The embedding problem of infinitely divisible probability measures on groups

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Abstract

A probability measure on a locally compact group is said to be infinitely divisible if it has convolution roots of all orders. Whether such a measure can be embedded in a continuous real one-parameter convolution semigroup, is known as the embedding problem. We will discuss the conditions on the measure, or on the group, under which this problem has been solved.
S. G. Dani has made major contributions towards resolution of the embedding problem. We will discuss techniques developed by Dani with his collaborators in solving the embedding problem on connected Lie groups. We will also discuss some related useful results of Dani, including convergence-of-types theorems, a structure theorem for the automorphism groups of certain Lie groups, concentration functions etc. (A survey of Dani's results in the area can be found in 'Dani's work on probability measures on groups' by F. Ledrappier and R. Shah, Contemporary Mathematics 631 (2015), 109–117).

Plan of Talk

- Introduction to the Embedding Problem
  Infinitely Divisible Measures
  Embeddable Measures
- Early Results
- Embedding on Lie Groups - Dani's contribution
- Techniques: Convergence-of-Types Theorems, Fourier Analysis, Concentration Functions of Measures, Structure Theory of Lie Groups and Algebraic Groups
- Embedding on Locally Compact Groups

Infinitely divisible probability measures

Let G be a locally compact Hausdorff topological group.

Let $P(G)$ denote the convolution semigroup of probability measures endowed with the weak* topology.

A measure $\mu \in P(G)$ is said to be infinitely divisible (i.d.) if for every natural number n, μ has an nth convolution root in $P(G)$.

i.e. $\forall\ n \in \mathbb{N}$, there exists $\lambda_n \in P(G)$, such that

$$\mu = \lambda_n^n = \lambda_n * \cdots * \lambda_n \ (n \text{ times}).$$

Examples: Dirac measures on $\mathbb{R}$, $\mathbb{R}^n$, S^1 or $\mathbb{Q}_p$, exponential measures.

Embeddable measures

A measure $\mu \in P(G)$ is said to be embeddable if there exists a continuous (real) one-parameter convolution semigroup $\{\mu_t\}_{t\geq 0}$ in $P(G)$ such that $\mu_1 = \mu$.

i.e. $t \mapsto \mu_t$ from $\mathbb{R}_+ \to P(G)$ is a continuous homomorphism $(\mu_{t+s} = \mu_t\mu_s)$ and $\mu_1 = \mu$.

Examples: Poisson (exponential) measures:
$\exp t\lambda = \delta_e + \sum_{n=1}^{\infty} \frac{t^n}{n!}\lambda^n$ (suitably normalised),
where δ_e denotes the Dirac measure at the identity e,

Gaussian measures, Dirac measures on $\mathbb{R}^n$, on S^1, or more generally on any connected nilpotent Lie group, Dirac measures supported on elements coming from one-parameter subgroups.

Note that every embeddable measure is infinitely divisible, for if μ is embeddable in $\{\mu_t\}_{t\geq 0}$, then $\mu = \mu_1 = (\mu_{\frac{1}{n}})^n$, for all n.

The embedding problem

Conversely, is every infinitely divisible probability measure embeddable (in a continuous one-parameter convolution semigroup)?

This was raised as a question by K. R. Parthasarathy in 1967 for connected Lie groups and it was gradually viewed as a conjecture when some evidence was gathered.S. G. Dani has made major contributions towards resolution of the embedding problem.

Why the embedding problem?

On $\mathbb{R}^n$, any embeddable measure is a convolution of a Dirac measure, a Gausssian measure and a Poisson measure. In general the structure of a continuous one-parameter convolution semigroup is well understood. Also, embeddable measures can be approximated by Poisson (exponential) measures. Infinitely divisible/embeddable measures appear in the central limit theory as limits of commutative infinitesimal triangular systems of measures.

Early results

Early on, the embedding problem was solved for the following:

$\mathbb{R}^n$ (Lévy); in fact every infinitely divisible measure on $\mathbb{R}^n$ has a unique nth root, and since the root set of a measure on $\mathbb{R}^n$ is a relatively compact set, there is an embedding.

Compact Lie groups

Connected nilpotent Lie group (Burrell and McCrudden - 1974). Every divisible point x in a connected Lie group is exponential (McCrudden - 1981).

On a connected Lie group G, if the support of μ is 'large', i.e. the closed subgroup $G(\mu)$, generated by the support of μ, is the whole of G, and if μ is infinitely divisible, then it is embeddable (McCrudden - 1981).

