NOISE SENSITIITY IN CONTINUUM PERCOLATION VIA RANDOMIZED ALGORITHMS AND SPECTRAL SAMPLES.

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https://www.isibang.ac.in/~d.yogesh/Talks/SSP/ ;

https://www.isibang.ac.in/~d.yogesh/Talks/SSP.pdf arXiv:2101.07180 arXiv:2407.13502

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"There is a general philosophy that if f is a function of `high complexity' then the support of the Fourier transform has to be spread out"

– Jean Bourgain.

- E. Friedgut KKL's Influence on me. Proc. ICM 2022.
- C. Garban and J. Steif Noise Sensitivity of Boolean functions and Percolation. 2014.
- R. O'Donnell Analysis of Boolean functions. 2014.
- C. Garban Oded's Schramm Contributions to Noise Sensitivity. Ann. Probab. 2011.

Discrete Fourier analysis : Key objects

- $f : \Omega = \{\pm 1\}^n \to \{\pm 1\}$; $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \sim$ Uniform $\{\pm 1\}^n$; I.I.D. symmetric random bits.
- Characters / Characteristic functions: $\chi_s : x = (x_1, \ldots, x_n) \mapsto \prod_{i \in S} x_i$, $S \subset [n]$; $\chi_\varnothing = 1$.
- Fourier-Walsh Expansion: $f = \sum_{S \subset [n]} \hat{f}(S) \chi_S$. $\hat{f}(S) := \mathbb{E} f(\varepsilon) \chi_S(\varepsilon) = 2^{-n} \sum_{x \in \Omega} f(x) \prod_{i \in S} x_i$
- Fourier Energy: $E_f(k) := \sum_{S : \# S = k} \hat{f}^2(S)$, $k = 0, \ldots, n.$ $\sum_{k=0}^{n} E_f(k) = 1$ (Plancherel's)
- Spectral sample: $\mathcal{S} \subset [n]$; $\mathbb{P}(\mathcal{S} = S) = \hat{f}^2(S)$, $S \subset [n]$. $\mathbb{P}(\# \mathcal{S} = k) = E_f(k)$, $k \geq 0$.
- t re-sampling: $\mathbb{E} f(\varepsilon) f(\varepsilon^t) = \mathbb{E} e^{-t \# \mathcal{S}} = \sum_{k=0}^{n} e^{-tk} E_f(k)$; Bits re-sampled w. Prob. $1 - e^{-t}$
- Sensitive to Noise / Perturbations $\leftrightarrow$ Fourier spectrum concentrated on `high frequencies'.

Overview and Motivation

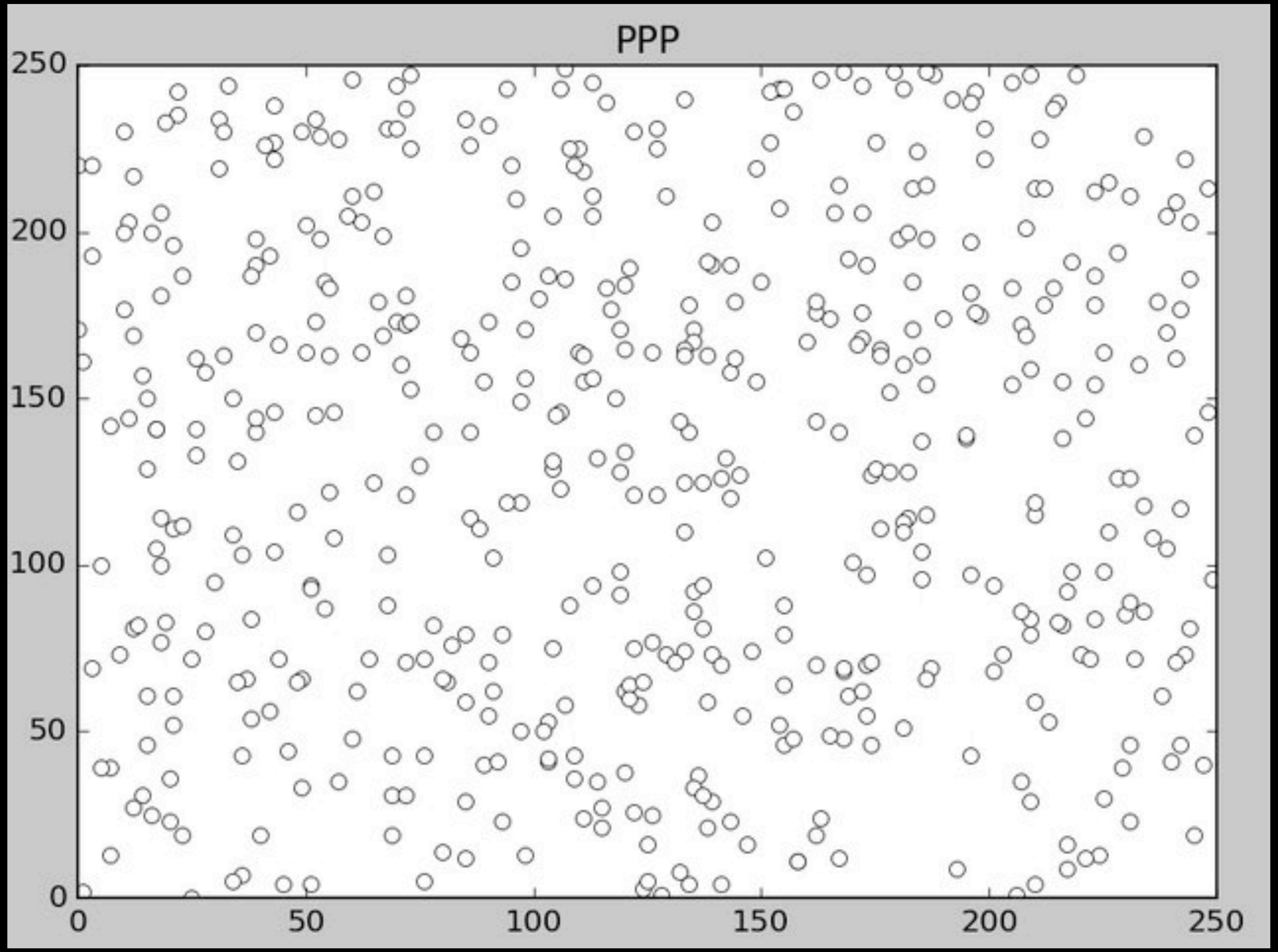
- Percolation theory - Spectacular progress in recent decades. Robust and diverse toolkit.
- Dynamical percolation - less explored. Introduced by Haggstrom, Peres & Steif (1996)
- Noise sensitivity / stability - Noise as dynamics for short-time intervals;
- Benjamini, Kalai and Schramm (1999) - Hypercontractivity & randomized algorithms to show noise sensitivity but non-quantitative. Continuum results via discretisation by Ahlberg, Broman, Griffith and Morris (2014).
- Schramm & Steif (2010)- Randomized algorithms to control Fourier coefficients & gives quantitative but not sharp noise sensitivity. Continuum results via discretisation by Ahlberg and Baldasso (2018) ; Pure continuum analogue by Last, Peccati and Y. (2023).
- Garban, Pete & Schramm (2010) - Spectral sample approach to sharp noise sensitivity. Discretisation is restrictive ! Voronoi percolation by Vanneuville (2019, 2021).

