\text{open}}{\subset} \mathsf{H}(F) \backslash \mathsf{G}(F)$$

(for good p)

$\Rightarrow$ have $\theta_{\pi,\ell,\ell^\vee}$ on $\mathfrak{p}(F)_r$ (descended via exp from $\Theta_{\pi,\ell,\ell^\vee}$ on $\mathsf{H}(F) \backslash \mathsf{G}(F)$).

Hope: $\theta_{\pi,\ell,\ell^\vee}(f) = \sum_{\mathcal{O} \in \mathcal{N}(\mathfrak{p}^*)/\mathsf{H}(F)} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}(f), \quad \forall f \in C_c^\infty(\mathfrak{p}(F)_r),$

whenever $r > \rho(\pi) :=$ the depth of π.

i.e., the range of validity of the character expansion contains $\mathfrak{p}(F)_{\rho(\pi)+}$.

Rough idea of the results

We cannot yet prove this in the most general case (even with $p \gg 0$):

- Can show (for $p \gg 0$) the analogue of $\dim_{\mathbb{C}} \tilde{J}(r)|_{\mathcal{D}_r} \leqslant \#\mathcal{O}^{\mathfrak{p}^*}(0)$: essentially following the 'group case'.
- But $\dim_{\mathbb{C}} J(\mathcal{N}(\mathfrak{p}^*))$ may be strictly less than $\#\mathcal{O}^{\mathfrak{p}^*}(0)$:
    - There may not be an invariant measure on some $\mathcal{O} \in \mathcal{O}^{\mathfrak{p}^*}(0)$: stabilizers may no longer be unimodular.
    - Even if $\exists$ inv. meas. $\mu_{\mathcal{O}}$ on $\mathcal{O}$, $\mu_{\mathcal{O}}(f)$ may not convg. for $f \in C_c^\infty(\mathfrak{p}^*(F))$: e.g., $\mathsf{G} = \mathsf{PGL}_2 \supset \mathbb{G}_m = \mathsf{H}$, $\mathfrak{p} = F^2$, $h \cdot (x, y) = (hx, h^{-1}y)$.
- Where these two issues don't arise, and $p \gg 0$, have the expected range of validity result.
- In rank one situations — $\mathsf{SL}_n / \mathsf{GL}_{n-1}$, $\mathsf{SO}_{2n} / \mathsf{SO}_{2n-1}$, $\mathsf{SO}_{2n+1} / \mathsf{SO}_{2n}$, $\mathsf{Sp}_{2n} / \mathsf{Sp}_2 \times \mathsf{Sp}_{2n-2}$, F_4 / Spin_9 — can establish the expectd range of validity result using dirct computation + features particular to rank one.
- So one hopes to establish the desired range of validity (i.e., $\mathfrak{p}(F)_{\rho(\pi)+}$) in general, but we don't have a general approach to proving it.

Some ingredients

- Moy-Prasad theory: A lot of the results regarding lattices, nilpotents etc. hold in somewhat general settings: a "rational" representation V of H together with a suitable collection of lattices $V_{x,r}$..., but to connect to harmonic analysis on $\mathrm{G}(F)$ (e.g., to relate 'degenerate cosets' for $\mathrm{H}(F) \circlearrowright \mathfrak{p}^*(F)$ and $\mathrm{G}(F) \circlearrowright \mathfrak{g}^*(F)$) we of course do make use of our special situation.
- Classification of "nilpotent orbits" using the building: Known by the work of Ricardo Portilla, involves $\mathcal{B}(\mathrm{H})$ with polysimplicial structure inherited from $\mathcal{B}(\mathrm{G})$. This is a crucial input for us.
- Like Portilla, one works a lot with normalized $\mathfrak{sl}_2$-triplets (Y, M, X), where the 'neutral element' M belongs to $\mathfrak{h}(F)$, the 'nilpositive element' X belongs to $\mathfrak{p}(F)$.