ICTS - Bangalore

Recent advances on control theory of PDE systems

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Feedback stabilization of FSI problems

Lecture 4

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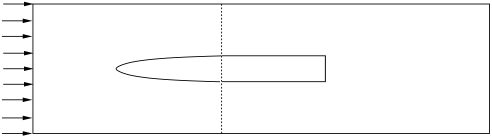
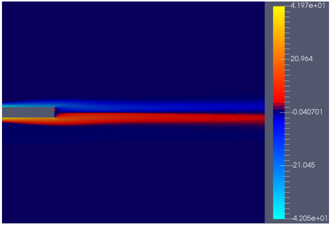
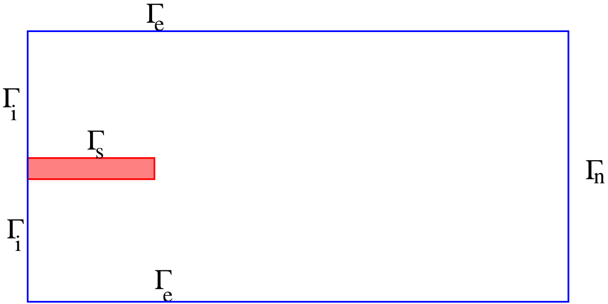
Motivation - Stabilization of a fluid-plate model





The geometrical configuration

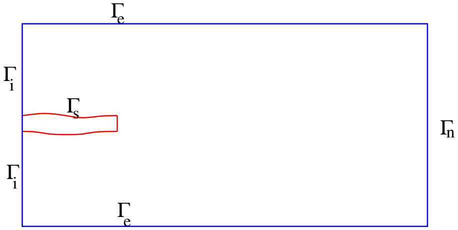


with Dirichlet and Neumann boundary conditions.

$$(\nu(\nabla u + \nabla u^T) - p\, I)n = \sigma(u,p)n = 0 \quad \text{on} \quad \Gamma_n \times (0,\infty),$$

$$\text{Dirichlet B.C.} \quad \text{on} \quad (\Gamma \setminus \Gamma_n) \times (0,\infty).$$

Due to the displacement $\eta(t)$ of the structure, the fluid equation is written in a time dependent geometrical domain



At time $t > 0$, the fluid domain and the fluid-structure interface depend on $\eta(t)$

$$\Omega_f(t) = \Omega_{\eta(t)} \quad \text{and} \quad \Gamma_s(t) = \Gamma_{\eta(t)}.$$

Motivations

- Stabilization of a fluid-structure-interaction model around an unstable stationary solution, with a control acting in the structure equation.

- Prove that the results for the stabilization of abstract parabolic systems apply.

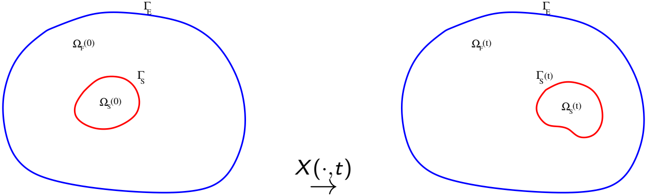
- Prove that the strategy based on spectral projections applies.

- Error estimates for the F.E. approximation of feedback gains can be obtained.

Outline

- Part I - Motivations and goals
- Part II - Local/maximal in time solutions to FSI systems
- Part III - Stabilization of FSI systems using spectral projections
- Part IV - Incursion in the numerical approximation
- Part V - Numerical experiments

Part II - Well-posedness of some FSI problems



Initial and deformed configurations

The fluid equations

$$\frac{\partial u}{\partial t} + (u \cdot \nabla_x)u - \mathrm{div}_x\, \sigma(u,p) = 0, \quad \mathrm{div}_x\, u = 0 \quad \text{in } \Omega_f(t),\ t \in (0,T),$$

$$u = 0 \quad \text{on} \quad \Sigma_e^T = \Gamma_e \times (0,T), \quad u(0) = u_0 \text{ in } \Omega_f,$$

$$u(X(y,t),t) = \partial_t w(y,t) \quad \text{for } (y,t) \in \Sigma_s^T = \Gamma_s \times (0,T),$$

The deformation

$$X(y,t) = y + w(y,t) \quad \text{for } (y,t) \in \Sigma_s^\infty, \quad X(y,0) = y.$$

The structure equation: Need the stress tensor of the fluid on Γ_s.