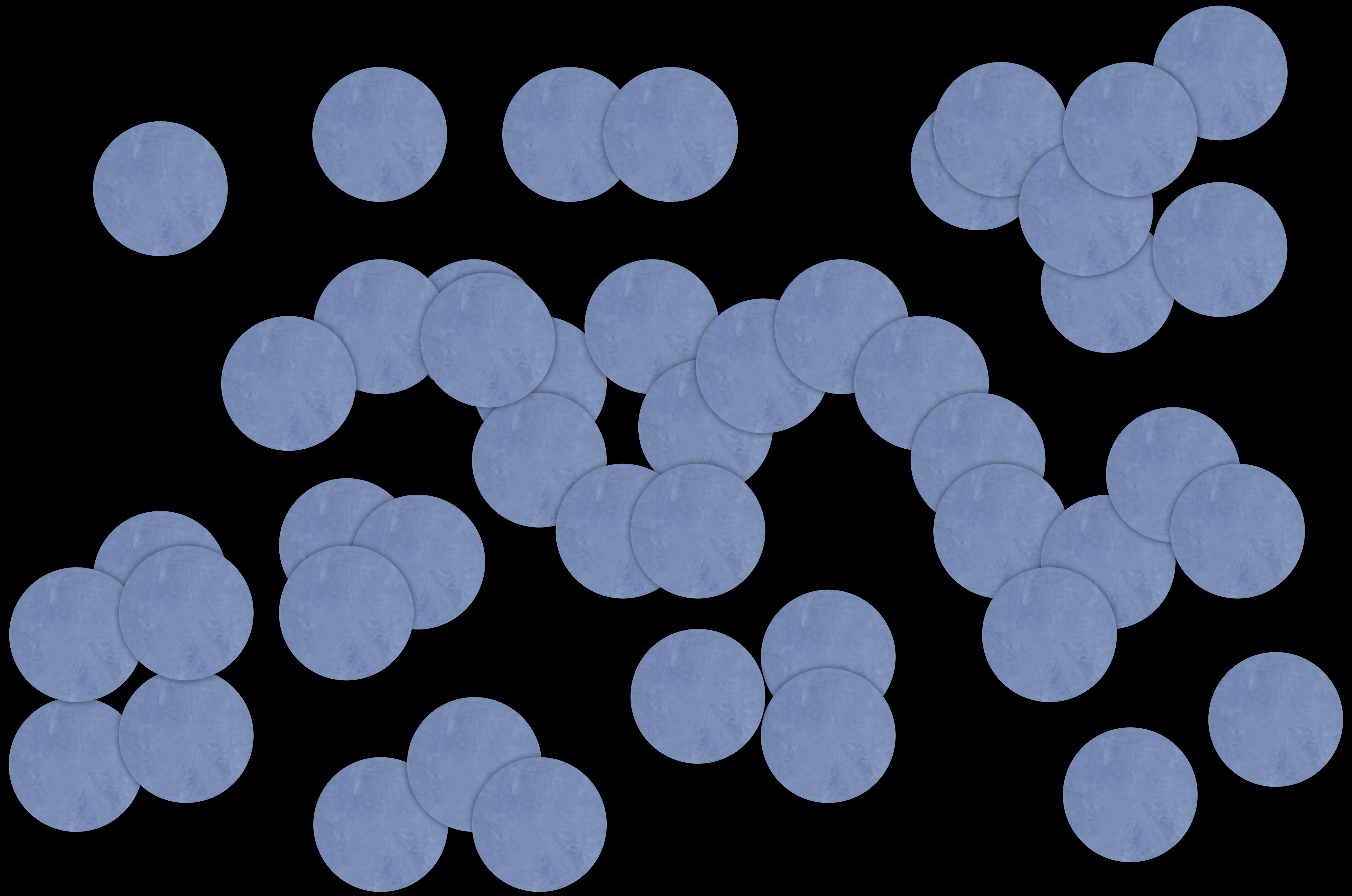
Planar Continuum percolation

$\eta = \sum_{i \geq 1} \delta_{X_i} = \{X_i\}_{i \geq 1}$ Stationary Poisson process in $\mathbb{R}^2$ with intensity $\lambda > 0$;

Poi (λn^d) points in $[0,n]^d$ uniformly and independently and $n \to \infty$.

$\mathcal{O}(\eta) := \cup_{X_i \in \eta} B_1(X_i)$ - Occupied region.



