Many-body chaos in a thermalised fluid

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Turbulence: Problems at the Interface of Mathematics and Physics

Bangalore (*Virtual)*, India

Background

Preamble

INTERNATIONAL CENTRE *for* THEORETICAL SCIENCES
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Understanding quantum chaos through the prism of the classical butterfly effect

Roberts and Stanford, Phys. Rev. Lett. (2015)
Maldacena, Shenker, and Stanford, J. High Energy Phys. (2016)
Aleiner, Faoro, and Ioffe, Ann. Phys. (2016)
Dora and Moessner, Phys. Rev. Lett. (2017)
Shenker and Stanford, J. High Energy Phys. (2014)
Roberts, Stanford and Susskind, J. High Energy Phys. (2016)
Kitaev (2014)
Leichenhauer, Phys. Rev. D (2014)

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Understanding quantum chaos through the prism of the classical butterfly effect

Consider Hermitian operators *V* and *W*: What is the effect on *W* of a perturbation on *V*?

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Consider Hermitian operators *V* and *W*: What is the effect on *W* of a perturbation on *V*?

$$\boxed{F(t) = -\langle [V, W(t)]^2 \rangle_\beta}$$

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Chaotic Systems

Large

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$F(t)$ → λ and v_B

Growth and spread of perturbations in a quantum many-body system

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Chaotic Systems ↓

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$$F(t) \longrightarrow \lambda \text{ and } v_B$$

Growth and spread of perturbations in a quantum many-body system

Applications: Information scrambling, extremal black holes

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OTOC

Consider Hermitian operators *V* and *W*: What is the effect on *W* of a perturbation on *V*?

$$F(t) = -\langle [V, W(t)]^2 \rangle_\beta$$

Chaotic Systems

Large

$F(t) \longrightarrow \lambda \text{ and } v_B$

Growth and spread of perturbations in a quantum many-body system

Out-of-time-ordered-commutators (OTOCs) are *the* diagnostic for quantum chaos

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OTOC

Out-of-time-ordered-commutators (OTOCs) are *the* diagnostic for quantum chaos

A bound on chaos

Juan Maldacena,[a] Stephen H. Shenker[b] and Douglas Stanford[a]

[a] *School of Natural Sciences, Institute for Advanced Study, 1 Einstein Drive, Princeton, NJ, U.S.A.*

[b] *Stanford Institute for Theoretical Physics and Department of Physics, Stanford University, 382 Via Pueblo Mall, Stanford, CA, U.S.A.*

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ABSTRACT: We conjecture a sharp bound on the rate of growth of chaos in thermal quantum systems with a large number of degrees of freedom. Chaos can be diagnosed using an out-of-time-order correlation function closely related to the commutator of operators separated in time. We conjecture that the influence of chaos on this correlator can develop no faster than exponentially, with Lyapunov exponent $\lambda_L \leq 2\pi k_B T/\hbar$. We give a precise mathematical argument, based on plausible physical assumptions, establishing this conjecture.

For classical systems, what is this bound?

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OTOC

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A bound on chaos

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Abstract: We conjecture a sharp bound on the rate of growth of chaos in thermal quantum systems with a large number of degrees of freedom. Chaos can be diagnosed using an out-of-time-order correlation function closely related to the commutator of operators separated in time. We conjecture that the influence of chaos on this correlator can develop no faster than exponentially, with Lyapunov exponent $\lambda_L \leq 2\pi k_B T/\hbar$. We give a precise mathematical argument, based on plausible physical assumptions, establishing this conjecture.

For classical systems, what is this bound?

What is the classical analogue of such OTOCs?

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Classical Analogue

How should we define the classical analogue?

$$\{f, g\} \to [1/(\imath\hbar)][f, g]$$

Decorrelators

PHYSICAL REVIEW LETTERS **121,** 024101 (2018)

Light-Cone Spreading of Perturbations and the Butterfly Effect in a Classical Spin Chain

Avijit Das,[1,*] Saurish Chakrabarty,[1,†] Abhishek Dhar,[1] Anupam Kundu,[1] David A. Huse,[2] Roderich Moessner,[3]
Samriddhi Sankar Ray,[1] and Subhro Bhattacharjee[1]
[1]*International Centre for Theoretical Sciences, Tata Institute of Fundamental Research, Bengaluru 560089, India*
[2]*Physics Department, Princeton University, Princeton, New Jersey 08544, USA*
[3]*Max-Planck Institute for the Physics of Complex Systems, 01187 Dresden, Germany*

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Decorrelators

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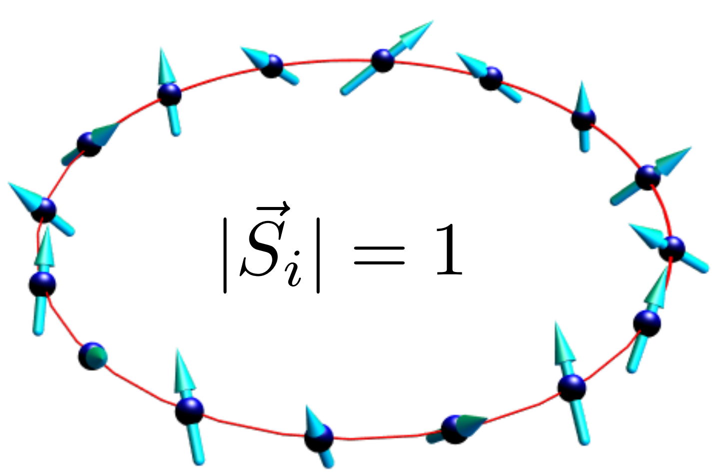
Classical Heisenberg spin chain at infinite temperature

Hamiltonian	Dynamics
Nearest-neighbour Ferromagnet	Spin-precession dynamics
$\mathcal{H} = -J \sum_{x=0}^{N-1} \mathbf{S}_x \cdot \mathbf{S}_{x+1}$	$\frac{d\mathbf{S}_x}{dt} = J\mathbf{S}_x \times (\mathbf{S}_{x-1} + \mathbf{S}_{x+1}) = \{\mathbf{S}_x, H\}$



Classical Heisenberg Model

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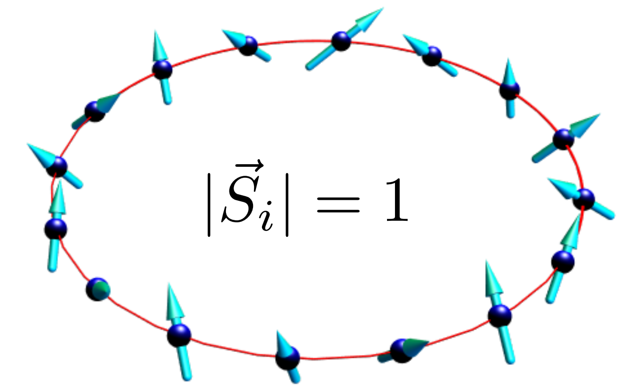
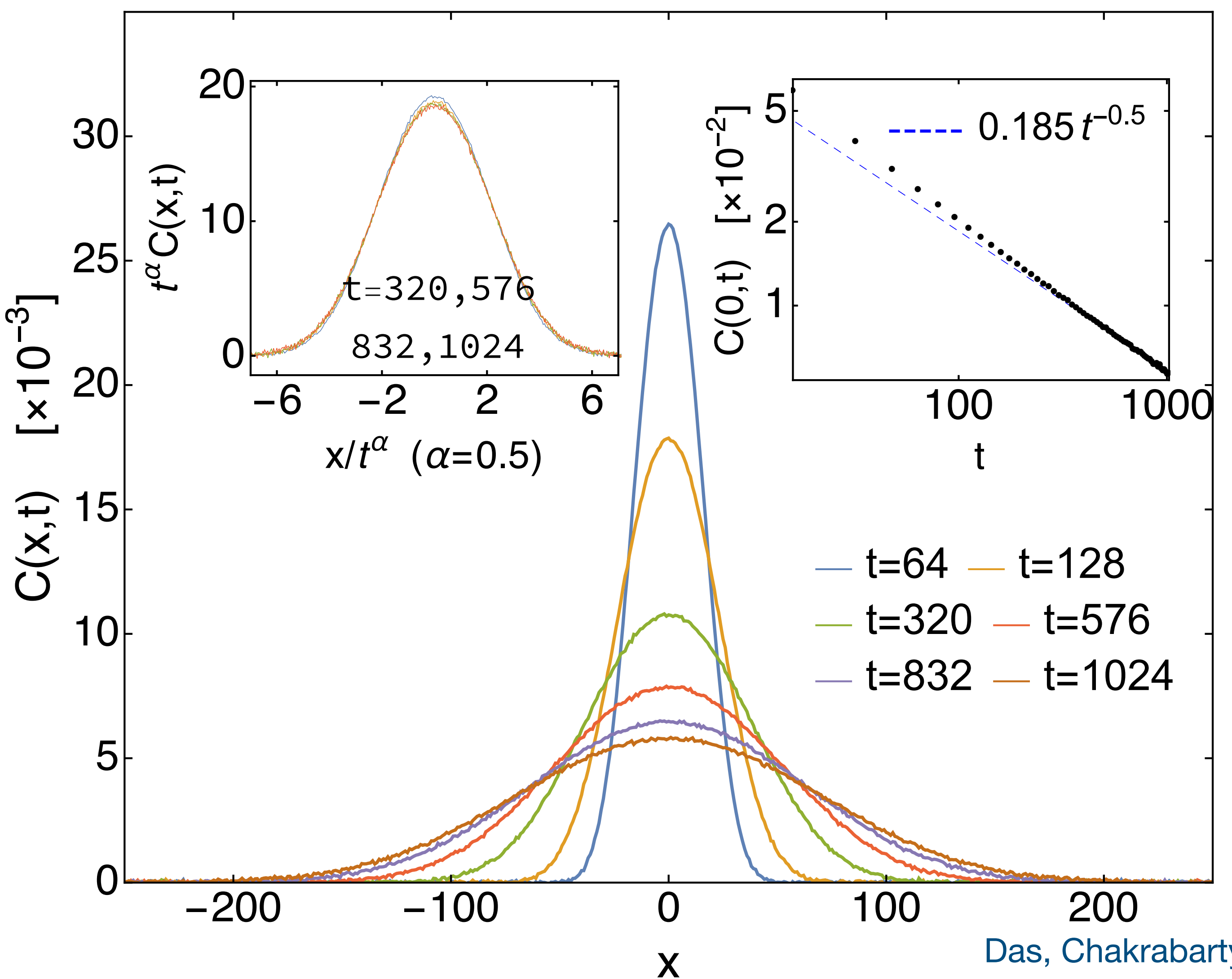


Infinite Temperature

2-point dynamical spin-correlator: $C(x,t) = \langle \mathbf{S}_x(t) \cdot \mathbf{S}_0(0) \rangle \sim \frac{e^{-x^2/t}}{\sqrt{t}}$



Diffusive and not ballistic

Das, Chakrabarty, Dhar, Kundu, Huse, Moessner, **Ray**, and Bhattacharjee, Phys. Rev. Lett. (2018)

Decorrelators: Heisenberg Model

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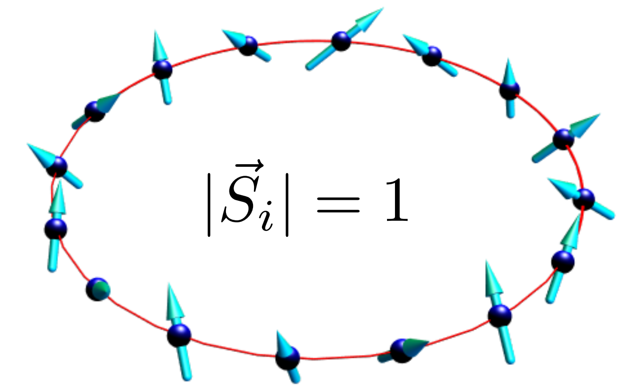

Infinite Temperature

Strategy

- Generate a random spin configuration (infinite temperature: Gibbs *ensemble*)

Das, Chakrabarty, Dhar, Kundu, Huse, Moessner, **Ray**, and Bhattacharjee, Phys. Rev. Lett. (2018)

Decorrelators: Heisenberg Model

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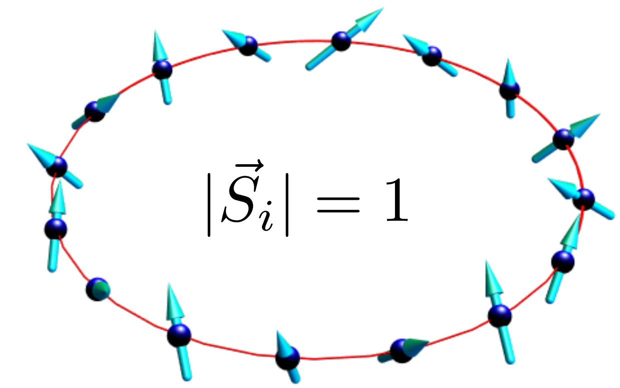
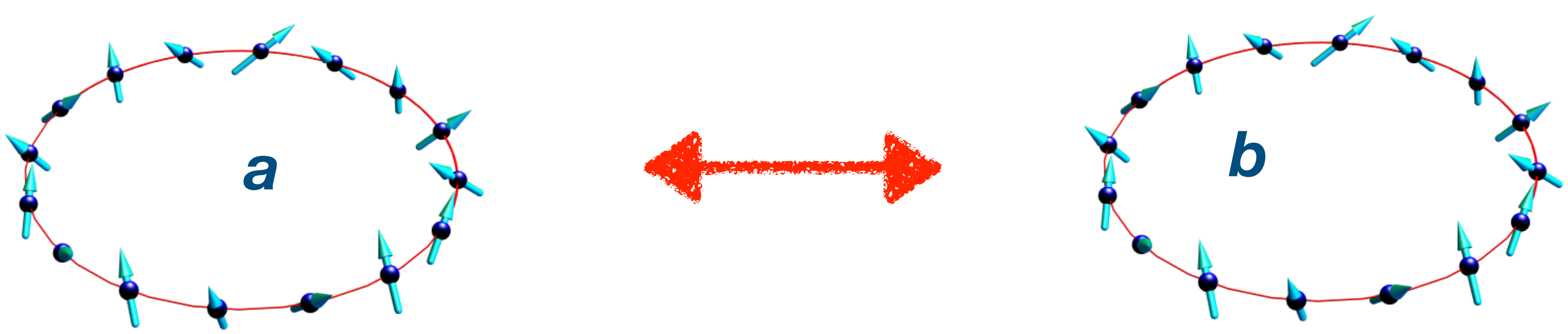