More on the embedding

In 1985, Dani met McCrudden at Oberwolfach in a conference of the series: *Probability measures on Groups*. They started working together there and continued working by sending letters through post (no email in those days). Their first paper appeared in the proceedings of this conference in 1986, where they proved the embedding for i.d. measures on semisimple groups with finite centre whose support is not contained in any parabolic subgroup of G. This was just a beginning.

Then we have the Dani effect on the embedding problem! In a series of papers with McCrudden, Dani solved the embedding problem for several class of Lie groups:
Closed linear groups, simply connected Lie groups, connected semsimple Lie groups.
More generally, a connected Lie group which has a linear representation with discrete kernel (Dani-McCrudden 1992). In fact, later in 2007 they labelled this class of Lie groups as class $\mathcal{C}$.

Techniques

Let $R(\mu) := \{\lambda^m \mid \lambda \in P(G), \lambda^n = \mu \text{ for some } n \in \mathbb{N}, 1 \leq m \leq n\}$ denote the root set of μ and let $F(\mu) := \{\lambda \in P(G) \mid \lambda\nu = \nu\lambda = \mu \text{ for some } \nu \in P(G)\}$ denote the set of factors of μ.

$R(\mu) \subset F(\mu)$. If $R(\mu)$ is relatively compact, then one can find a rational embedding $f : \mathbb{Q}_+^* \to P(G)$, with $f(1) = \mu$, which is locally tight, i.e. $f(]0,1[\cap\mathbb{Q})$ is relatively compact. This can be extended to a real embedding in case of Lie groups.

If $\lambda_n \in F(\mu)$, then there exists $\{x_n\}$ in G, such that $\{\lambda_n x_n = \lambda_n * \delta_{x_n}\}$ and $\{\lambda_n x_n^{-1}\}$ are relatively compact. This implies that $\{x_n \mu x_n^{-1}\}$ and $\{x_n^{-1} \mu x_n\}$ are relatively compact. If one wants to show that $F(\mu)$ is compact, one needs to show that $\{x_n\}$ is relatively compact.

Techniques...

For a linear group $G \subset GL(V) \subset M(V)$, the vector space of linear maps $V \to V$, the conjugation action of G extends to that of $GL(V)$ and hence to $M(V)$. Then there is a simple technique developed by Dani to study when a sequence $\{\tau_n\}$ of linear maps on a vector space is relatively compact in terms of behaviour of its orbits. This, in particular, led to convergence-of-types theorems: If $\mu_n \to \mu$ and $\tau_n(\mu_n) \to \gamma$, then $\{\tau_n\}$ is relatively compact under certain conditions. e.g. if $\operatorname{supp} \mu$ and $\operatorname{supp} \gamma$ are 'large' on a connected Lie group which has no compact central subgroup of positive dimension.

In the above, scenario, it will give us the following:

On Algebraic groups

For a measure μ on an algebraic group, it follows that $\{x_n\}$ can be chosen to be in $Z(\mu)$, the centraliser of $\operatorname{supp}\mu$. Then $R(\mu)$ is relatively compact, if $Z(\mu)$ is compact. If $G(\mu)$ is Zariski dense in G, then $Z(\mu)$ is the center of G and it follows by a result of McCrudden that $R(\mu)$ is relatively compact, and μ is embeddable.

For a general i.d. measure μ on an algebraic group, Dani and McCrudden find a subgroup H of G such that μ is i.d. in $P(H)$ and there exists a simply connected nilpotent subgroup N of $Z(\mu)$ such that for any sequence of roots $\{\lambda_n\}$ of μ, $\{\lambda_n x_n\}$ is relatively compact for some $x_n \in N$. Now Dani and McCrudden magically turn these $\lambda_n x_n$ into roots of μ. namely, they show that there exists $z_n \in Z(\mu) \cap N$, such that $z_n \lambda_n z_n^{-1} = \lambda_n x_n$. As $\lambda_n^m = \mu$ for some m, then $(z_n \lambda_n z_n^{-1})^m = z_n \lambda_n^m z_n^{-1} = z_n \mu z_n^{-1} = \mu$. Since one has a relatively compact sequence of roots, one can find a rational, and hence, a real embedding of μ.

More techniques

Dani and McCrudden showed in 1988 that any homomorphism $f : \mathbb{Q}_+^* \to P(G)$ is locally tight for a connected Lie group G. So it is enough to find such a rational homomorphism.

But how does one find such a rational embedding?