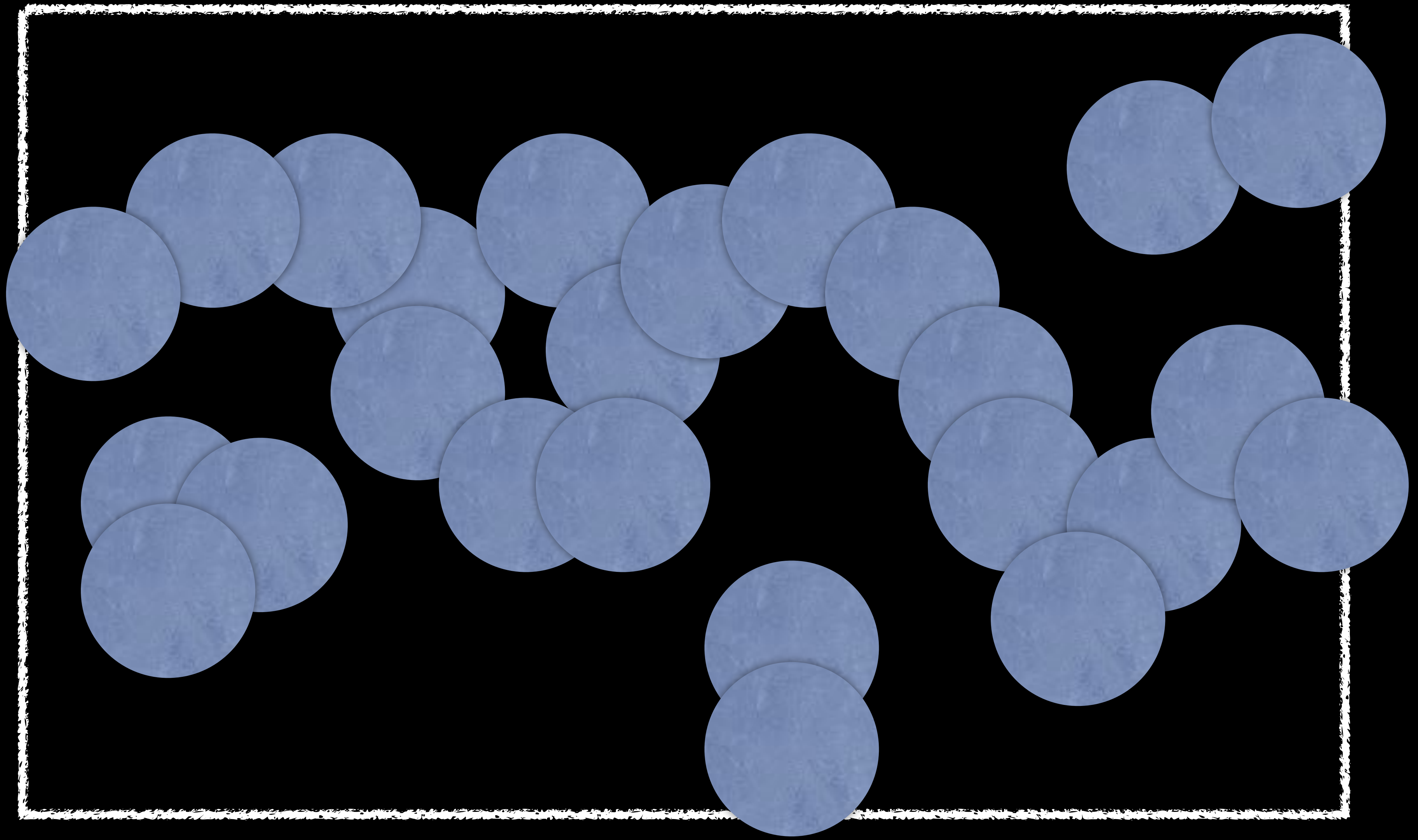


Planar Continuum percolation : Phase transition

$\eta = \eta(\lambda) = \sum_{i\geq 1} \delta_{X_i} = \{X_i\}_{i\geq 1}$ Stationary Poisson process in $\mathbb{R}^2$ with intensity $\lambda > 0$;

$\mathcal{O}(\eta) := \cup_{X_i \in \eta} B_1(X_i)$ - Occupied region.

$Cross_L(\eta) :=$ L-R Crossing of $\mathcal{O}(\eta) \cap W_L$; $W_L = [-L, L]^2$



Sub-critical

$\mathbb{P}(Cross_L(\eta)) \rightarrow 0$ if $\lambda < \lambda_c$

$\mathbb{P}(Cross_L(\eta)) \xrightarrow{\text{RSW}} [a, 1-a]$ if $\lambda = \lambda_c$

$\mathbb{P}(Cross_L(\eta)) \rightarrow 1$ if $\lambda > \lambda_c$

Super-critical

Sharp noise instability / sensitivity

$\eta = \{X_i\}_{i\geq 1}$; Stationary Poisson(λ) process in $\mathbb{R}^2$; $\mathcal{O}(\eta) := \cup_{X_i \in \eta} B_1(X_i)$ - Occupied region.

$Cross_L(\eta)$:= L-R Crossing of $\mathcal{O}(\eta) \cap W_L$; $W_L = [-L, L]^2$

η_t - Stationary dynamics for Poisson process.

Noise Stable

$$\lim_{t\to 0} \sup_L \mathbb{P}(Cross_L(\eta) \neq Cross_L(\eta_t)) = 0.$$

Noise Sensitive

$$\lim_{L\to\infty} \mathbb{COV}(Cross_L(\eta), Cross_L(\eta_t)) = 0, \forall t > 0.$$

Sharp Noise Instable $\exists A_L \to \infty$, Noise stable for $t_L A_L \to 0$ & not for $t_L A_L \to \infty$.

$$\boxed{\mathbb{P}(Cross_L(\eta) \neq Cross_L(\eta_{t_L})) \to 0.}$$

$$\boxed{\liminf_{L\to\infty} \mathbb{P}(Cross_L(\eta) \neq Cross_L(\eta_{t_L})) > 0.}$$

Sharp noise instability / sensitivity : Predictions

$\eta = \{X_i\}_{i\geq 1}$; Stationary Poisson(λ) process in $\mathbb{R}^2$; $\mathcal{O}(\eta)$ - Occupied region; $\mathcal{V}(\eta) = \mathcal{O}^c$, Vacant region.

$Cross_L(\eta) :=$ L-R Crossing of $\mathcal{O}(\eta) \cap W_L$; $W_L = [-L, L]^2$

Sub-critical

$$\mathbb{P}(Cross_L(\eta)) \to \begin{cases} 0 & \text{if } \lambda < \lambda_c \\ [a, 1-a] & \text{if } \lambda = \lambda_c \\ 1 & \text{if } \lambda > \lambda_c \end{cases}$$