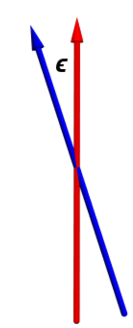
Infinite Temperature

Strategy

- Generate a random spin configuration (infinite temperature: Gibbs *ensemble*)
- Make a copy but with the spin at *x = 0* rotated infinitesimally



$$\{\mathbf{S}_x^a(t=0)\} - \{\mathbf{S}_x^b(t=0)\} = \delta\mathbf{S}_0$$



$$\delta\mathbf{S}_0 = \varepsilon[\hat{\mathbf{n}} \times \mathbf{S}_0]$$

Das, Chakrabarty, Dhar, Kundu, Huse, Moessner, **Ray**, and Bhattacharjee, Phys. Rev. Lett. (2018)

Decorrelators: Heisenberg Model

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Hamiltonian

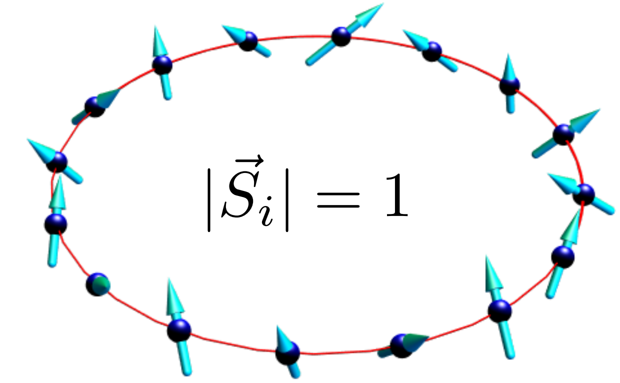
Nearest-neighbour Ferromagnet

$$\mathcal{H} = -J \sum_{x=0}^{N-1} \mathbf{S}_x \cdot \mathbf{S}_{x+1}$$

Dynamics

Spin-precession dynamics

$$\frac{d\mathbf{S}_x}{dt} = J\mathbf{S}_x \times (\mathbf{S}_{x-1} + \mathbf{S}_{x+1}) = \{\mathbf{S}_x, H\}$$



Infinite Temperature

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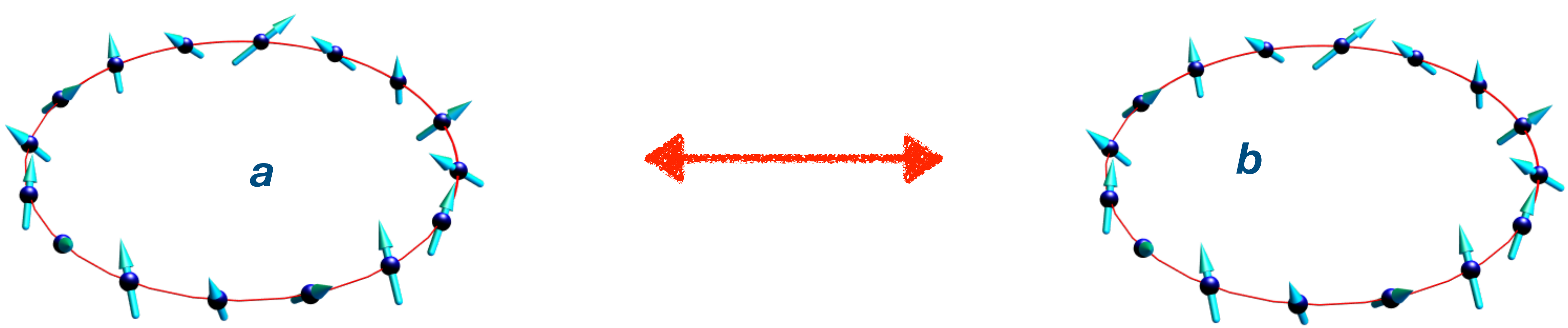
- Generate a random spin configuration (infinite temperature: Gibbs *ensemble*)
- Make a copy but with the spin at *x = 0* rotated infinitesimally
- Evolve the two systems independently and measure how fast they decorrelate



$$\{\mathbf{S}_x^a(t=0)\} - \{\mathbf{S}_x^b(t=0)\} = \delta\mathbf{S_0}$$

Das, Chakrabarty, Dhar, Kundu, Huse, Moessner, **Ray**, and Bhattacharjee, Phys. Rev. Lett. (2018)

Decorrelators: Heisenberg Model

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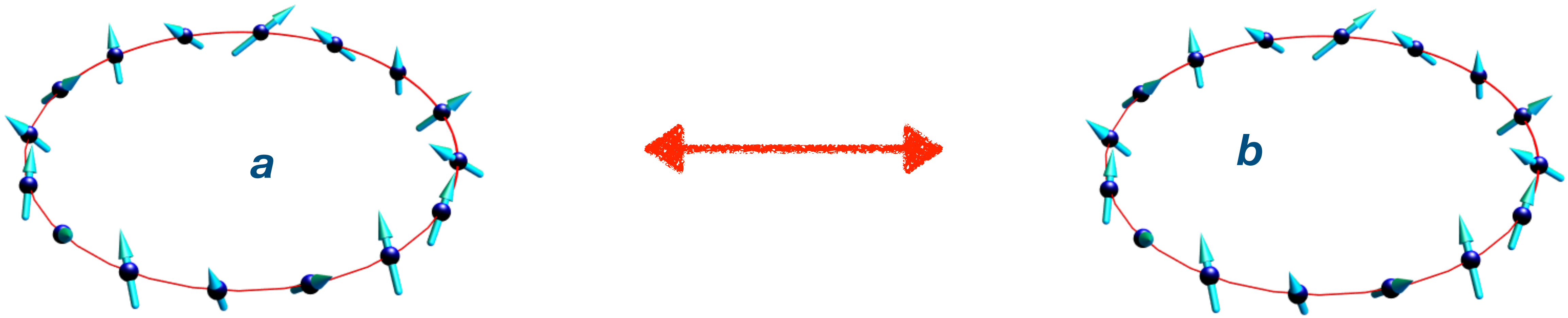


$$D(x,t) = (1 - \langle \mathbf{S}_x^a(t) \cdot \mathbf{S}_x^b(t) \rangle) = \frac{1}{2} \langle (\mathbf{S}_x^a(t) - \mathbf{S}_x^b(t))^2 \rangle$$

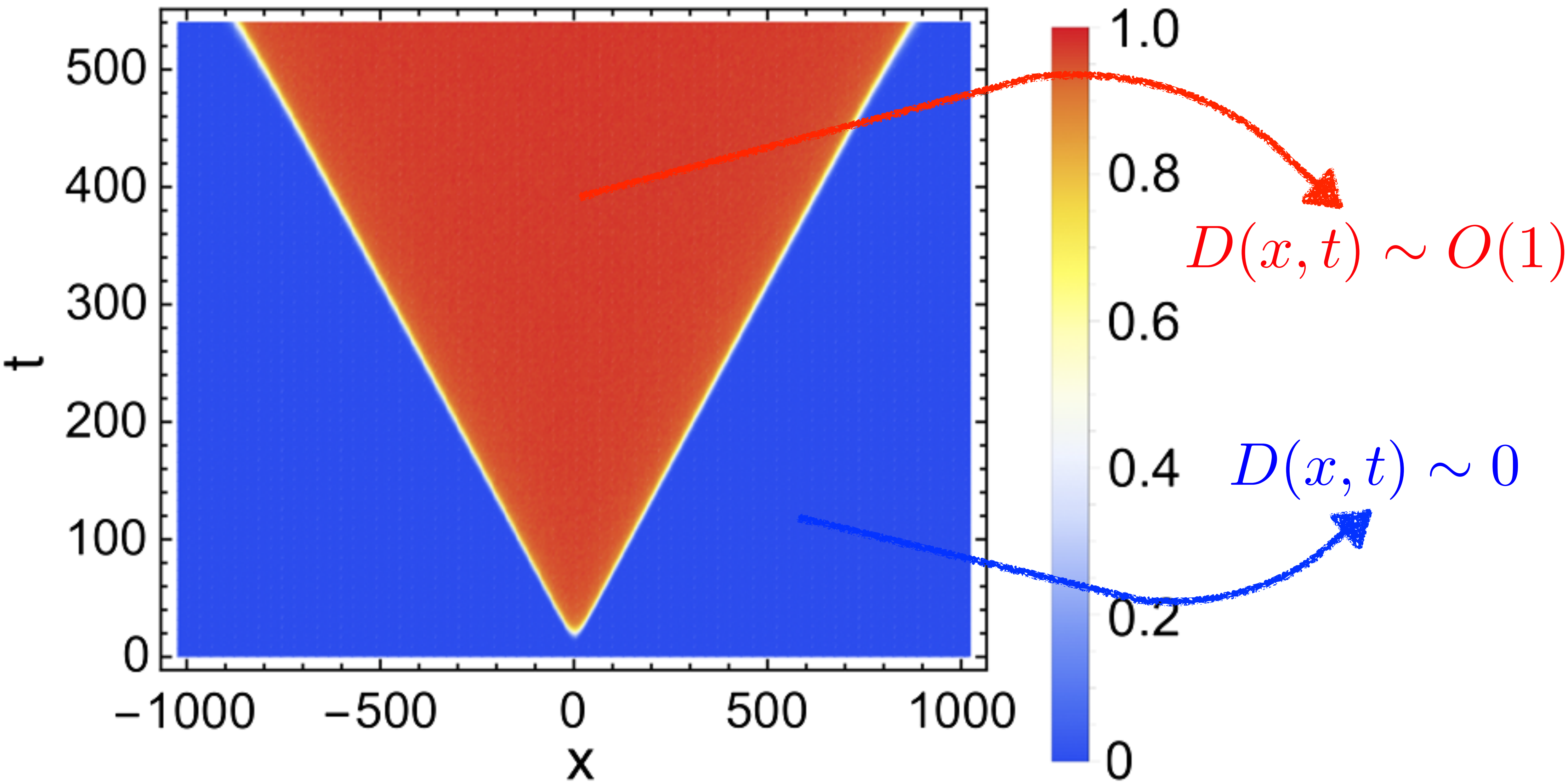
Rigorously shown to be the classical limit of the OTOC

$$D(x,t) \to -(\epsilon^2/\hbar^2)\mathrm{Tr}[\rho_T([\mathbf{S}_x(t), \hat{\mathbf{n}} \cdot \mathbf{S}_0(0)])^2] \to OTOC$$

Das, Chakrabarty, Dhar, Kundu, Huse, Moessner, **Ray**, and Bhattacharjee, Phys. Rev. Lett. (2018)

Decorrelators

$$D(x,t) = (1 - \langle \mathbf{S}_x^a(t) \cdot \mathbf{S}_x^b(t) \rangle) = \frac{1}{2}\langle (\mathbf{S}_x^a(t) - \mathbf{S}_x^b(t))^2 \rangle$$



Inside the *light cone*, spins are uncorrelated

Das, Chakrabarty, Dhar, Kundu, Huse, Moessner, **Ray**, and Bhattacharjee, Phys. Rev. Lett. (2018)

Decorrelators

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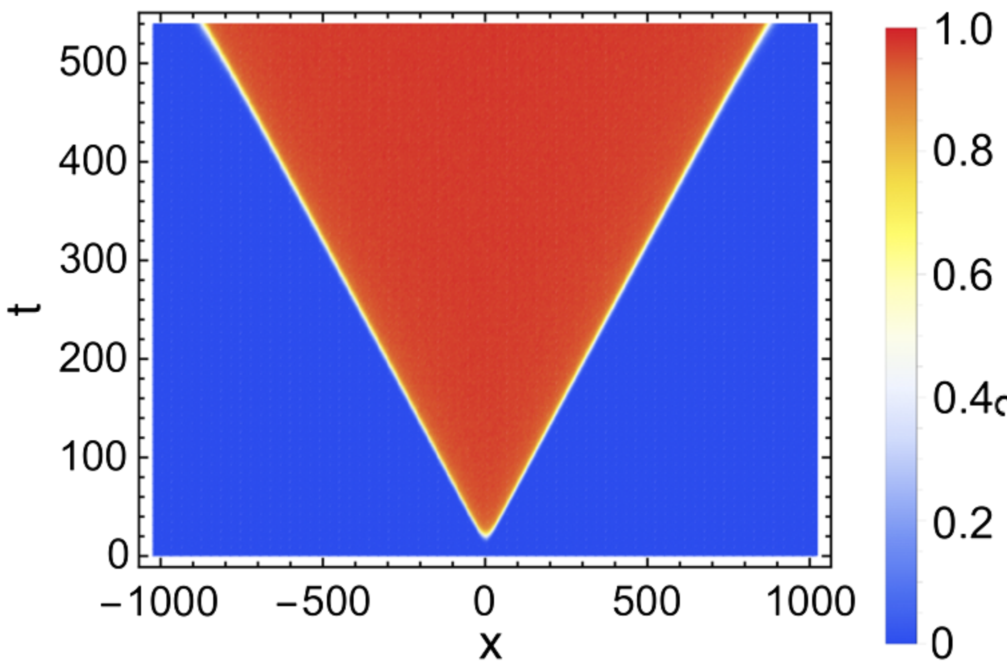