Transformation into the fixed initial domain

We consider a C^1 diffeomorphism $X(\cdot,t)\ :\ \Omega_f=\Omega_f(0)\longmapsto\Omega_f(t)$ (not nec. the ref. config.).

$$Y(X(y,t),t)=y,\quad y\in\Omega_f\quad\text{and}\quad X(Y(x,t),t)=x,\quad x\in\Omega_f(t).$$

We make the change of variables

$$\widetilde{u}(y,t)=u((X(y,t),t)),\quad \widetilde{p}(y,t)=p((X(y,t),t)).$$

We have

$$\nabla_x u(x,t)=\nabla_y\widetilde{u}(Y(x,t),t)J_Y(x,t),\quad x\in\Omega_f(t),\ t\in[0,T],$$

$$\nabla_x p(x,t)=J_Y(x,t)^T\nabla_y\widetilde{p}(Y(x,t),t),\quad x\in\Omega_f(t),\ \text{for } t\in[0,T],$$

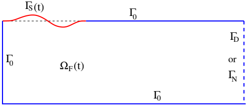
where $J_Y(x,t)=(J_X(Y(x,t),t))^{-1}$. We obtain

$$\frac{\partial\widetilde{u}}{\partial t}-\nu\sum_{k=1}^{3}\frac{\partial}{\partial y_k}\left(\nabla\widetilde{u}\,J_Y\right)\nabla_x Y_k+J_Y^T\nabla\widetilde{p}=0,\quad \text{in } Q_f^T=\Omega_f\times(0,T),$$

$$\nabla\widetilde{u}:J_Y^T=0\quad\text{in } Q_f^T,$$

$$\widetilde{u}=0\quad\text{on } \Sigma_e^T=\Gamma_e\times(0,T),\quad \widetilde{u}=\partial_t w\quad \text{on } \Sigma_s^T.$$

Choice of X



- X can be defined analytically $(w(y,t) = (0, \eta(y_1,t))^T)$

$$X(y,t) = (x_1, x_2)^T = (y_1, y_2(1+\eta(y_1,t),t))^T.$$

- X can be defined by a Lagrangian transformation

$$\frac{\partial X}{\partial t}(y,t) = u(X(y,t),t), \quad X(y,t) = y + \int_0^t \widetilde{u}(y,\tau)\, d\tau.$$

The system in the initial configuration - Beam or Shell

$$
\begin{aligned}
&\widetilde{u}_t - \operatorname{div}\sigma(\widetilde{u},\widetilde{p}) = \mathcal{F}[\widetilde{u},\widetilde{p},\eta_1,\eta_2] - \nu\nabla(\mathcal{G}[\widetilde{u},\eta_1]),\\
&\operatorname{div}\widetilde{u} = \mathcal{G}[\widetilde{u},\eta_1] \quad \text{in } Q_T = \Omega_f \times (0,T),\\
&\widetilde{u} = \eta_2\,\vec{n}\,\chi_{\Gamma_s} \text{ on } \Sigma_d^T = \Gamma_d \times (0,T),\\
&\sigma(\widetilde{u},\widetilde{p})n = 0 \text{ on } \Sigma_n^T = \Gamma_n \times (0,T), \quad \widetilde{u}(0) = u_0 \text{ in } \Omega,\\
&\eta_{1,t} = \eta_2 \quad \text{on } \Sigma_s^T,\\
&\eta_{2,t} + \alpha\Delta_s^2\eta_1 - \delta\Delta_s\eta_2 = \mathcal{H}[\widetilde{u},\widetilde{p},\eta_1] + f\,\chi_{\Gamma_c} \quad \text{on } \Sigma_s^T,\\
&\eta_1 = 0 \quad \text{and} \quad \frac{\partial\eta_1}{\partial n} = 0 \quad \text{on } \partial\Gamma_s \times (0,T),\\
&\eta_1(0) = \eta_1^0 = 0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \quad \text{on } \Gamma_s.
\end{aligned}
$$

$$
\begin{aligned}
& v_t - \operatorname{div}\sigma(v,q) = \mathcal{F}[\widehat{u},\widehat{p},\widehat{\eta}_1,\widehat{\eta}_2] - \nu\nabla(\mathcal{G}[\widehat{u},\widehat{\eta}_1]),\\
& \operatorname{div} v = \mathcal{G}[\widehat{u},\widehat{\eta}_1] \quad \text{in } Q_T = \Omega_f \times (0,T),\\
& v = \eta_2\,\vec{n}\,\chi_{\Gamma_s} \text{ on } \Sigma_d^T = \Gamma_d \times (0,T),\\
& \sigma(v,q)n = 0 \text{ on } \Sigma_n^T = \Gamma_n \times (0,T), \quad v(0) = u_0 \text{ in } \Omega,\\
& \eta_{1,t} = \eta_2 \quad \text{on } \Sigma_s^T,\\
& \eta_{2,t} + \alpha\Delta_s^2\eta_1 - \delta\Delta_s\eta_2 = q + \mathcal{H}_s[\widehat{u},\widehat{p},\widehat{\eta}_1] + f\,\chi_{\Gamma_c} \quad \text{on } \Sigma_s^T,\\
& \eta_1 = 0 \quad \text{and} \quad \frac{\partial\eta_1}{\partial n} = 0 \quad \text{on } \partial\Gamma_s \times (0,T),\\
& \eta_1(0) = \eta_1^0 = 0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \quad \text{on } \Gamma_s
\end{aligned}
$$

We consider the nonlinear mapping

$$
\mathcal{N} \;:\; (\widehat{u},\widehat{p},\widehat{\eta}_1,\widehat{\eta}_2) \longmapsto (v,q,\eta_1,\eta_2).
$$

Space of solutions, of I.C., of the RHS

- We look for solutions $(\widetilde{u}, \widetilde{p}, \widetilde{\eta}_1, \widetilde{\eta}_2)$, defined over a time intervall $(0, T)$, belonging to a Banach space E_T.