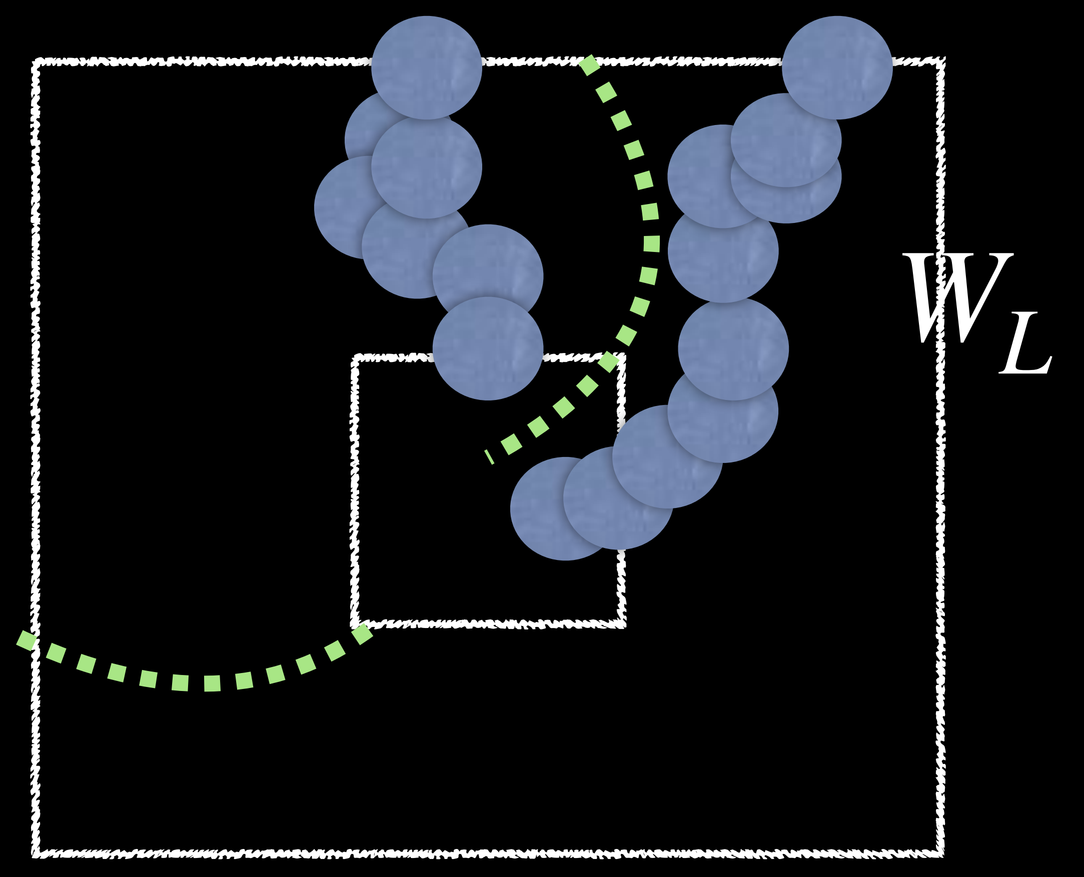
Super-critical

Noise stable and sensitive as degenerate.

Sharp noise instable & sensitive with

$A_L = L^2 \alpha_4(L)$

Noise stable and sensitive as degenerate.



$\alpha_4(L)$ = 4-arm probability

= Probability of 4 alternating arms from W_1 to W_L

$\mathbb{E}$[# Pivotal points for $Cross_L(\eta)$] $\approx L^2\alpha_4(L)$

Why sharp noise instability ?

$\eta = \{X_i\}_{i\geq 1}$; Stationary Poisson(λ) process in $\mathbb{R}^2$; $\mathcal{O}(\eta)$ - Occupied region; $\mathcal{V}(\eta) = \mathcal{O}^c$, Vacant region.

$Cross_L(\eta)$:= L-R Crossing of $\mathcal{O}(\eta) \cap W_L$; $W_L = [-L, L]^2$

Sub-critical

$$\mathbb{P}(Cross_L(\eta)) \to \begin{cases} 0 & \text{if } \lambda < \lambda_c \\ [a, 1-a] & \text{if } \lambda = \lambda_c \\ 1 & \text{if } \lambda > \lambda_c \end{cases}$$

Super-critical

No Exceptional Times & Not volatile.

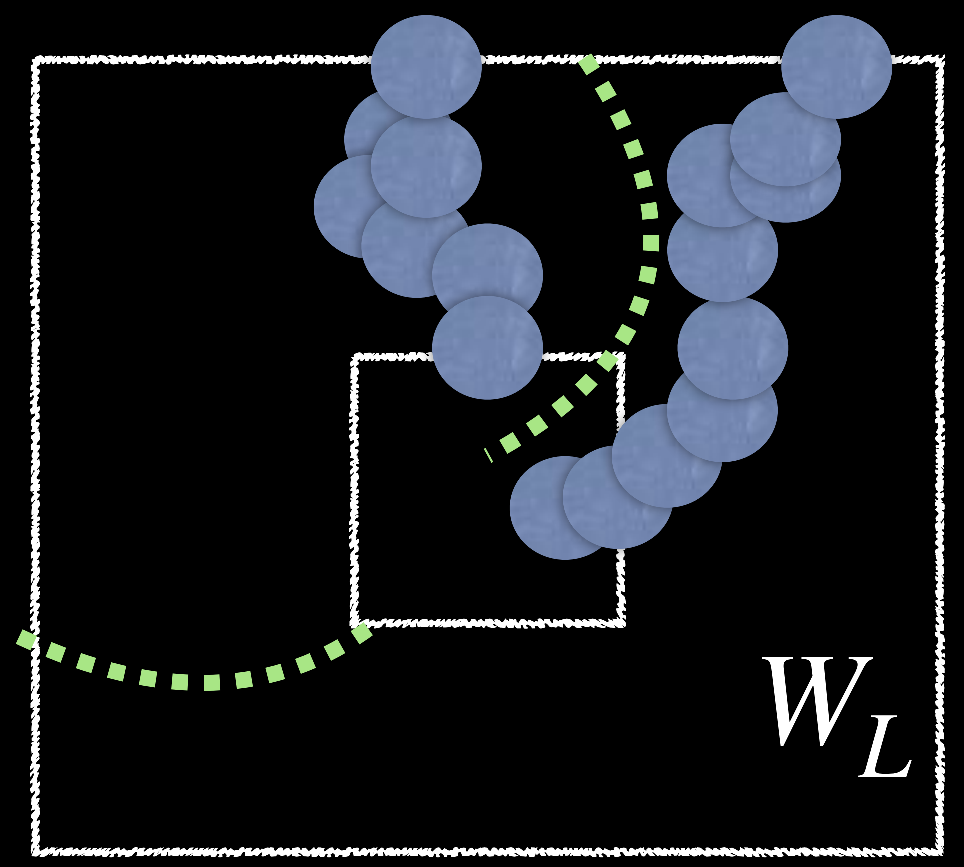
Volatile & Critical Window of Width A_L^{-1},

$$A_L = L^2 \alpha_4(L)$$

No Exceptional Times & Not volatile.

Critical Window of width A_L^{-1}

$$\mathbb{P}\big(Cross_L(\eta(\lambda_L))\big) \to \begin{cases} 0 & \text{if } \lambda_L = \lambda_c\,(1 - o(A_L^{-1})) \\ 1 & \text{if } \lambda_L = \lambda_c\,(1 + \omega(A_L^{-1})) \end{cases}$$



$\alpha_4(L)$

OU or Birth-Death Dynamics for Poisson process

η - Stationary Poisson process in $\mathbb{R}^2$ with intensity $\lambda > 0$;

Points are deleted at unit rate and new points are born at rate λ - η_t, $\quad t \geq 0$

$\eta_t \stackrel{d}{=} e^{-t} \circ \eta + (1 - e^{-t}) \circ \eta' \stackrel{d}{=} \eta$; $\quad \eta, \eta'$ are independent Poisson (λ) processes.