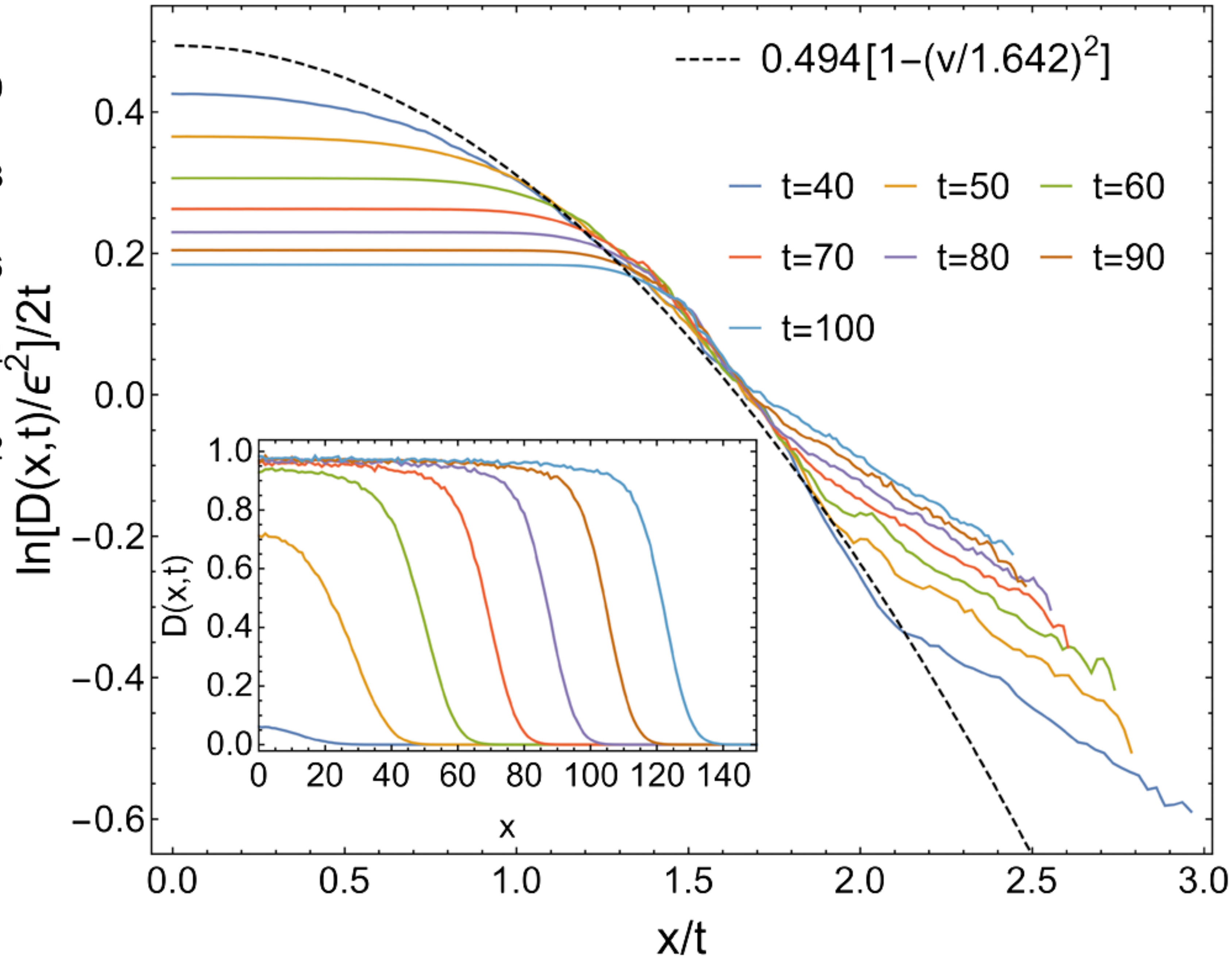

Velocity-dependent Lyapunov exponent

Vastano and Swinney, Phys. Rev. Lett. (1988)
Lepri, Politi, and Torcini, J. Stat. Phys. (1996)
Gaspard *et al.,* Nature (1998)
Grassberger, Nature (1999)

Deissler and Kaneko, Phys. Lett. (1987)
Lepri, Politi, and Torcini, J. Stat. Phys. (1997)

Das, Chakrabarty, Dhar, Kundu, Huse, Moessner, **Ray**, and Bhattacharjee, Phys. Rev. Lett. (2018)

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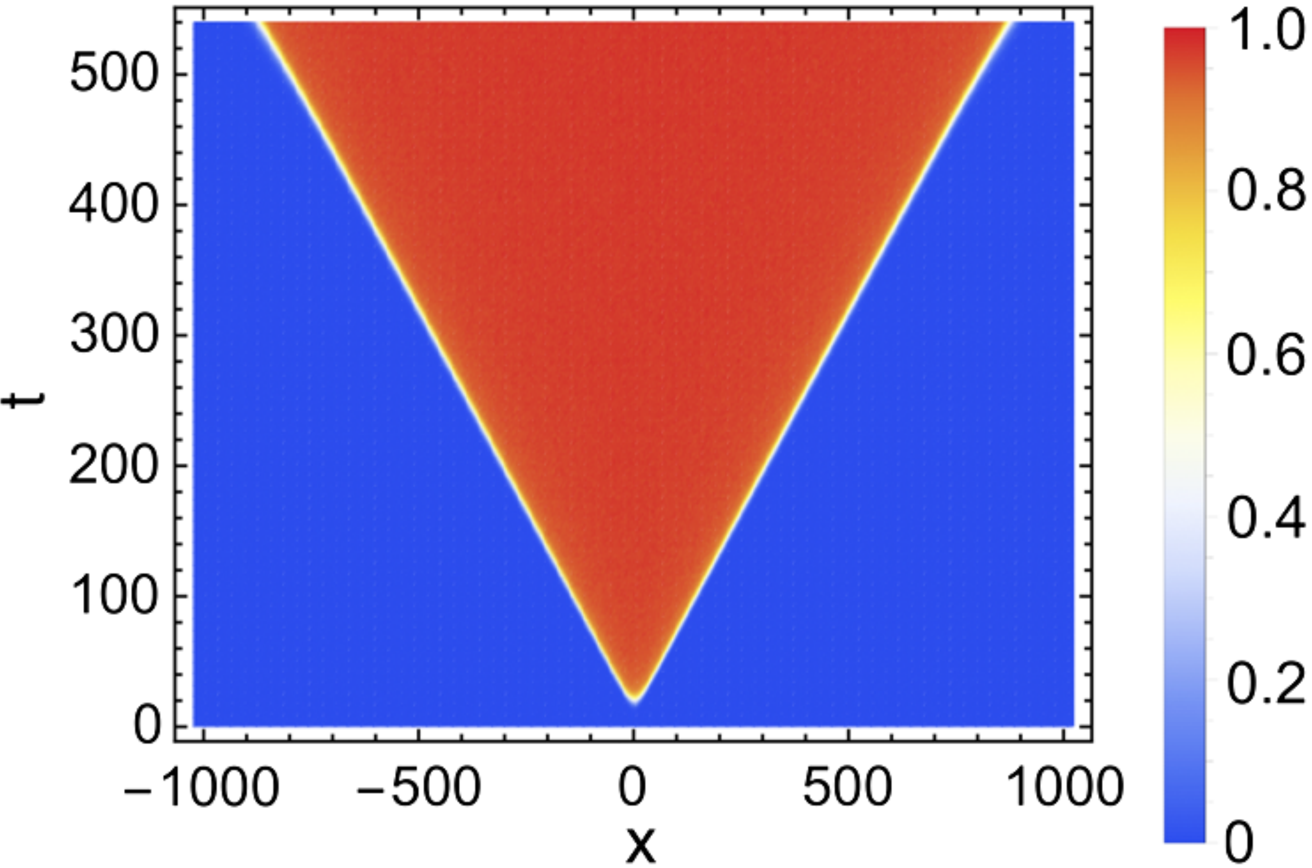

Summary

- Exponential growth and ballistic spread of an initially localised perturbation
- Characterised by a Lyapunov exponent and butterfly speed
- Connection of the growth, spread and propagation of the perturbation to the Kardar-Parisi-Zhang equation

Other Examples

Quantum Phase Transitions: Banerjee and Altman, Phys. Rev. B (2017)
Bose-Hubbard Chain: Shen, Zhang, Fan, and Zhai, Phys. Rev. B (2017)
Diffusive Metals: Patel, Chowdhury, Sachdev, and Swingle, Phys. Rev. X (2017)
Scrambling: Khemani, Vishwanath, and Huse, Phys. Rev. X (2018)

Das, Chakrabarty, Dhar, Kundu, Huse, Moessner, **Ray**, and Bhattacharjee, Phys. Rev. Lett. (2018)

Classical Analogue to OTOC

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Why bother?

Nature of intrinsic differences/universality between classical and quantum many-body systems

Connection between thermodynamic variables with the time scales of chaos

Connection between chaos and transport in strongly correlated systems

A unified mechanism at the heart of *thermalisation, ergodicity* and *equilibration*

Significance of these chaotic length and time scales in dynamics

Can capture ballistic spread even when two-point functions are diffusive

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Recent Studies

Spin Liquids: Bilitewski, Bhattacharjee, and Moessner, Phys. Rev. Lett. (2018)
Duffing Chain: Chatterjee, Kundu and Kulkarni, Phys. Rev. E (2020)
Butterfly effect/Spontaneous Stochasticity: Thalabard, Bec, and Mailybaev, Comm. Phys. (2020)
Heissenberg Magnets: Bilitewski, Bhattacharjee, and Moessner, ArXiv: 2011.04700
2D Anisotropic XXZ model: Ruidas and Banerjee, ArXiv: 2007.12708

Classical Analogue to OTOC

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Why bother?

The nature of intrinsic differences between classical and quantum many-body systems

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Connection between chaos and transport in strongly correlated systems

A unified mechanism at the heart of *thermalisation* and *equilibration*

Significance of these chaotic length and time scales in dynamics

Can capture ballistic spread even when two-point functions are diffusive

Can we adapt these ideas within the framework of equations of hydrodynamics?

Finding the Right Model

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The Temptation

Put these ideas to the test in the most natural and well-known example of *chaotic* systems:
Turbulent flows

Finding the Right Model

The Temptation

Put these ideas to the test in the most natural and well-known example of *chaotic* systems: Turbulent flows

The Problem

Driven-dissipative systems lacking a Hamiltonian structure or a statistical description in terms of thermodynamic variables

Finding the Right Model

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The Temptation

Put these ideas to the test in the most natural and well-known example of *chaotic* systems: Turbulent flows

The Problem

Driven-dissipative systems lacking a Hamiltonian structure or a statistical description in terms of thermodynamic variables

Redemption

Look for variations (while being nonlinear and chaotic) which retains a Hamiltonian structure (energy conservation) and allows a statistical equilibria

Thermalised fluids

An *Ideal* Model: Thermalised Fluid

Inviscid, three-dimensional Euler or the one-dimensional Burgers equation with a finite number of Fourier modes through Galerkin truncation

$$\text{Finite Truncation Wavenumber } K_G \longrightarrow \text{ Finite Number of Modes } N_G$$

An *Ideal* Model: Thermalised Fluid

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Inviscid, three-dimensional Euler or the one-dimensional Burgers equation with a finite number of Fourier modes through Galerkin truncation

<u>Inviscid equation</u>

1D Burgers

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0$$

3D Euler

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla P$$

Galerkin Truncation

$$\mathrm{P}_{\mathrm{K_G}} u(x) = \sum_{|k|\le K_G} \mathrm{e}^{\imath k x}\hat{u}_k$$

$$\text{Finite Truncation Wavenumber } K_G \longrightarrow \text{ Finite Number of Modes } N_G$$

An *Ideal* Model: Thermalised Fluid

Inviscid, three-dimensional Euler or the one-dimensional Burgers equation with a finite number of Fourier modes through Galerkin truncation

Inviscid equation

1D Burgers

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} = 0$$

3D Euler

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla P$$

Galerkin Truncated Inviscid equation

$$\frac{\partial v}{\partial t} + \mathrm{P}_{\mathrm{K_G}}\frac{1}{2}\frac{\partial v^2}{\partial x} = 0$$

$$\frac{\partial \hat{v}_\alpha(\mathbf{k})}{\partial t} = -\frac{\imath}{2}\mathcal{P}_{\alpha\beta\gamma}(\mathbf{k})\sum_{\mathbf{p}}\hat{v}_\beta(\mathbf{p})\hat{v}_\gamma(\mathbf{k}-\mathbf{p})$$

Finite Truncation Wavenumber K_G $\longrightarrow$ Finite Number of Modes N_G

An *Ideal* Model: Thermalised Fluid

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Inviscid, three-dimensional Euler or the one-dimensional Burgers equation with a finite number of Fourier modes through Galerkin truncation

↓

Chaotic, thermalised solutions within a finite time

An *Ideal* Model: Thermalised Fluid

Inviscid, three-dimensional Euler or the one-dimensional Burgers equation with a finite number of Fourier modes through Galerkin truncation

↓

Chaotic, thermalised solutions within a finite time

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Statistical Equilibria

Energy equipartition
Gibbs distributions
Strict notion of temperature

An *Ideal* Model: Thermalised Fluid

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Thermalised Solutions

Remarkable statistical behavior for truncated Burgers–Hopf dynamics

PNAS (2000)

Andrew J. Majda* and Ilya Timofeyev
Courant Institute of Mathematical Sciences, New York University, New York, NY 10012
Contributed by Andrew J. Majda, September 11, 2000

PRL **95**, 264502 (2005) PHYSICAL REVIEW LETTERS week ending 31 DECEMBER 2005

Effective Dissipation and Turbulence in Spectrally Truncated Euler Flows

Cyril Cichowlas,[1] Pauline Bonaïti,[1] Fabrice Debbasch,[2] and Marc Brachet[1]
[1]*Laboratoire de Physique Statistique de l'Ecole Normale Supérieure, associé au CNRS et aux Universités, Paris VI et VII, 24 Rue Lhomond, 75231 Paris, France*
[2]*ERGA, CNRS UMR 8112, 4 Place Jussieu, F-75231 Paris Cedex 05, France*
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PHYSICAL REVIEW E **84**, 016301 (2011)

Resonance phenomenon for the Galerkin-truncated Burgers and Euler equations

Samriddhi Sankar Ray,[1,*] Uriel Frisch,[1] Sergei Nazarenko,[2] and Takeshi Matsumoto[3]
[1]*UNS, CNRS, OCA, Laboratoire Cassiopée, B.P. 4229, F-06304 Nice Cedex 4, France*
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Research

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The onset of thermalization in finite-dimensional equations of hydrodynamics: insights from the Burgers equation

Divya Venkataraman† and Samriddhi Sankar Ray
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SSR, 0000-0001-9407-0007

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Thermalised Fluid: Real Space

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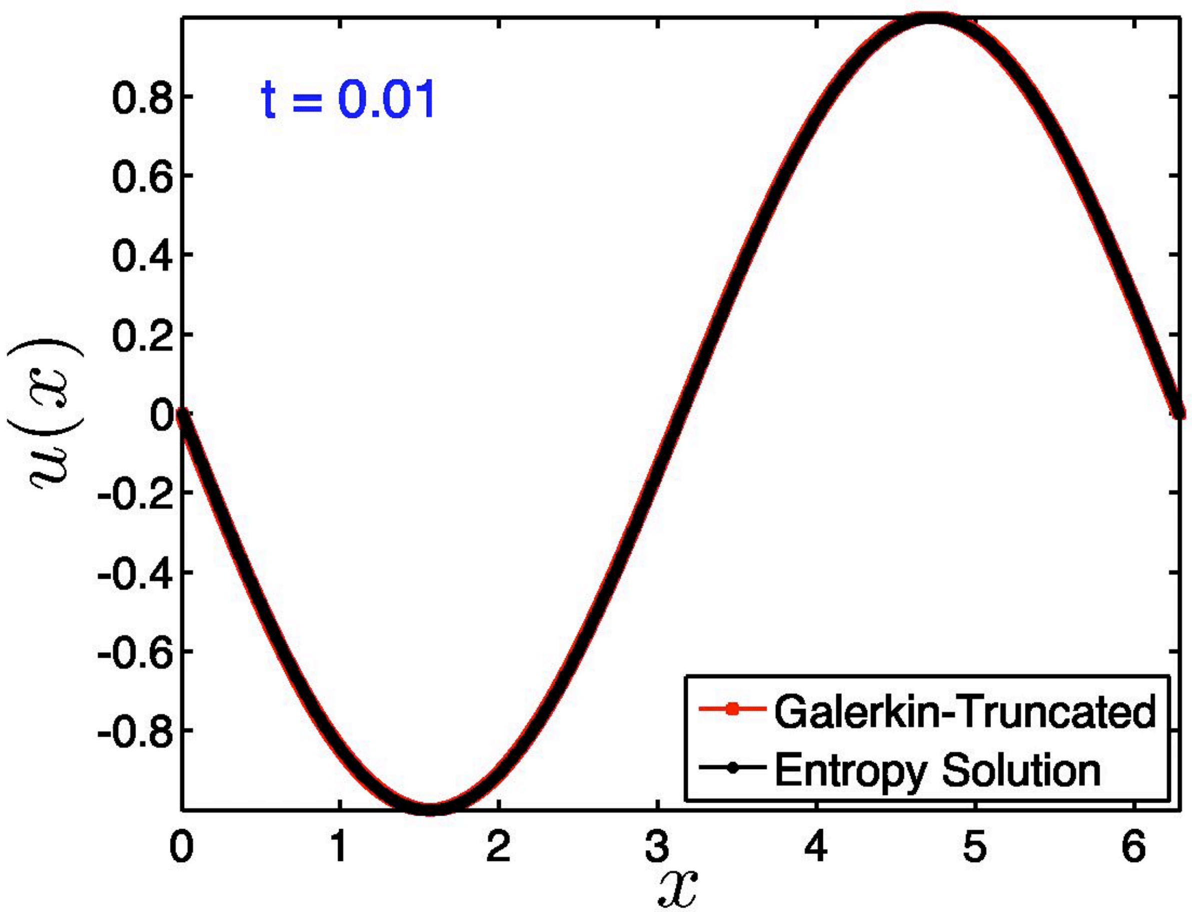
1D Burgers