It is sufficient to choose a particular set of nth roots which are contained in a compact set.
For example, if we have a symmetric i.d. measure μ whose all roots are also symmetric, then each root is unique and we have a rational embedding, which is locally tight from the above result and hence μ is embeddable.

Embedding problem on class $\mathcal{C}$ Lie groups

For G in class $\mathcal{C}$, there is a linear representation $\rho : G \to GL(n, \mathbb{R})$ with discrete kernel, for some n. (One can also change the representation to get that $\rho(G)$ is closed - Dani-Margulis). Then one can choose the roots of μ in such a way that its images are roots of $\rho(\mu)$ and are relatively compact. Since the kernel is a central subgroup, these roots of μ are also in a compact set and hence μ is embeddable.

Note that, for any connected Lie group G, the quotient group G/T is in class $\mathcal{C}$, where T is the maximal compact (connected) central subgroup.

Moreover, for any infinitely divisible measure μ on a connected Lie group G, there exists $\{z_n\}$ in G such that $z_n T$ centralises $\pi(\mu)$ in G/T, T as above, such that $z_n \mu z_n^{-1} \to \lambda$ and λ is embeddable. (This is called the week embedding of μ, Dani-McCrudden 2007).

Observe that $z_n(\mu * \omega_T)z_n^{-1} = \mu * \omega_T = \lambda * \omega_T$, and hence $\mu * \omega_T$ is embeddable, where ω_T is the Haar measure supported on T.

Embedding problem on new classes of groups

For which class of Lie groups was the embedding problem not known in 2007?

Any generic infinitely divisible measure on a connected Lie group if it is not nilpotent or it is not in class $\mathcal{C}$; i.e. if the closure of the commutator group $\overline{[R, R]}$ of the radical R contains a nontrivial central torus. (Of course, one can sometime get an embedding for a specific measure on any connected Lie group - e.g. i.d. measures with large support or i.d. symmetric measures with all symmetric roots as mentioned above).

Compact extensions of Heisenberg groups which have a non-trivial compact connected central subgroup T and any linear representation has T in the kernel.

For example, the group of this type with the smallest dimension possible for which the embedding problem was not solved is the so-called Walnut Group.

Walnut Group

The Walnut group is defined as follows:

Take the 3-dimensional Heisenberg group H with the one-dimensional center Z. Now take a discrete subgroup D in the center and take $N = H/D$. Then the center of N is $T = Z/D$ which is a one-dimensional torus.

There is a canonical action of $SL(2, R)$ on H which fixes elements of the center. So this action extends to the action on N. Let K be the subgroup of $SL(2, R)$ which is the group of rotations. Take $G = SL(2, \mathbb{R}) \ltimes N$ and take $W = K \ltimes N \subset G$. This W is called the Walnut group, it is a four dimensional solvable group. It turns out that any linear representation of G or W will have T in its kernel.

Neither G nor W belongs to class $\mathcal{C}$, since $[N, N] = T$.

Embedding Theorems (Dani with Guivarc'h & R. Shah)

Theorem 1: Let G be a semidirect product of $SL(2,\mathbb{R})$ and N as above. Let M be a closed connected subgroup of G. Then any infinitely divisible measure on M is embeddable in $P(M)$. In particular, every infinitely divisible measure on the Walnut group is embeddable.

This follows as a special case of the following general result which is somewhat technical in terms of some conditions on the group and on the measure.

A general theorem

Main Theorem (Dani-Guivarc'h-R. Shah 2012): Let G be a Lie group admitting a surjective continuous homomorphism $p : G \to \tilde{G}$ onto an almost algebraic group $\tilde{G}$, such that $\ker p$ is contained in the center of G and $(\ker p)^0$ is compact. Let $T = (\ker p)^0$ and let $q : G \to G/T$ be the natural quotient homomorphism. Let μ belong to $P(G)$ be such that $q(\mu)$ has no non-trivial idempotent factor. If μ is infinitely divisible, then μ is embeddable.

The first condition is satisfied if $G/\overline{[G,G]}$ is compact or more generally, if $G/\overline{[G,G]N}$ is compact, where N is the nilradical of G; the latter condition is equivalent to the condition that the solvable radical of G is a compact extension of the nilradical. The second condition is satisfied (for example) if the support of μ is contained in the nilradical of N, (as N/T is simply connected and has no non-trivial compact subgroups). It can also be bypassed in some cases.