$e^{-t} \circ \eta -$ Poisson process with points retained independently with probability e^{-t}

Reversible dynamics w.r.t. η

Noise stability w.r.t OU Dynamics implies Stability w.r.t. CONSERVATIVE Dynamics.

Conserve number of points but move them according to a Brownian motion ;

Points die and relocate ; ………

Noise sensitivity w.r.t CONSERVATIVE Dynamics implies sensitivity w.r.t. OU Dynamics.

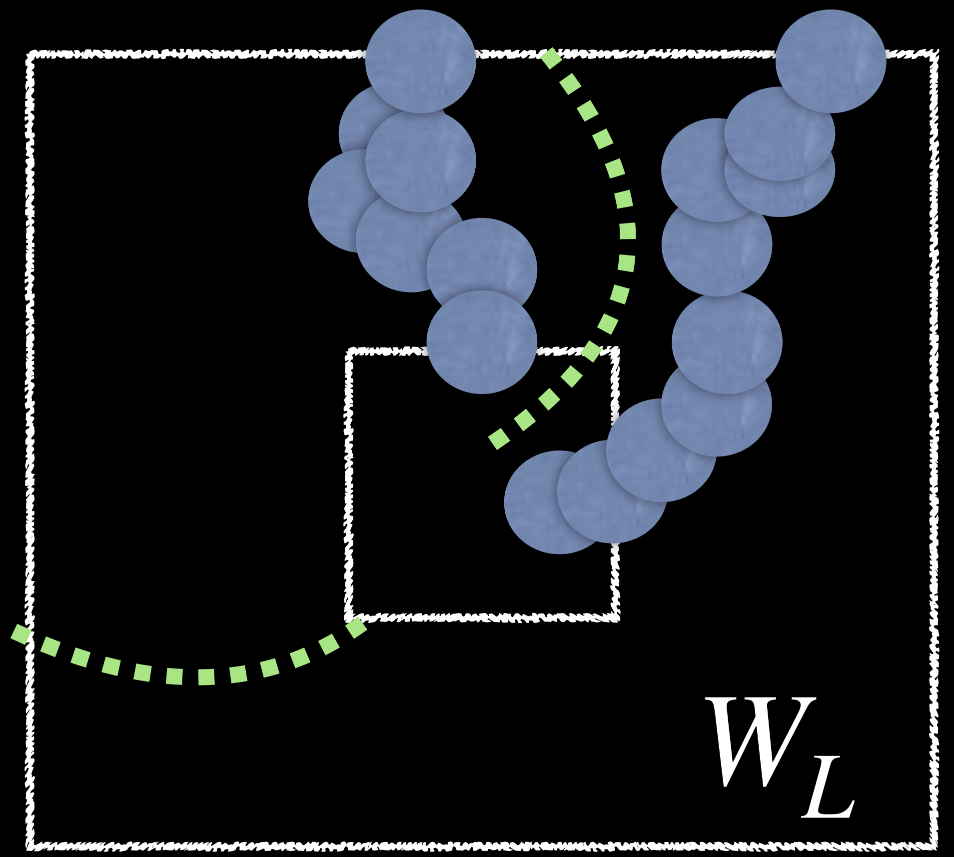
Sharp noise instability for Boolean Percolation

$\eta = \{X_i\}_{i \geq 1}$; Stationary Poisson(λ_c) process in $\mathbb{R}^2$; $\mathcal{O}(\eta)$ - Occupied region

$Cross_L(\eta) :=$ L-R Crossing of $\mathcal{O}(\eta) \cap W_L$; $W_L = [-L, L]^2$; η_t - OU DYNAMICS

$\mathbb{P}(Cross_L(\eta)) \xrightarrow{\text{RSW}} [a, 1-a]$ if $\lambda = \lambda_c$

$\alpha_4(L)$ = 4-arm probability



THEOREM (BPY 2024 and LPY 2023.)

Sharp noise instable at $A_L = L^2 \alpha_4(L) \to \infty$

Noise stable for $t_L A_L \to 0$ i.e., $\mathbb{P}(Cross_L(\eta) \neq Cross_L(\eta_{t_L})) \to 0$

AND

Not Noise stable for $t_L A_L \to \infty$ i.e., $\mathbb{P}(Cross_L(\eta) \neq Cross_L(\eta_{t_L})) > 0$

Noise sensitive for $t_L^{1+c} A_L \to \infty$ for some c > 0.

$$\liminf_{L \to \infty} \mathbb{COV}(Cross_L(\eta), Cross_L(\eta_{t_L})) = 0.$$

Spectral approach to sharp noise sensitivity

- $\eta = \{X_i\}_{i \geq 1}$; Stationary Poisson(λ) process in $\mathbb{R}^2$; $F(\eta) \in \{-1, +1\}$ - Functional.
- $\gamma_F \subset \mathbb{R}^2$ - Spectral Point process; $E_F(k) = \mathbb{P}(\# \gamma_F = k)$ - Energy.
- $\mathbb{E}\, F(\eta)\, F(\eta_t) = \sum_{k=0}^{\infty} e^{-kt}\, E_F(k) = \mathbb{E}\, e^{-t\, \# \gamma_F} \geq e^{-t\, \mathbb{E} \# \gamma_F}$.
- "TO UNDERSTAND CONCENTRATION OF $\# \gamma_F$"
- $\mathscr{P}_F := \{X_i : F(\eta) \neq F(\eta - X_i)\}$ - Pivotal Point process. For ex., $\mathbb{E} \# \mathscr{P}_{Cross_L} \approx L^2 \alpha_4(L)$.
- $\mathbb{E} \# \gamma_F \cap B = \mathbb{E} \# \mathscr{P}_F \cap B$, B bounded subset ; Not true for higher moments !!!
- $\# \gamma_{Cross_L} \overset{???}{\approx} \mathbb{E} \# \gamma_{Cross_L} = \mathbb{E} \# \mathscr{P}_{Cross_L} \approx L^2 \alpha_4(L)$; Concentration of γ_{Cross_L} is DIFFICULT !!!