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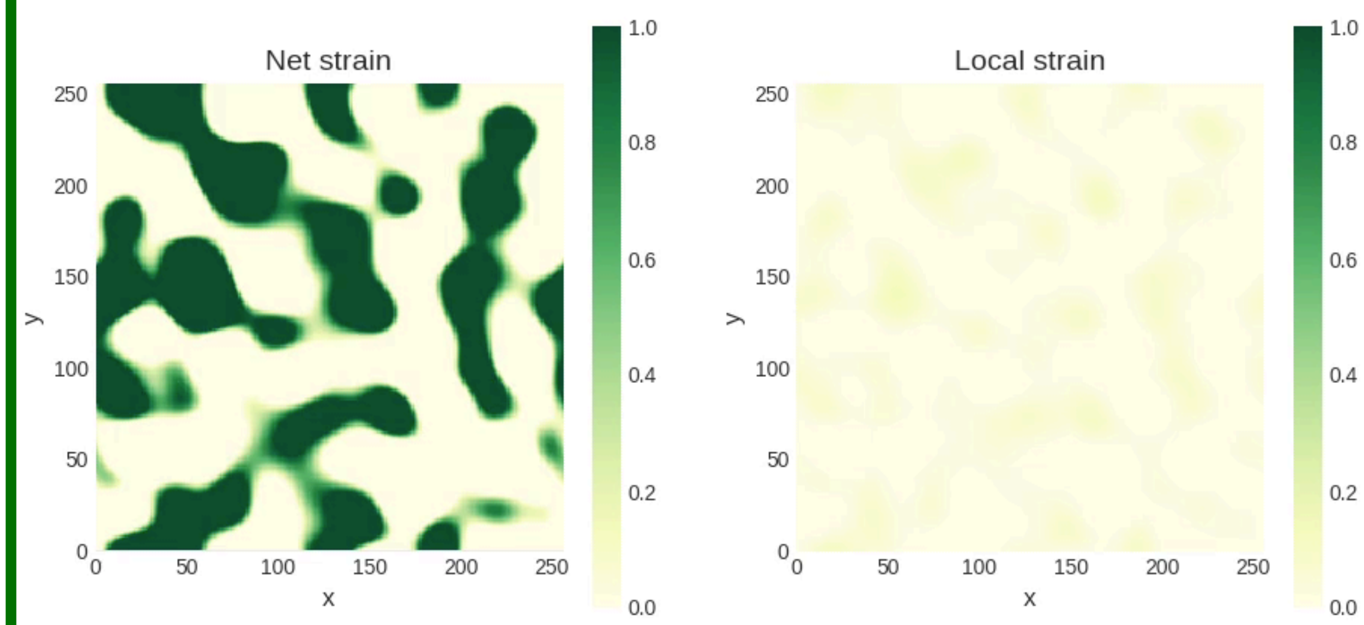
3D Euler



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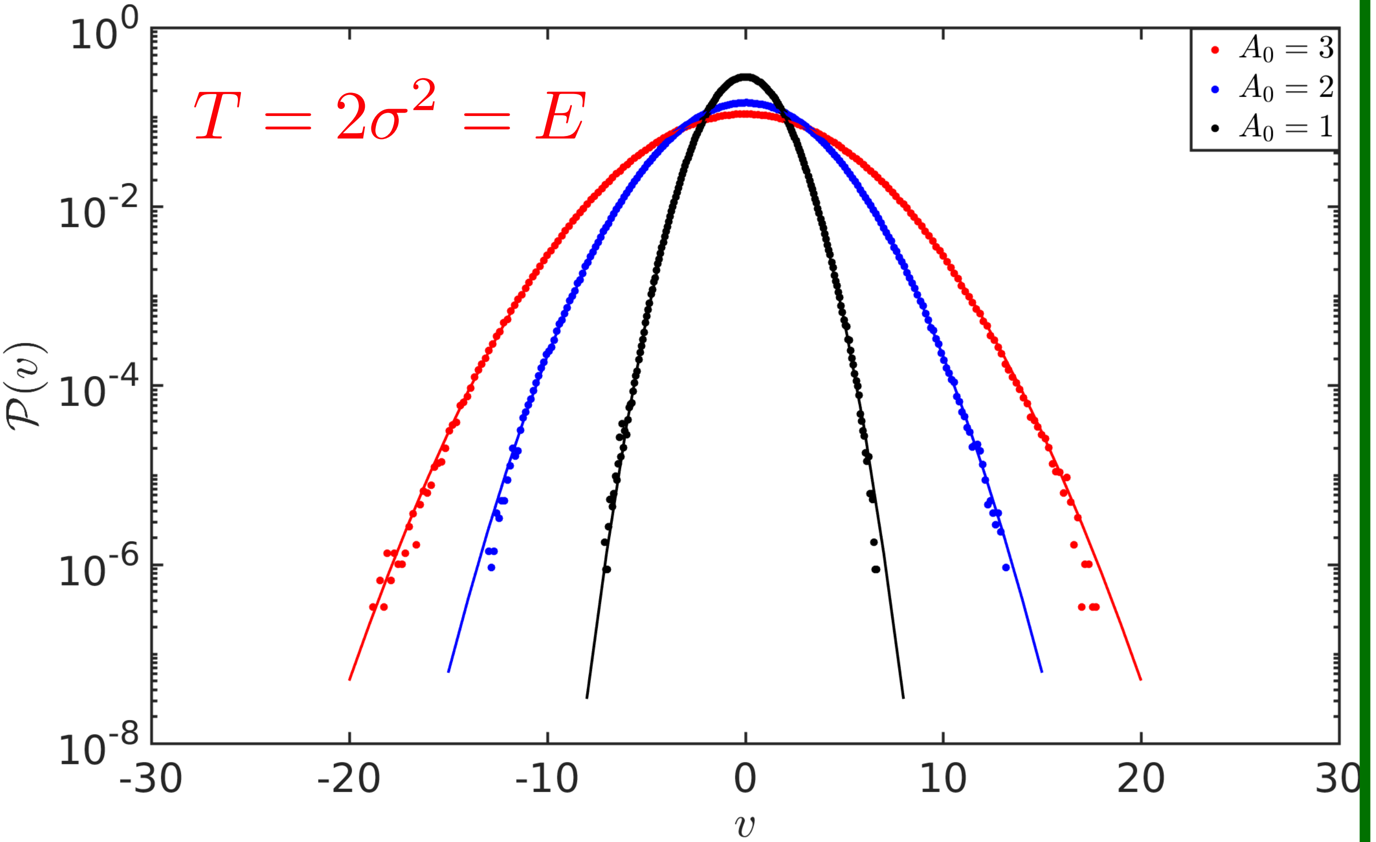
Thermalised Fluid: Gibbs Distributions

1D Burgers



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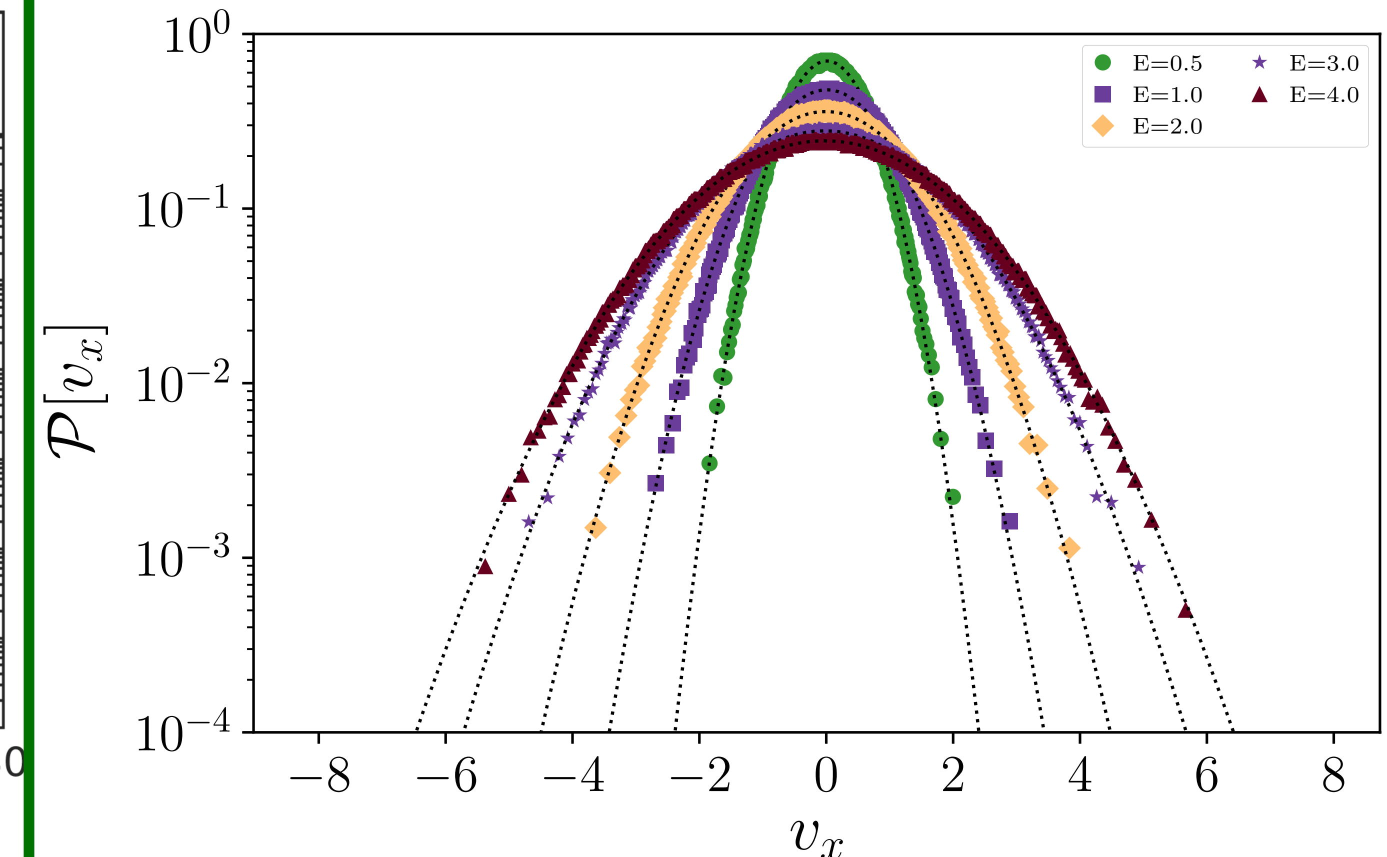

Vary temperature by varying the amplitude of the initial conditions since energy is conserved

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Thermalised Fluid: Energy Equipartition