Using this general theorem, the following was shown:

Theorem 2: An infinitely divisible measure μ on a connected Lie group G is embeddable if (1) the support of μ is contained in the nilradical of G or (2) if $G(\mu)/\overline{[G(\mu), G(\mu)]}$ is compact.

Condition (2) is satisfied for example if the group $G(\mu)$ has property T.

The proof uses various techniques developed by Dani with McCrudden and some more techniques developed for the general theorem, a result on the support of a specific factor of a measure (R. Shah 2009) derived from the work on concentration functions by Dani with R. Shah (1997) and the following result:

Theorem 3: Let K be a compact connected subgroup of $GL(d, \mathbb{R})$, $d \geq 2$, such that the K-action on $\mathbb{R}^d$ has no nonzero fixed point. Let λ be a K-invariant probability measure on $\mathbb{R}^d$ such that $\lambda(V) = 0$ for any proper subspace V of $\mathbb{R}^d$. Then λ^d has a density in $\mathbb{R}^d$. In particular, $\hat{\lambda}$ vanishes at infinity.

Embedding...

So far we have got the following for any infinitely divisible measure μ on any connected Lie group G:

1. $\mu * \omega_T$ is embeddable, where T is the maximal central torus (of G or of $\overline{[R, R]}$), and
2. there exists z_n in the centraliser of $\mathrm{supp}(\mu * \omega_T)$ such that $z_n \mu z_n^{-1} \to \lambda$ and λ is embeddable.
3. μ is embeddable if G is nilpotent or G has a linear representation with discrete kernel.
4. μ is embeddable if $G(\mu)$, the closed subgroup generated by $\mathrm{supp}\,\mu$, satisfies one of the special properties mentioned earlier.

Some easy consequences

From the embedding of $\mu * \omega_T$, which uses the images of roots of μ in G/T, one can conclude that μ is infinitely divisible on $H = N^0(\mu * \omega_T)G(\mu)$. Here, $N^0(\mu * \omega_T)$ (resp. $Z^0(\mu * \omega_T)$) is the connected component of the identity in the normaliser (resp. centraliser) of $G(\mu * \omega_T)$. Then $H = N^0(\mu * \omega_T)$ if $G(\mu)$ is connected. Also, $H = Z^0(\mu * \omega_T)G(\mu)$, if $G(\mu)$ is discrete.

Any infinitely divisible probability measure on a connected Lie group is embeddable if any one of the following holds:

1. $G(\mu)$ is contained in the center of G.
2. If $G(\mu)$ is connected and nilpotent.
3. $G(\mu)N$ is dense in G, where N is the nilradical of G.
4. If $G(\mu)$ is discrete.
5. μ is i.d. on $G(\mu)$, where $G(\mu)$ need not be connected.

Embedding on locally compact groups

There are also results by Dani which discuss the embedding problem on a general locally compact group.

(Dani with K. Schmidt 2002): μ is infinitely divisible on G such that $\operatorname{supp}\mu$ is contained in $\mathbb{R}^d$, then it is infinitely divisible on $\mathbb{R}^d$, and hence it is embeddable.

Any infinitely divisible measure on a discrete finitely generated subgroup of $GL(n, \mathbb{A})$ is embeddable (Poisson), where $\mathbb{A}$ is the field of algebraic numbers (Dani with R. Shah 1993). The proof uses results on real linear groups and the following:

Any measure on a p-adic algebraic group is infinitely divisible if and only if there exists a unipotent element x in $Z(\mu)$ such that $x\mu$ is embeddable (R. Shah 1991). One of the techniques developed by Dani for real algebraic groups was useful here.

Automorphism group of a connected Lie group

There is a result by D. Wigner that for a connected Lie group G, the connected component of the identity in $\mathrm{Aut}(G)$ is an almost algebraic subgroup of $\mathrm{GL}(\mathcal{G})$, where $\mathcal{G}$ is the Lie algebra of G. This does not hold for $\mathrm{Aut}(G)$ in general as $\mathrm{Aut}(\mathbb{T}^n)$ is isomorphic to $\mathrm{GL}(n, \mathbb{Z})$, $n \in \mathbb{N}$.

For a certain subclass of Lie groups, Dani proved the following result which is very useful:

(Dani 1992) Let G be a connected Lie group without any compact central subgroup of positive dimension. Then $\mathrm{Aut}(G)$ is almost algebraic as a subgroup of $\mathrm{GL}(\mathcal{G})$, where $\mathcal{G}$ is the Lie algebra of G. In particular, $\mathrm{Aut}(G)$ has finitely many connected components.

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Happy Birthday Dani!