Sharp noise instability for Boolean Percolation

$\eta = \{X_i\}_{i\geq 1}$; Stationary Poisson(λ_c) process in $\mathbb{R}^2$; $\mathcal{O}(\eta)$ - Occupied region

$Cross_L(\eta) :=$ L-R Crossing of $\mathcal{O}(\eta) \cap W_L$; $W_L = [-L, L]^2$; η_t - OU DYNAMICS

$\mathbb{P}(Cross_L(\eta)) \xrightarrow{\text{RSW}} [a, 1-a]$ if $\lambda = \lambda_c$

$\alpha_4(L)$ = 4-arm probability

THEOREM Sharp noise instable at $A_L = L^2\alpha_4(L) \to \infty$

Noise stable for $t_L A_L \to 0$ i.e., $\mathbb{P}(Cross_L(\eta) \neq Cross_L(\eta_{t_L})) \to 0$

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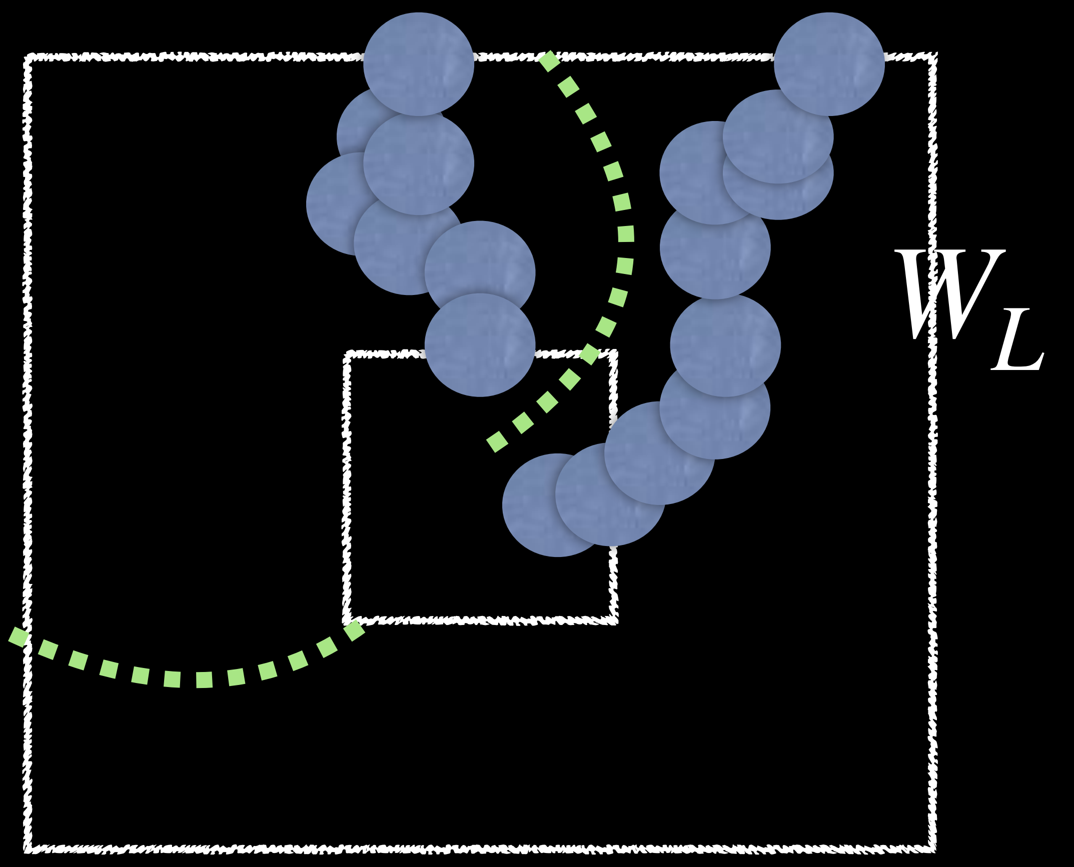
Not Noise stable for $t_L A_L \to \infty$ i.e., $\mathbb{P}(Cross_L(\eta) \neq Cross_L(\eta_{t_L})) > 0$

Noise sensitive for $t_L^{1+c} A_L \to \infty$ for some c > 0.

$$\liminf_{L\to\infty} \mathbb{COV}(Cross_L(\eta), Cross_L(\eta_{t_L})) = 0.$$



Schramm-Steif inequality and Stopping sets

- $\eta = \{X_i\}_{i \geq 1}$; Stationary Poisson(λ) process in $\mathbb{R}^2$; $F(\eta) \in \{-1, +1\}$ - Functional.

- $\gamma_F \subset \mathbb{R}^2$ - Spectral Point process; $E_F(k) = \mathbb{P}(\# \gamma_F = k)$ - Energy.

- $$\mathbb{E}\, F(\eta)\, F(\eta_t) = \sum_{k=0}^{\infty} e^{-kt}\, E_F(k) = \mathbb{E}\, e^{-t \# \gamma_F}.$$ Mehler's Formula.

- $Z(\eta) \subset \mathbb{R}^2$ Randomised stopping set. $\{Z(\eta) \subset B\}$ is determined by $\eta \cap B$

- Z determines F i.e., $F(\eta) = F(\eta \cap Z(\eta))$.

- Continuum Schramm-Steif inequality: $E_F(k) \leq k\, \delta(Z)$; $\delta(Z) := \sup_{x \in \mathbb{R}^2} \mathbb{P}(x \in Z)$