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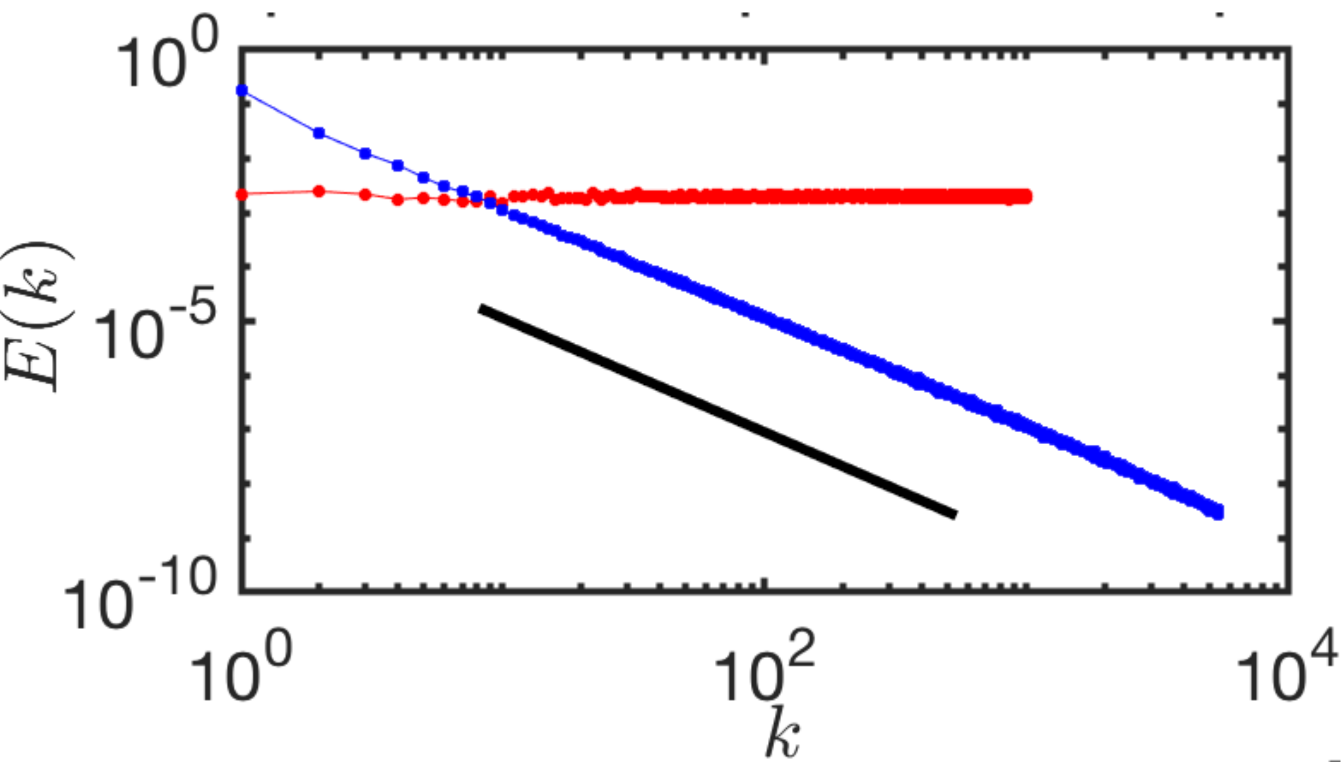
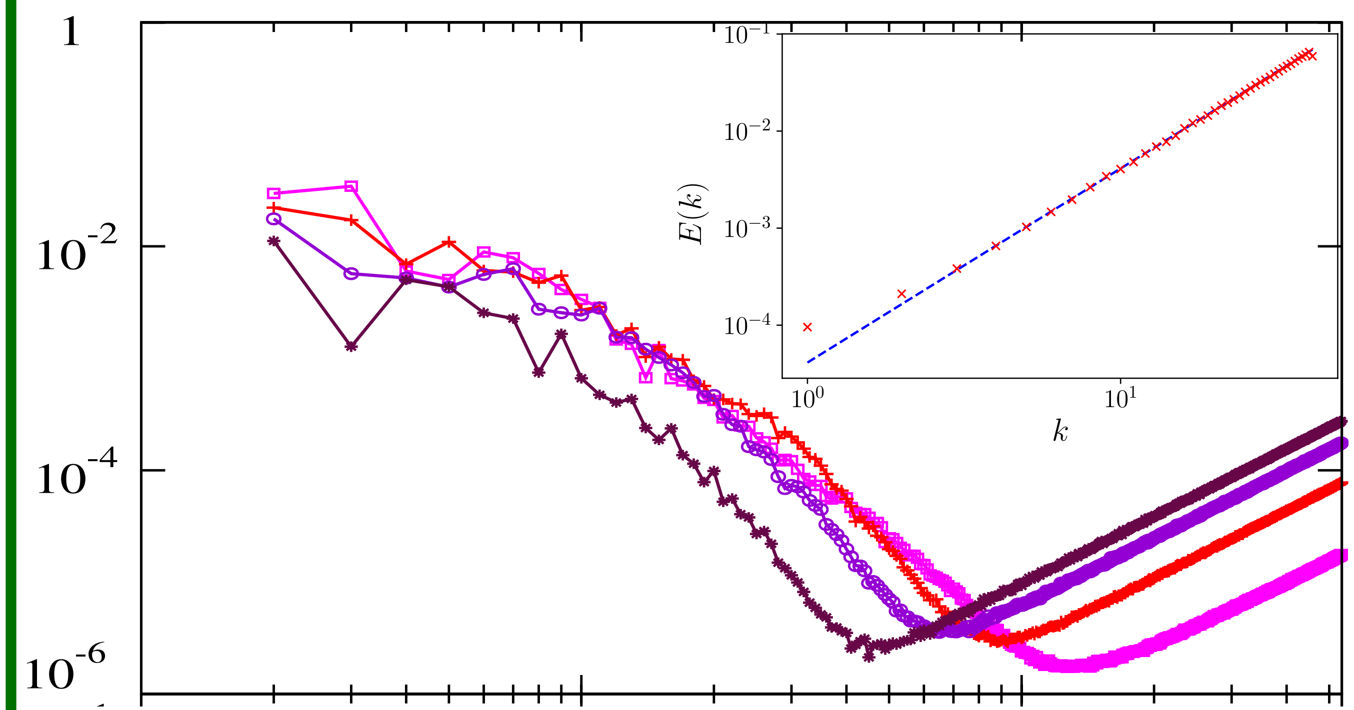
1D Burgers Energy equipartition : $E(k) = k^{d-1}$ **3D Euler**





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Understanding many-body chaos in such thermalised fluids

Sugan D. Murugan

Dheeraj Kumar

Subhro Bhattacharjee

Murugan, Kumar, Bhattacharjee, and **Ray**, *in preparation* (see also: ArXiv: 1906.00016)

Many-body Chaos: The Burgers Equation

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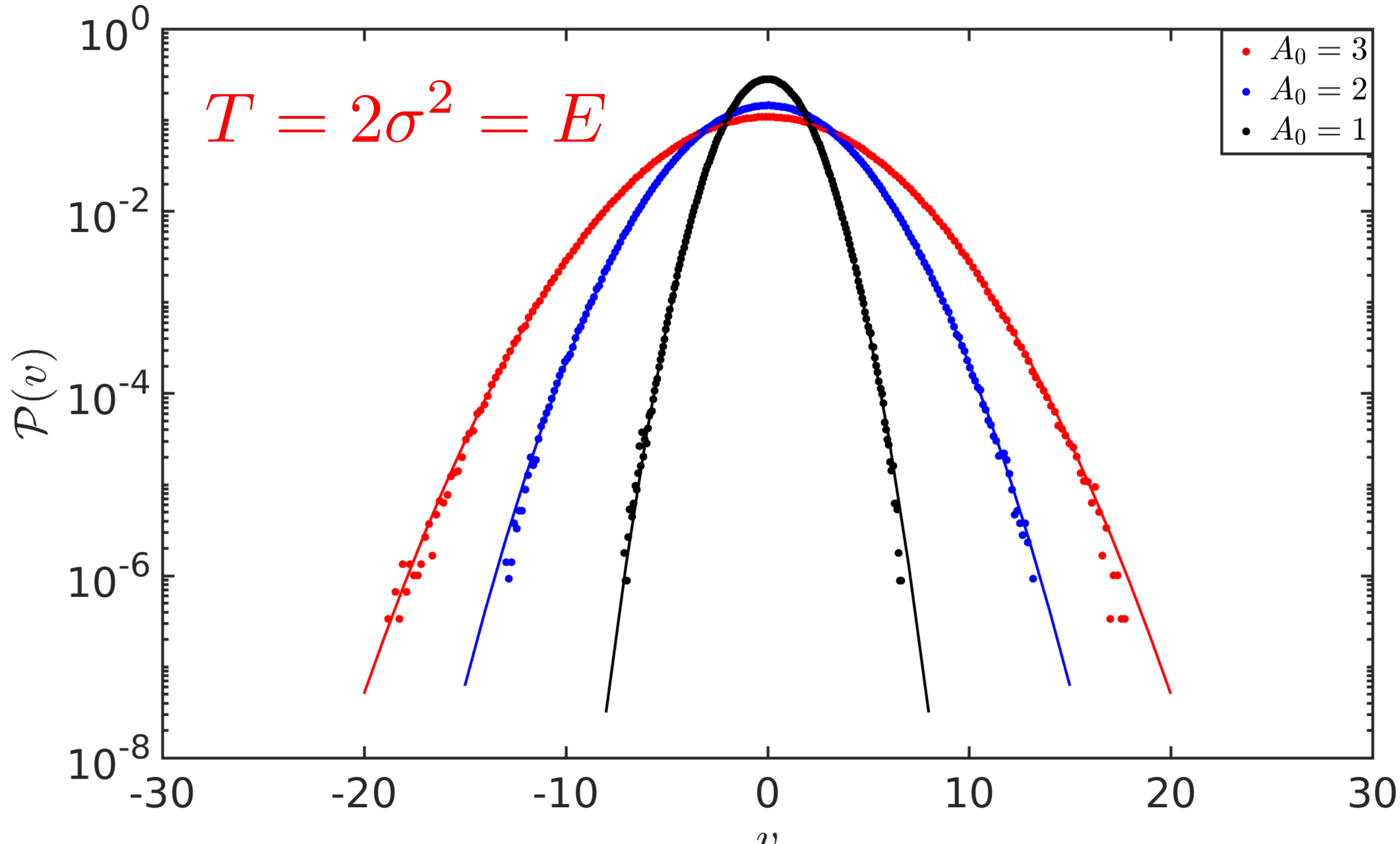
Galerkin-truncated Burgers (Spectral Methods) → Thermalises / Statistical Equilibria → $v^{\rm th}$



Vary temperature by varying the amplitude of the initial conditions since energy is conserved

Many-body Chaos: The Burgers Equation

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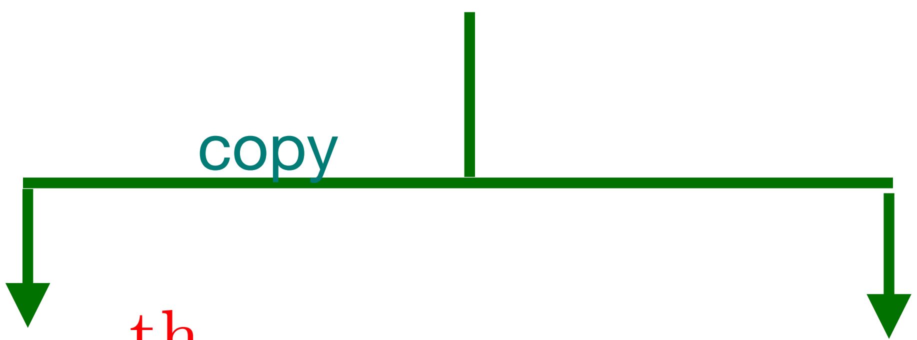
Strategy

$$v^{\mathrm{th}}$$



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Murugan, Kumar, Bhattacharjee, and **Ray**, *in preparation* (see also: ArXiv: 1906.00016)

Many-body Chaos: The Burgers Equation

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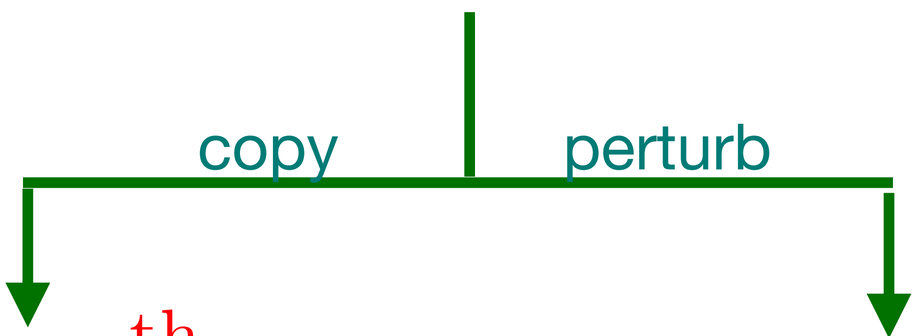

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Many-body Chaos: The Burgers Equation

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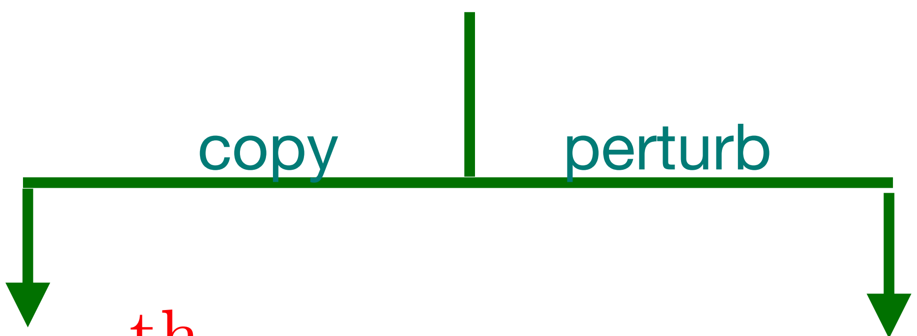

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Independent evolution of systems *a* and *b*

$$\frac{\partial v}{\partial t} + \mathrm{P}_{\mathrm{K_G}} \frac{1}{2}\frac{\partial v^2}{\partial x} = 0$$

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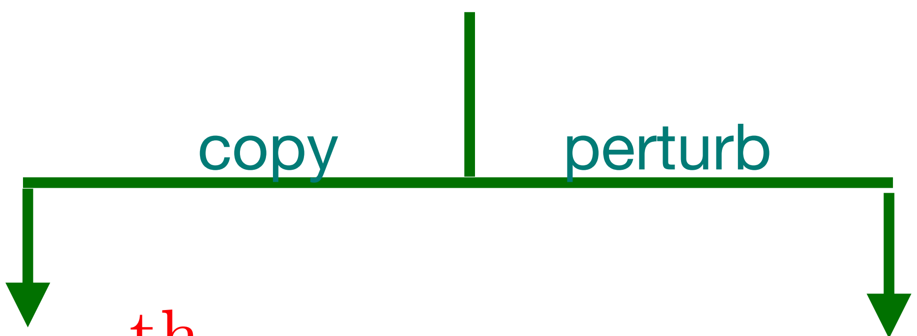

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Many-body Chaos: The Burgers Equation

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Strategy

$$v^{\mathrm{th}}$$

copy — perturb

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Theory

$$\frac{\partial \Delta}{\partial t} + P_{K_G}\left[\frac{\partial \Delta v}{\partial x} + \frac{1}{2}\frac{\partial \Delta^2}{\partial x}\right] = 0$$

Linearised Form

Trivial to solve under reasonable assumptions
(validated numerically)

Murugan, Kumar, Bhattacharjee, and **Ray**, *in preparation* (see also: ArXiv: 1906.00016)

Many-body Chaos: The Burgers Equation

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$$\lambda \sim N_G \sqrt{T}$$

Recall the Quantum bound: $\lambda \leq T/\hbar$

Maldacena, Shenker, and Stanford, JHEP (2016)
Kurchan, ArXiv: 1612.01278
Bilitewski, Bhattacharjee, and Moessner, Phys. Rev. Lett. (2018)

Many-body Chaos: The Burgers Equation

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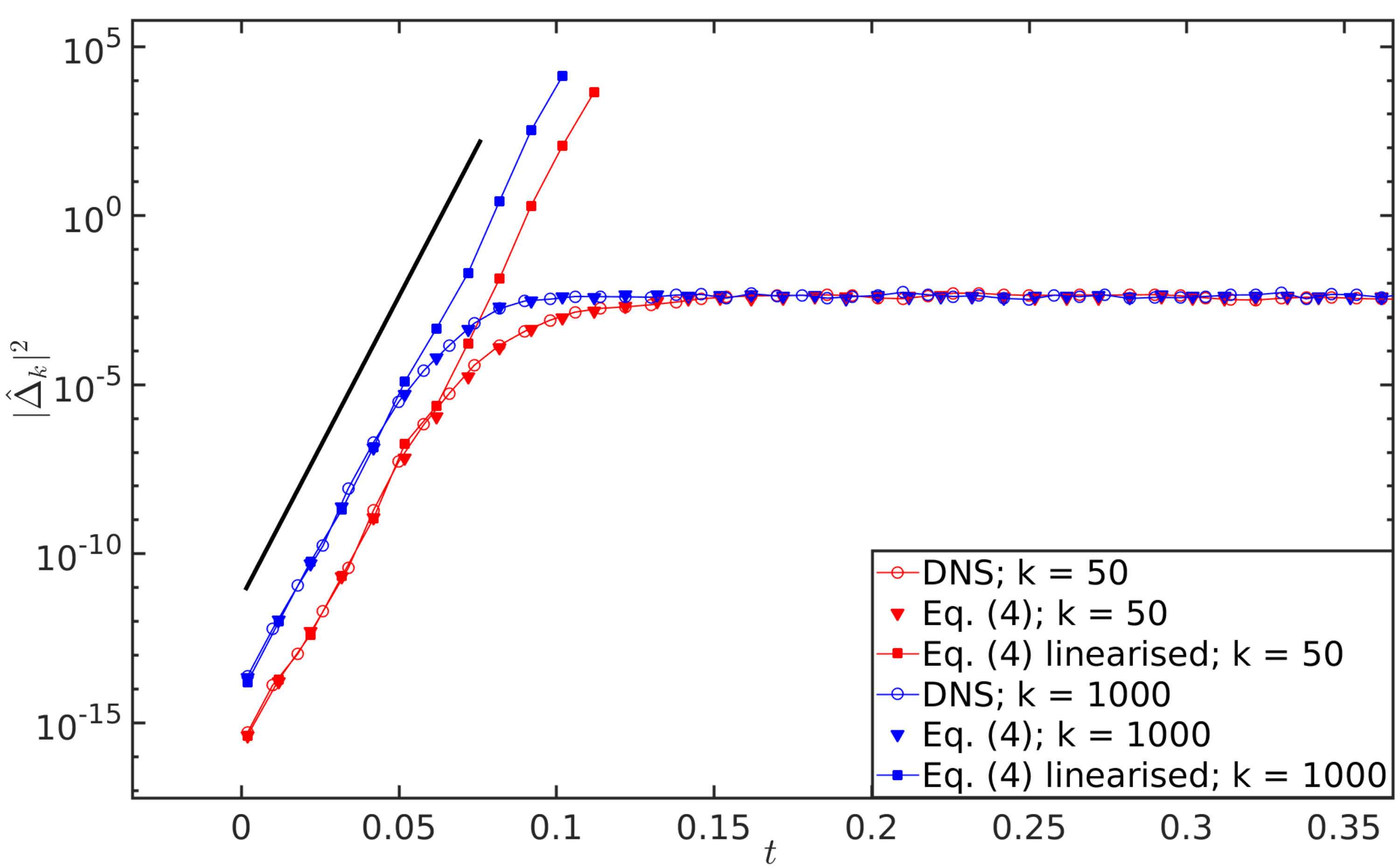
Direct Numerical Simulations



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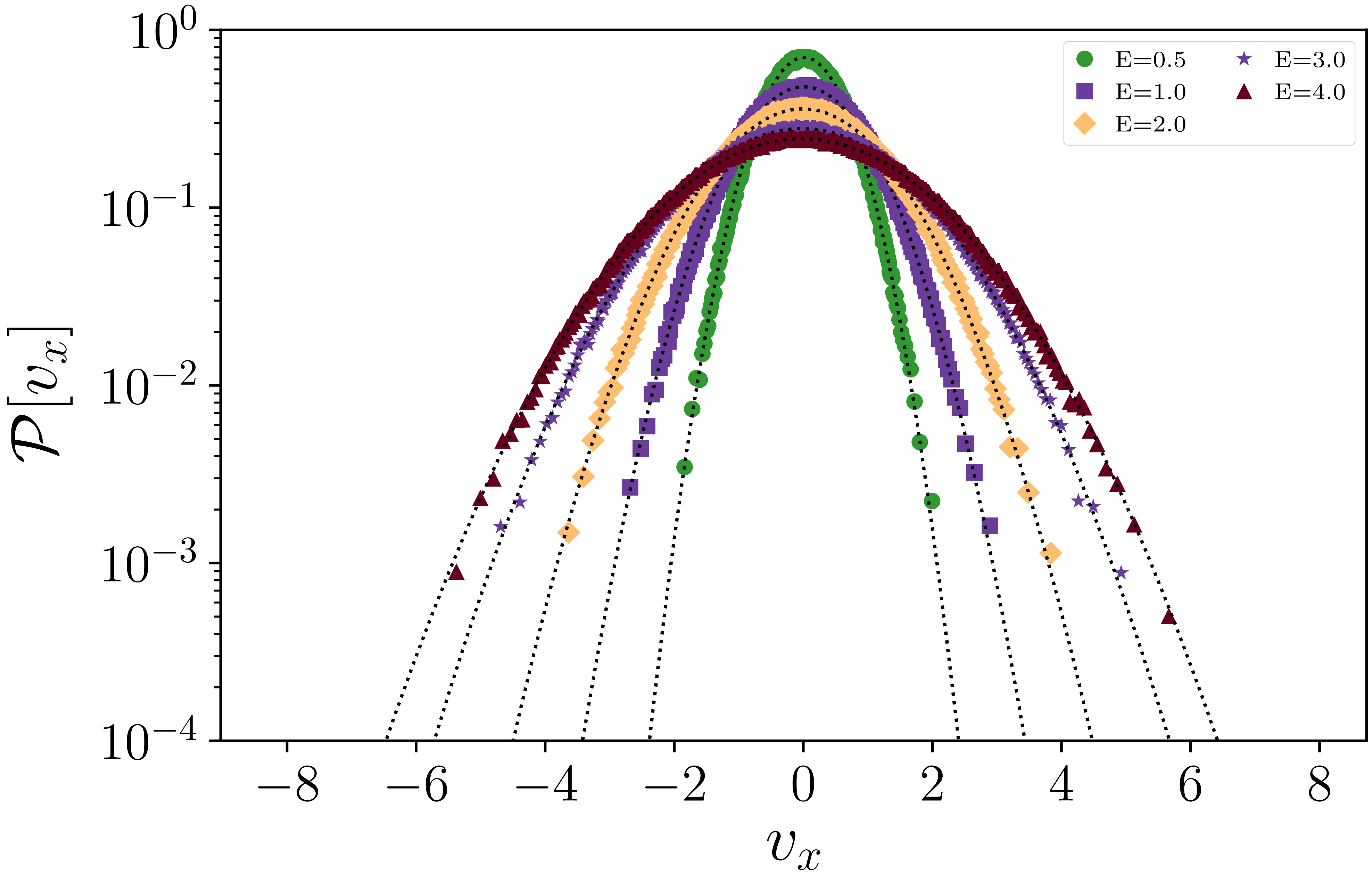

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Many-body Chaos: The Euler Equation

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Strategy

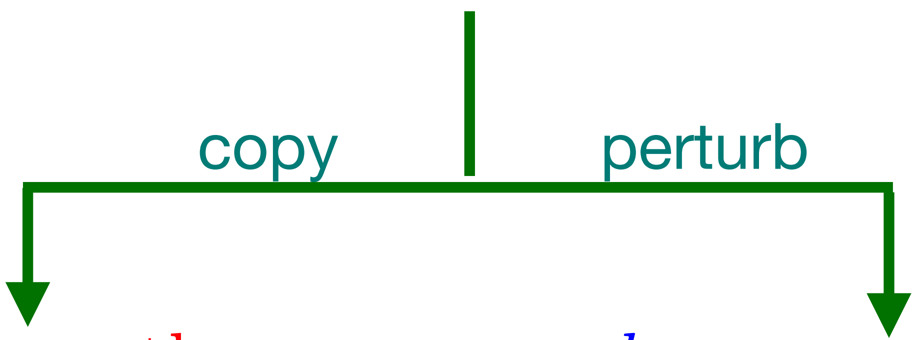
$$\mathbf{v}^{\mathrm{th}}$$

$$\mathbf{v}_0^a = \mathbf{v}^{\mathrm{th}}$$

Many-body Chaos: The Euler Equation

Strategy

$$\mathbf{v}^{\mathrm{th}}$$



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Murugan, Kumar, Bhattacharjee, and **Ray**, *in preparation* (see also: ArXiv: 1906.00016)

Many-body Chaos: The Euler Equation

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Strategy

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copy — perturb

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Independent evolution of systems *a* and *b*

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$$\delta\mathbf{u}(\mathbf{x}, t) \equiv \mathbf{v}^a(\mathbf{x}, t) - \mathbf{v}^b(\mathbf{x}, t)$$

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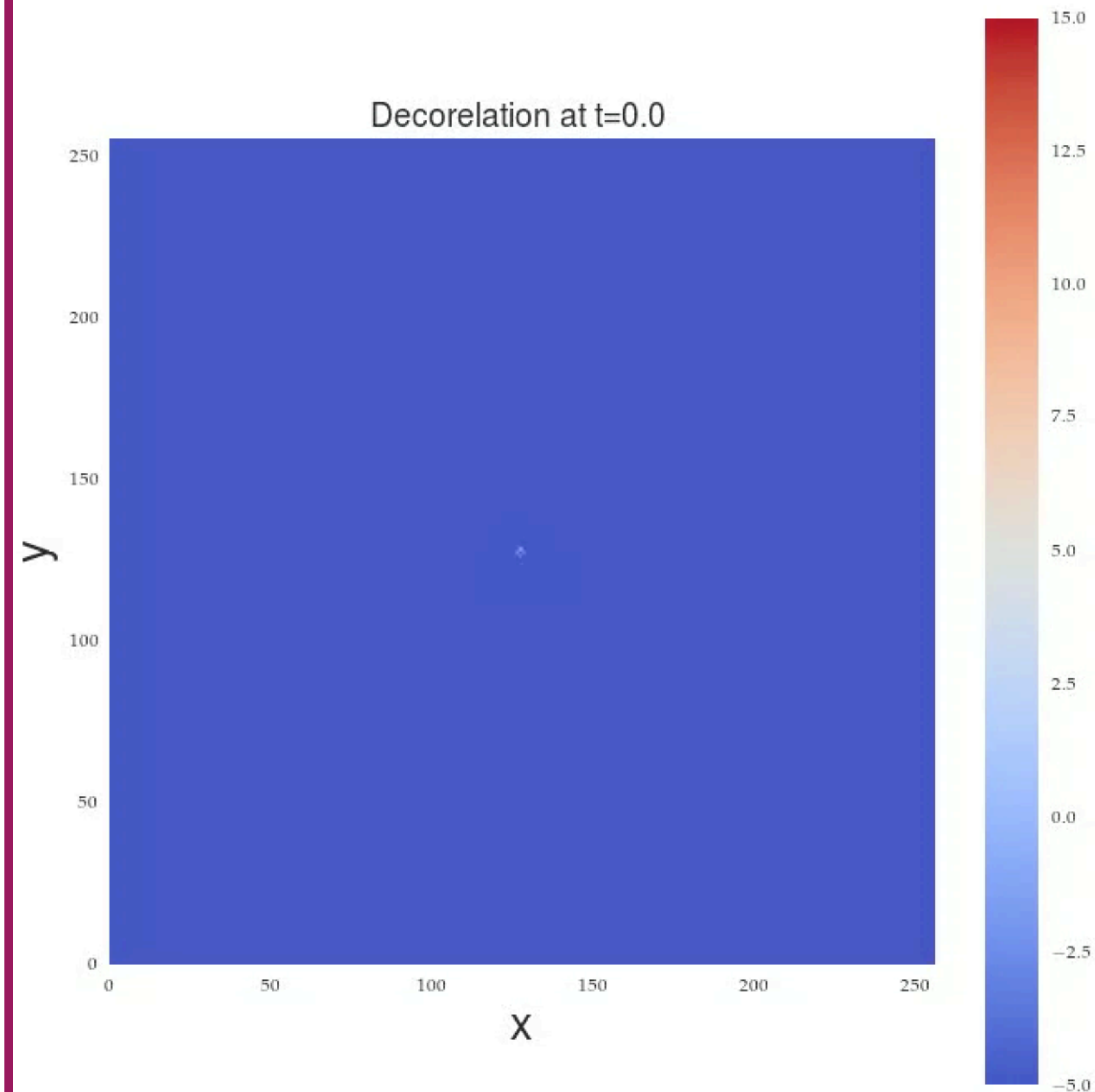

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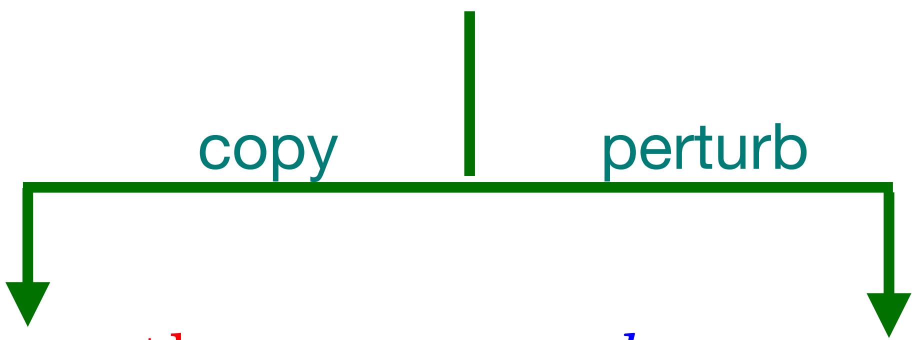

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Decorrelator

$$\phi(r,t) = \frac{1}{4\pi r^2}\int_{\mathcal{D}} d\mathbf{x}\, \frac{1}{2}|\boldsymbol{\delta}\mathbf{u}|^2 \boldsymbol{\delta}\left(|\mathbf{x}| - r\right) = \frac{1}{2}\left\langle |\boldsymbol{\delta}\mathbf{u}|^2 \right\rangle_r$$

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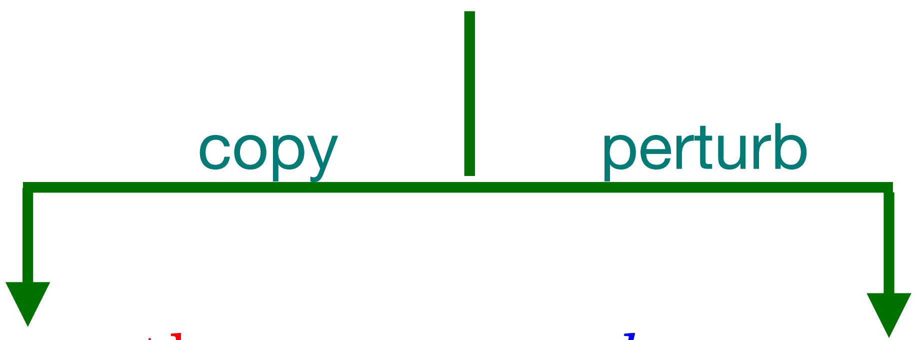

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Many-body Chaos: The Euler Equation