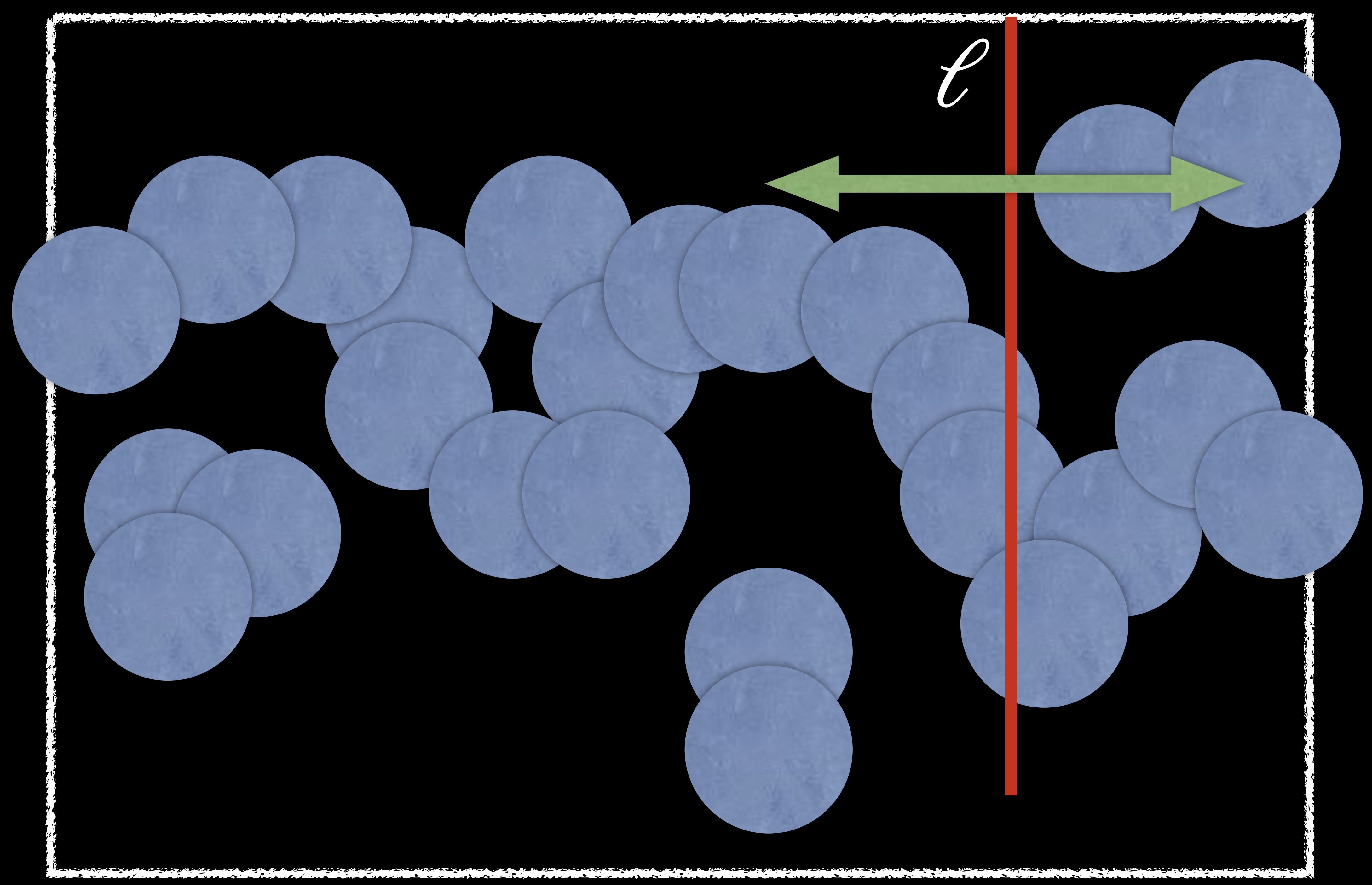
- If $\delta(Z_L) \to 0$ then $\liminf_{L \to \infty} \mathbb{C}OV(Cross_L(\eta)\,, Cross_L(\eta_{t_L})) = 0$ for $t_L >> \sqrt{\delta(Z_L)}$.

Stopping set for Crossings

$\eta = \sum_{i \geq 1} \delta_{X_i} = \{X_i\}_{i \geq 1}$ Stationary Poisson process in $\mathbb{R}^2$ with intensity $\lambda > 0$;

$\mathcal{O}(\eta) := \cup_{X_i \in \eta} B_1(X_i)$ - Occupied region. $Cross_L(\eta) :=$ L-R Crossing of $\mathcal{O}(\eta) \cap W_L$.

Start at a random vertical line ℓ and explore the occupied region to the left and right.



$$\mathbb{P}(x \in Z) \leq \mathbb{P}(B_1(x) \leftrightarrow \ell \text{ in } \mathcal{O}(\eta))$$

$$\leq \frac{2}{L} \int_0^L \alpha_1(s)\, \mathrm{d}s \approx L^{-c}$$

$\alpha_1(s) = 1$-arm crossing probability from W_1 to W_s.

Polynomial decay in many percolation models !

Sharp noise instability for Boolean Percolation

$\eta = \{X_i\}_{i \geq 1}$; Stationary Poisson(λ_c) process in $\mathbb{R}^2$; $\mathcal{O}(\eta)$ - Occupied region

$Cross_L(\eta) :=$ L-R Crossing of $\mathcal{O}(\eta) \cap W_L$; $W_L = [-L, L]^2$; η_t - OU DYNAMICS

$\mathbb{P}(Cross_L(\eta)) \xrightarrow{\text{RSW}} [a, 1-a]$ if $\lambda = \lambda_c$

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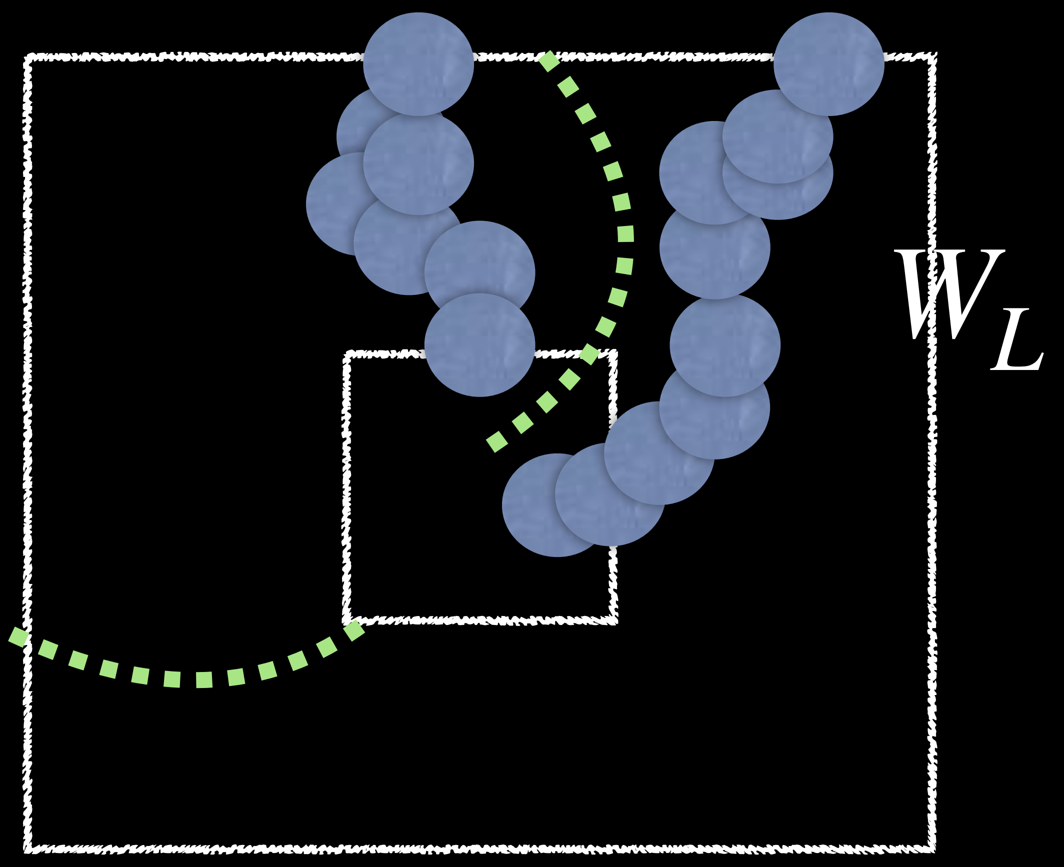