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Strategy

$\mathbf{v}^{\mathrm{th}}$

copy | perturb

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Theory

From the 3D Euler Equations

Linearised equation for $\delta\mathbf{u}$

$$\partial_t \delta u_i(\mathbf{x},t) \approx -v_j^{\mathrm{th}}\partial_j \delta u_i - \delta u_j(\partial_j u_i) + \partial_i T$$

$$T = 2\delta_{ij}^2 \int d\mathbf{y} G(|\mathbf{x}-\mathbf{y}|)\delta u_j(\mathbf{y}) v_k^{\mathrm{th}}(\mathbf{y})$$

Murugan, Kumar, Bhattacharjee, and **Ray**, *in preparation* (see also: ArXiv: 1906.00016)

Many-body Chaos: The Euler Equation

Strategy

$\mathbf{v}^{\rm th}$

copy | perturb

$$\mathbf{v}_0^a = \mathbf{v}^{\rm th} \qquad \mathbf{v}_0^b = \mathbf{v}_0^a + \delta\mathbf{v}_0$$

$$\delta\mathbf{v}_0 = \nabla \times \mathbf{A}; \quad A_i \epsilon_0 v_{\rm rms} e^{-\frac{r^2}{r_0^2}}$$

Independent evolution of systems *a* and *b*

Decorrelator

$$\phi(r,t) = \frac{1}{4\pi r^2}\int_{\mathcal{D}} d\mathbf{x}\, \frac{1}{2}|\boldsymbol{\delta}\mathbf{u}|^2 \boldsymbol{\delta}\left(|\mathbf{x}| - r\right) = \frac{1}{2}\left\langle |\boldsymbol{\delta}\mathbf{u}|^2 \right\rangle_r$$

$$\Phi(t) = \frac{1}{V}\int_{\mathcal{D}} d\mathbf{x}\, \frac{1}{2}|\delta\mathbf{u}(\mathbf{x},t)|^2 = \frac{1}{2}\left\langle |\boldsymbol{\delta}\mathbf{u}|^2 \right\rangle$$

Theory

From the 3D Euler Equations

↓

Linearised equation for $\delta\mathbf{u}$

$$\partial_t \delta u_i(\mathbf{x},t) \approx -v_j^{\rm th}\partial_j \delta u_i - \delta u_j(\partial_j u_i) + \partial_i T$$

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↓

$$\partial_t \Phi(t) \approx -\langle \delta u_i S_{ij} \delta u_j \rangle$$

$$S_{ij} = \frac{1}{2}(\partial_j v_i^{\rm th} + \partial_i v_j^{\rm th})$$

Murugan, Kumar, Bhattacharjee, and **Ray**, *in preparation* (see also: ArXiv: 1906.00016)

Many-body Chaos: The Euler Equation

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Strategy

$$\mathbf{v}^{\rm th}$$

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Independent evolution of systems *a* and *b*

Decorrelator

$$\phi(r,t) = \frac{1}{4\pi r^2} \int_{\mathcal{D}} d\mathbf{x}\, \frac{1}{2} |\boldsymbol{\delta}\mathbf{u}|^2 \boldsymbol{\delta}\left(|\mathbf{x}| - r\right) = \frac{1}{2} \left\langle |\boldsymbol{\delta}\mathbf{u}|^2 \right\rangle_r$$

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Theory

$$\partial_t \Phi(t) \approx -\langle \delta u_i S_{ij} \delta u_j \rangle$$

$$S_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$$

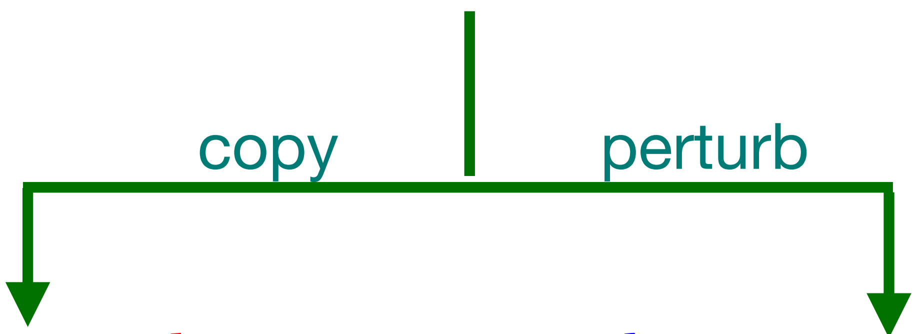
Diagonalising and solving in the basis of the eigenvectors of the strain-rate matrix

Murugan, Kumar, Bhattacharjee, and **Ray**, *in preparation* (see also: ArXiv: 1906.00016)

Many-body Chaos: The Euler Equation

Strategy

$$\mathbf{v}^{\rm th}$$



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Independent evolution of systems *a* and *b*

Decorrelator

$$\phi(r,t) = \frac{1}{4\pi r^2}\int_{\mathcal{D}} d\mathbf{x}\, \frac{1}{2}|\boldsymbol{\delta}\mathbf{u}|^2 \boldsymbol{\delta}\left(|\mathbf{x}| - r\right) = \frac{1}{2}\left\langle |\boldsymbol{\delta}\mathbf{u}|^2 \right\rangle_r$$

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Theory

$$\partial_t \Phi(t) \approx -\langle \delta u_i S_{ij} \delta u_j \rangle$$

$$S_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$$

Diagonalising and solving in the basis of the eigenvectors of the strain-rate matrix

Perturbations align with the compressional direction

$$\partial_t \Phi(t) \propto -\lambda_c \Phi$$

Murugan, Kumar, Bhattacharjee, and **Ray**, *in preparation* (see also: ArXiv: 1906.00016)

Many-body Chaos: The Euler Equation

Strategy

$$\mathbf{v}^{\mathrm{th}}$$

copy | perturb

$$\mathbf{v}_0^a = \mathbf{v}^{\mathrm{th}} \qquad \mathbf{v}_0^b = \mathbf{v}_0^a + \delta\mathbf{v}_0$$

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Independent evolution of systems *a* and *b*

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Murugan, Kumar, Bhattacharjee, and **Ray**, *in preparation* (see also: ArXiv: 1906.00016)

Theory

Linearised Solution

$$\partial_t \Phi(t) \approx -\langle \delta u_i S_{ij} \delta u_j \rangle$$

$$S_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$$

Short time $\Phi(t) \sim e^{\lambda t}$

Long time $\Phi(t) \sim E/N_G$

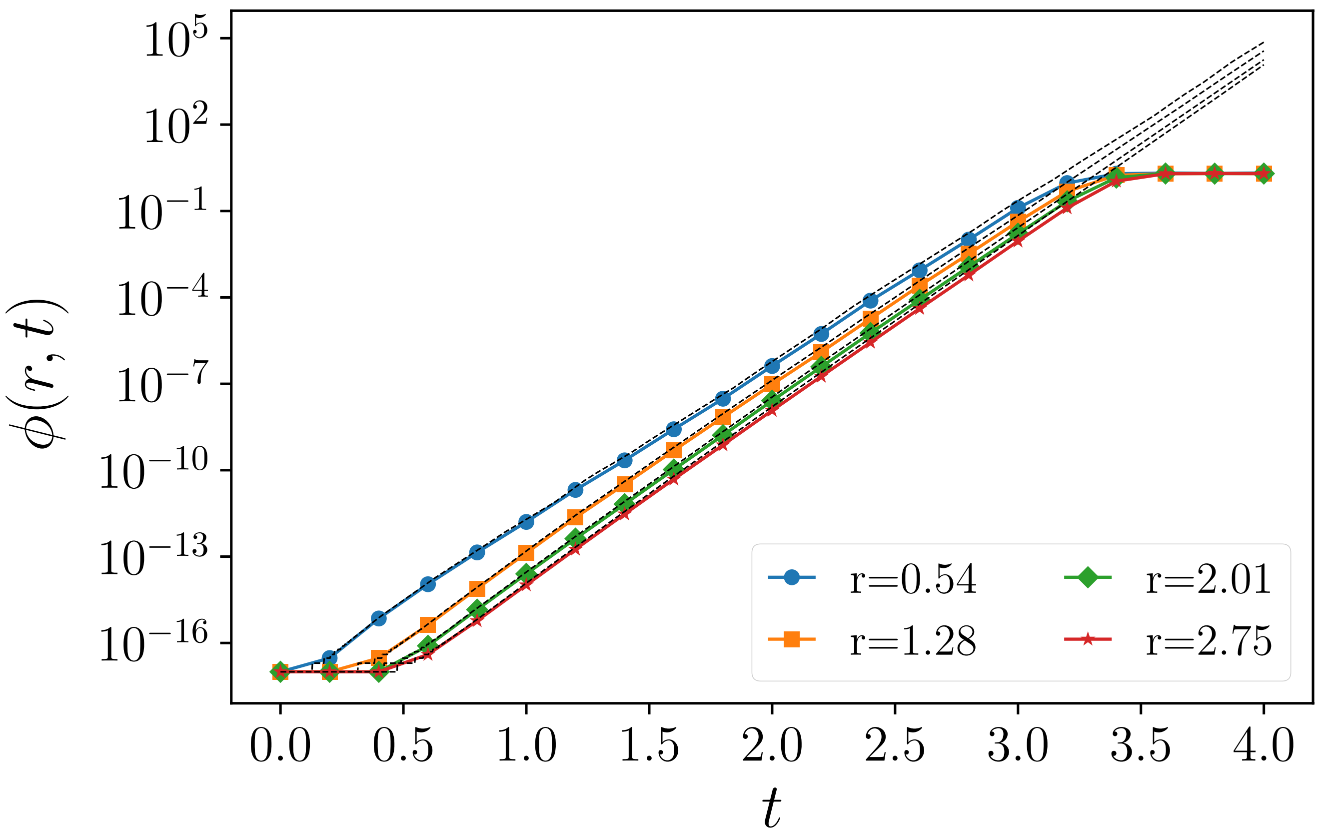
$$\lambda \sim N_G\sqrt{T}$$

Recall the Quantum bound: $\lambda \leq T/\hbar$

Maldacena, Shenker, and Stanford, JHEP (2016)
Kurchan, ArXiv: 1612.01278
Bilitewski, Bhattacharjee, and Moessner, Phys. Rev. Lett. (2018)

Many-body Chaos: The Euler Equation

Numerical Results



Theory

Decorrelator

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Predictions

Short time $\Phi(t) \sim e^{\lambda t}$

Long time $\Phi(t) \sim E/N_G$

Murugan, Kumar, Bhattacharjee, and **Ray**, *in preparation* (see also: ArXiv: 1906.00016)

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Many-body Chaos: The Euler Equation

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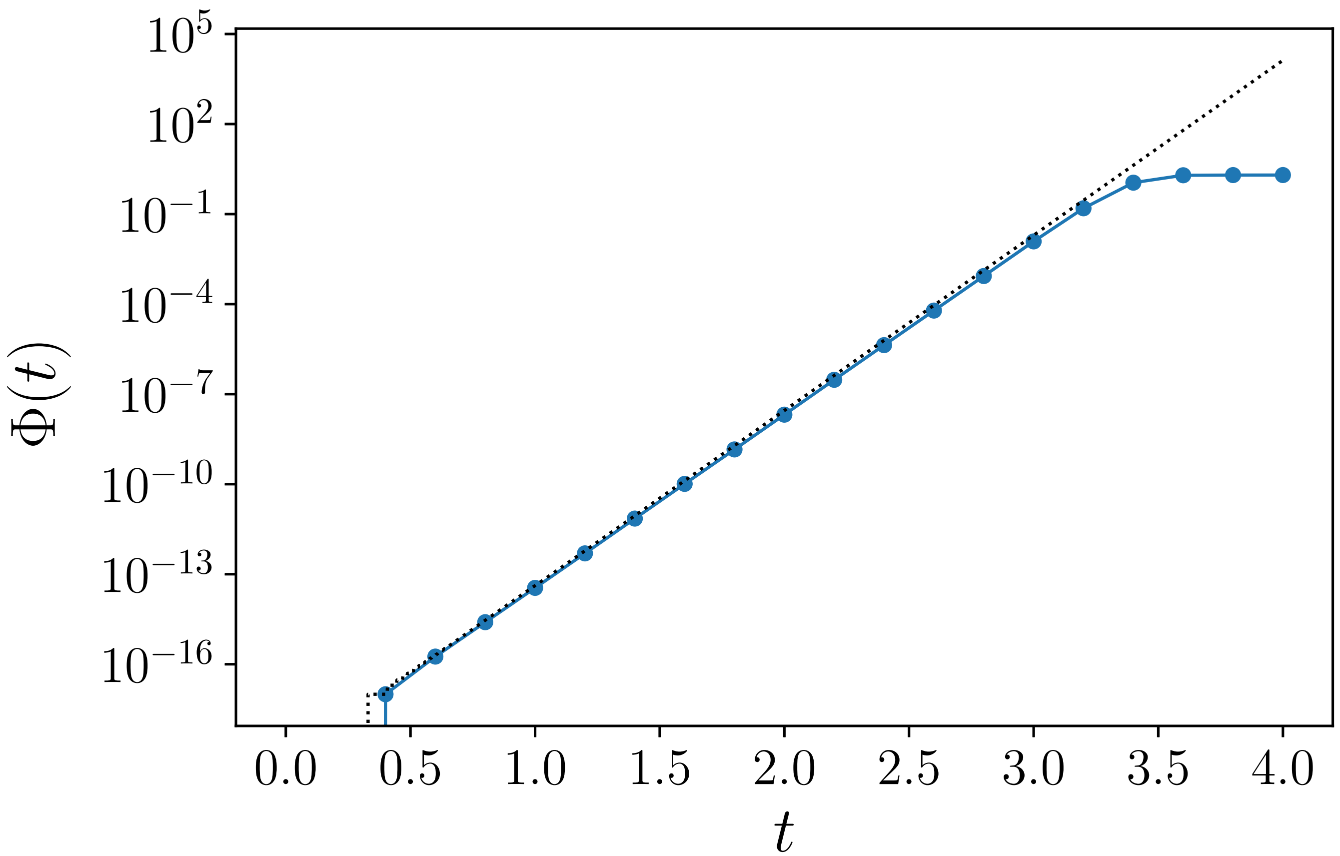

Murugan, Kumar, Bhattacharjee, and **Ray**, *in preparation* (see also: ArXiv: 1906.00016)

Theory

Decorrelator

$$\phi(r,t) = \frac{1}{4\pi r^2}\int_{\mathcal{D}} d\mathbf{x}\, \frac{1}{2}|\boldsymbol{\delta}\mathbf{u}|^2 \boldsymbol{\delta}\left(|\mathbf{x}| - r\right) = \frac{1}{2}\left\langle|\boldsymbol{\delta}\mathbf{u}|^2\right\rangle_r$$