Sharp noise instability for Boolean Percolation

THEOREM (BPY 2024)

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AND Not Noise stable for $t_L A_L \to \infty$ i.e., $\mathbb{P}(Cross_L(\eta) \neq Cross_L(\eta_{t_L})) \to 0.$

Key Ingredients -

- Lower bound on 4-arm probabilities ; $\epsilon L^\epsilon \lesssim L^2\alpha_4(L)$ (Manolescu-Santoro '22)
- Quasi-multiplicativity - $\alpha_4(s,t) \approx \alpha_4(s,r)\alpha(r,t)$ for $1 \leq s \leq r \leq t$; à la Kesten.
- Expected Pivotal points - $\mathbb{E}\#\mathscr{P}_{Cross_L} \approx L^2\alpha_4(L)$
- Asymptotic Independence.

Spectral and Pivotal point process for Poisson functionals

- $\eta = \{X_i\}_{i \geq 1}$; Stationary Poisson(λ) process in $\mathbb{R}^2$; $F(\eta) \in \{-1, +1\}$ - Functional.

- $\mathscr{P}_F := \{X_i : F(\eta) \neq F(\eta - X_i)\}$ - Pivotal Point process. For ex., $\mathbb{E}\#\mathscr{P}_{Cross_L} \approx L^2\alpha_4(L)$.

  kth Correlation functions are $\mathbb{E}D_{x_1}F(\eta)^2 \ldots D_{x_k}F(\eta)^2$ - Geometric and easier to analyse !!!

- $F(\eta) = \sum_{k=0} I_k(u_k)$ Wiener-Itô Chaos expansion. $k!\, u_k(x_1, \ldots, x_k) = \mathbb{E}D^k_{x_1,\ldots,x_k}F(\eta)$, Diff. Operators

- $1 = \mathbb{E}F(\eta)^2 = k! \sum_{k=0} \mathbb{E}\, I_k(u_k)^2 = \sum_{k=0}^{\infty} E_F(k)$

- $\mathbb{P}(\#\gamma_F = k) = E_F(k) = k!\ \mathbb{E}I_k(u_k)^2$ &. if $\#\gamma_F = k$, joint density = $\dfrac{u_k^2}{\mathbb{E}I_k(u_k)^2}$

- γ_F - Spectral Point process; kth Correlation functions are $\mathbb{E}D^k_{x_1,\ldots,x_k}F(\eta)^2$.

Spectral process and Noise Sensitivity

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- γ_F - Spectral Point process; kth Correlation functions are $\mathbb{E}D^k_{x_1,\ldots,x_k}F(\eta)^2$.

- First correlation functions <u>ALONE</u> coincide $\longrightarrow$ $\mathbb{E}\#\gamma_F \cap B = \mathbb{E}\#\mathscr{P}_F \cap B$, B bounded.

- $\mathbb{E}F(\eta)F(\eta_t) = \sum_{k=0} e^{-kt}\,\mathbb{E}I_k(u_k)^2 = \mathbb{E}e^{-t\#\gamma_F}$; Concentration of γ_F yields Sharp Noise sensitivity.

- Analyse $\mathbb{E}(\#\gamma_F)^2$ using its second correlation function and partial concentration results !

Sharp noise instability for Boolean Percolation

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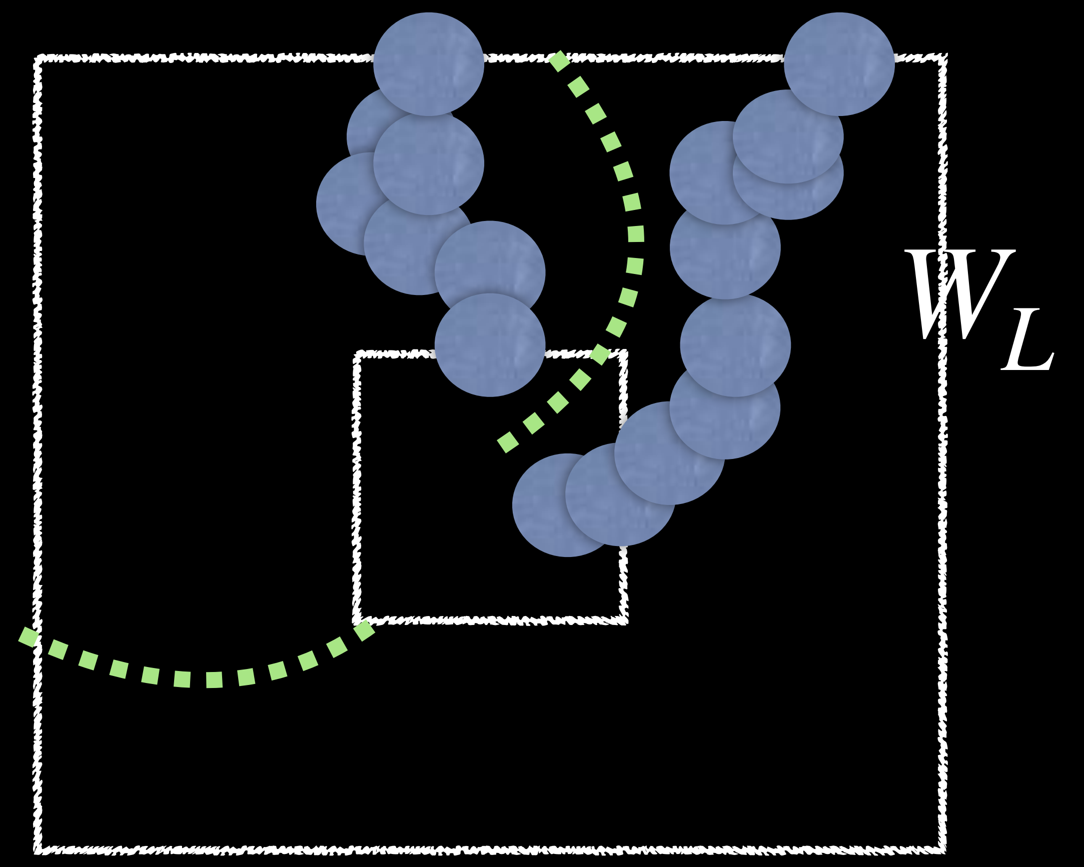

Sharp noise instability and sensitivity for Voronoi percolation as well via Sharp noise sensitivity for Frozen dynamics by Hugo Vanneuville.