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Predictions

Short time	$\Phi(t) \sim e^{\lambda t}$
Long time	$\Phi(t) \sim E/N_G$

Kurchan, ArXiv: 1612.01278
Bilitewski, Bhattacharjee, and Moessner, Phys. Rev. Lett. (2018)

The Classical Bound of the Lyapunov Exponent

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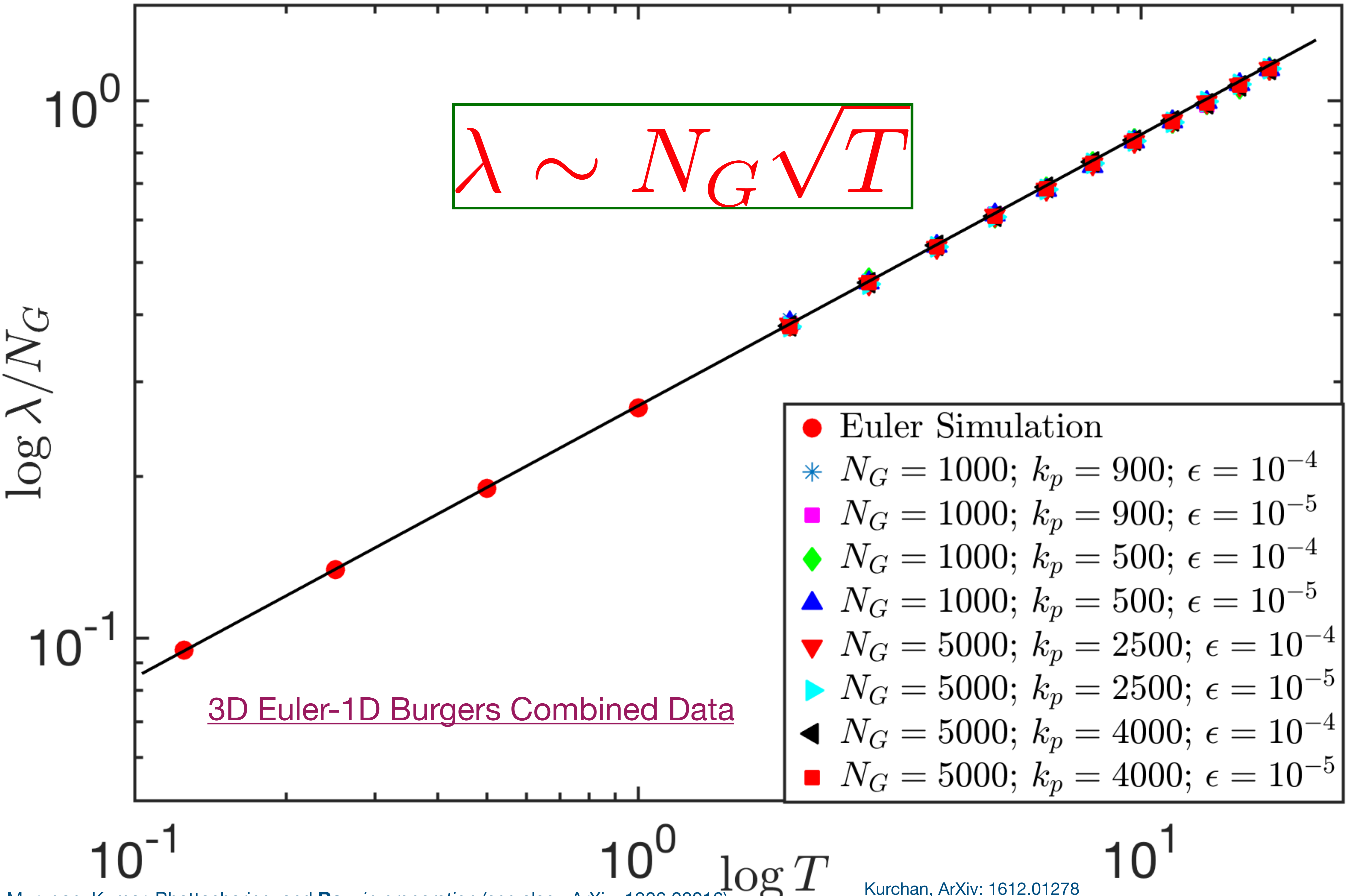

Murugan, Kumar, Bhattacharjee, and **Ray**, *in preparation* (see also: ArXiv: 1906.00016)

Kurchan, ArXiv: 1612.01278
Bilitewski, Bhattacharjee, and Moessner, Phys. Rev. Lett. (2018)

Lyapunov Exponent and Decorrelation Time: 3D Euler

Autocorrelation function (3D Euler)

$$C(t) = \frac{\langle \mathbf{v}^{\rm th}(\mathbf{x}, t_0) \cdot \mathbf{v}^{\rm th}(\mathbf{x}, t_0 + t) \rangle}{\langle \mathbf{v}^{\rm th^2} \rangle} \Longrightarrow \tau$$

Autocorrelation Function: 3D Euler

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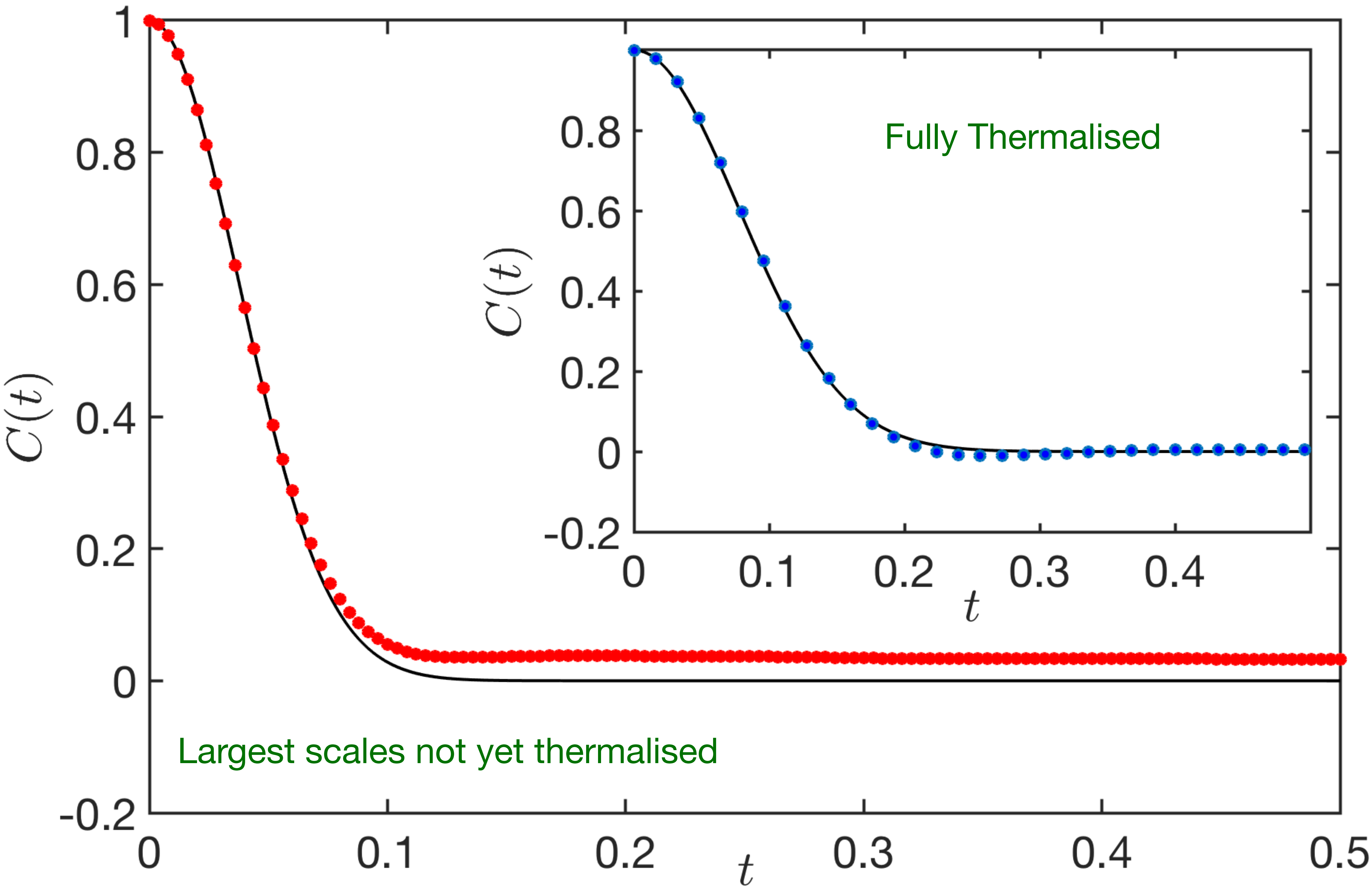
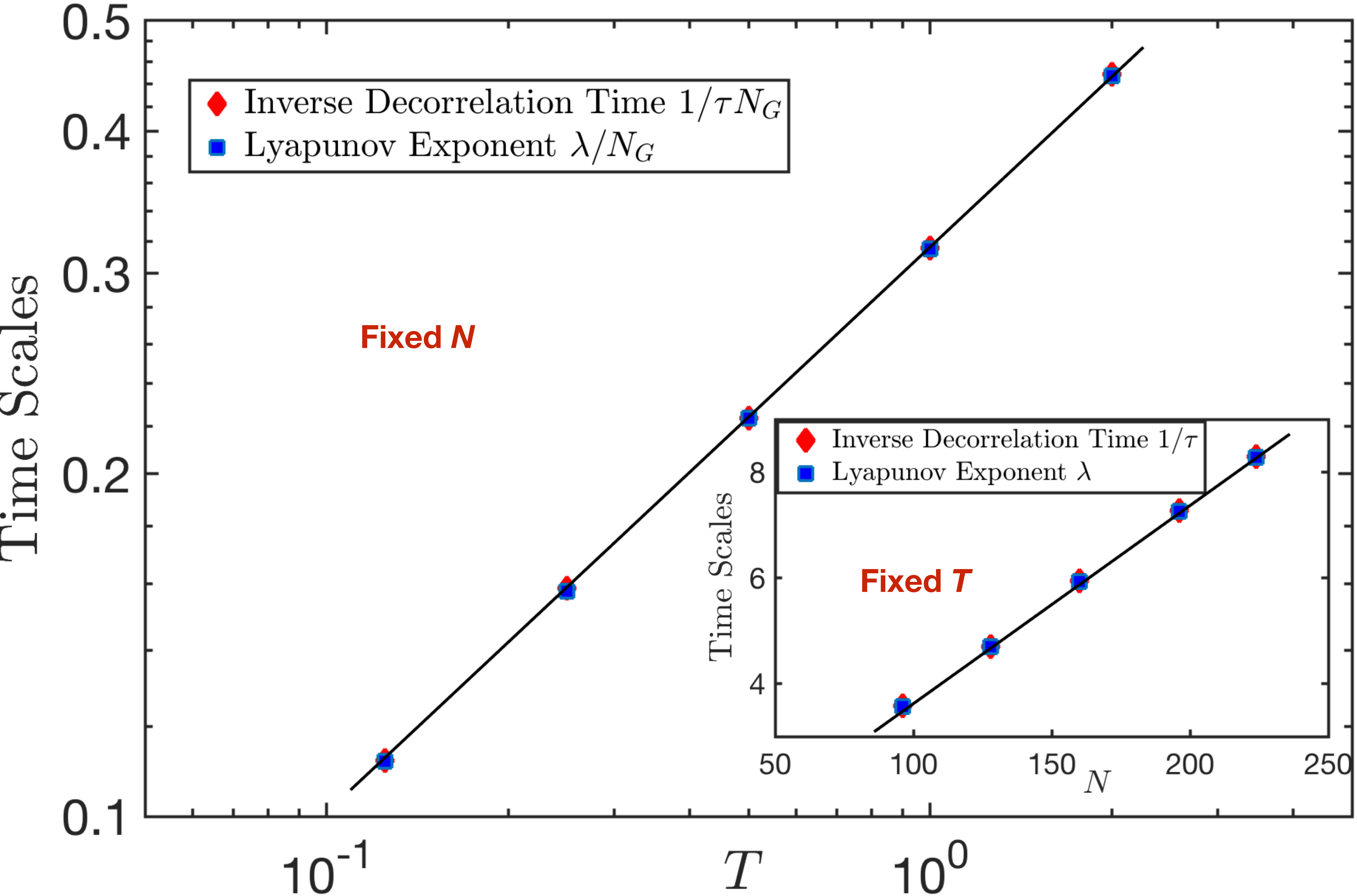
$$C(t) = \frac{\langle \mathbf{v}^{\rm th}(\mathbf{x},t_0)\cdot\mathbf{v}^{\rm th}(\mathbf{x},t_0+t)\rangle}{\langle \mathbf{v}^{\rm th^2}\rangle} \Longrightarrow \tau$$



Murugan, Kumar, Bhattacharjee, and **Ray**, *in preparation* (see also: ArXiv: 1906.00016)

Lyapunov Exponent and Decorrelation Time: 3D Euler



Murugan, Kumar, Bhattacharjee, and **Ray**, *in preparation* (see also: ArXiv: 1906.00016)

Perspective

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Ongoing Work

- *Spin Systems*: Long-range interactions [Murugan, **Ray**, and Bhattacharjee]
- *Hydrodynamics*: Gross-Pitaevskii equation [Shukla, Bhattacharjee, and **Ray**]
- *Dynamical Systems*: Shell models [Bhattacharjee and **Ray**]
- *Driven-Dissipative Systems*: Active Turbulence [Singh, Mukherjee, James, and **Ray**]

Epilogue

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PHYSICAL REVIEW RESEARCH **2**, 033202 (2020)

Suppressing thermalization and constructing weak solutions in truncated inviscid equations of hydrodynamics: Lessons from the Burgers equation

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[2] *Université Côte d'Azur, CNRS, OCA, Laboratoire J.-L. Lagrange, Nice, France*
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Finite-dimensional, inviscid equations of hydrodynamics, obtained through a Fourier-Galerkin projection, thermalize with an energy equipartition. Hence, numerical solutions of such inviscid equations, which typically must be Galerkin-truncated, show a behavior at odds with the parent equation. An important consequence of this is an uncertainty in the measurement of the temporal evolution of the distance of the complex singularity from the real domain leading to a lack of a firm conjecture on the finite-time blow-up problem in the incompressible, three-dimensional Euler equation. We now propose, by using the one-dimensional Burgers equation as a testing ground, a numerical recipe, named *tyger purging*, to arrest the onset of thermalization and hence recover the true dissipative solution. Our method, easily adapted for higher dimensions, provides a tool to not only tackle the celebrated blow-up problem but also to obtain weak and dissipative solutions—conjectured by Onsager and numerically elusive thus far—of the Euler equation.

Numerically obtaining weak (dissipative) solutions from Galerkin-truncated inviscid equations