- The initial condition $(u_0, \eta_1^0, \eta_2^0) = (u_0, 0, \eta_2^0)$ belongs to a Hilbert space E_0 which includes some compatibility conditions.

- The space of the RHS is such that if $(\widetilde{u}, \widetilde{p}, \widetilde{\eta}_1, \widetilde{\eta}_2)$ belongs to E_T, the RHS

$$(\mathcal{F}[\widehat{u}, \widehat{p}, \widehat{\eta}_1, \widehat{\eta}_2], \mathcal{G}[\widetilde{u}, \eta_1], \mathcal{H}_s[\widehat{u}, \widehat{p}, \widehat{\eta}_1]) \in F_T.$$

The linear nonhomogeneous system

We need regularity results for the linear system

$$v_t - \mathrm{div}\,\sigma(v,q) = F_f - \nu\nabla F_{\mathrm{div}}, \quad \mathrm{div}\, v = F_{\mathrm{div}} \quad \text{in } Q_T,$$

$$v = \eta_2 \vec{n}\chi_{\Gamma_s} \text{ on } \Sigma_d^T, \quad v(0) = u_0$$

$$\eta_{1,t} = \eta_2 \quad \text{on } \Sigma_s^T,$$

$$\eta_{2,t} + \alpha\Delta_s^2\eta_1 - \delta\Delta_s\eta_2 = H_s + q \quad \text{on } \Sigma_s^T,$$

$$\eta_1 = 0 \quad \text{and} \quad \frac{\partial\eta_1}{\partial n} = 0 \quad \text{on } \partial\Gamma_s \times (0,T),$$

$$\eta_1(0) = \eta_1^0 = 0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \quad \text{on } \Gamma_s.$$

We prove that, for all $T \in (0,1)$, we have

$$\|(v,q,\eta_1,\eta_2)\|_{E_T} \le C_0(\|(u_0,\eta_1^0,\eta_2^0)\|_{E_0} + \|(F_f, F_{\mathrm{div}}, H_s)\|_{F_T}).$$

We assume that $v_0 \in H^1_{\Gamma_i}(\Omega;\mathbb{R}^2)$, $\mathrm{div}\, v_0 = 0$,
$\eta_1^0 \in H^3(\Gamma_s) \cap H^2_0(\Gamma_s)$, $\eta_2^0 \in H^1_0(\Gamma_s)$, $v_0 = \eta_2^0 \vec{e}_2$ on Γ_s,
and that
$(F_f, F_{\mathrm{div}}, H_s) \in$
$L^2(0,T;L^2) \times L^2(0,T;H^1) \cap H^1(0,T;(H^1_\delta)') \times L^2(0,T;L^2(\Gamma_s))$.

The solution to the linearized system belongs to

$$v \in L^2(0,T;H^2_\delta(\Omega;\mathbb{R}^2)) \cap H^1(0,T;L^2(\Omega;\mathbb{R}^2)) = H^{2,1}_\delta(Q_T;\mathbb{R}^2),$$
$$p \in L^2(0,T;H^1_\delta(\Omega;\mathbb{R}^2)),$$
$$\eta_1 \in L^2(0,T;H^4(\Gamma_s)) \cap H^2(0,T;L^2(\Gamma_s)) = H^{4,2}(\Sigma_s^T),$$
$$\eta_2 \in L^2(0,T;H^2(\Gamma_s)) \cap H^1(0,T;L^2(\Gamma_s)) = H^{2,1}(\Sigma_s^T).$$

$$\|v\|^2_{H^2_\delta(\Omega;\mathbb{R}^2)} = \sum_{|k|=0}^{2} \sum_{i=1}^{2} \int_\Omega \prod_{j\in\mathcal{J}_{d,n}\cup\mathcal{J}_{d,d}} r_j^{2\delta} |\partial_k v|^2 dx,$$

$$\|p\|^2_{H^1_\delta(\Omega)} = \sum_{|k|=0}^{1} \sum_{i=1}^{2} \int_\Omega \prod_{j\in\mathcal{J}_{d,n}\cup\mathcal{J}_{d,d}} r_j^{2\delta} |\partial_k p|^2 dx,$$

where r_j stands for the distance to the junction point $J_j \in \mathcal{J}_{d,n} \cup \mathcal{J}_{d,d}$.

Due to the right angles at the Neumann-Dirichlet junctions and Dirichlet-Dirichlet junctions, we have

$$H^2_\delta(\Omega;\mathbb{R}^2) \subset H^{3/2+\alpha_0}(\Omega;\mathbb{R}^2),$$

$$H^1_\delta(\Omega) \subset H^{1/2+\alpha_0}(\Omega;\mathbb{R}^2), \quad \text{for some } \alpha_0 > 0.$$

For a tubular shell of radius 1 or a 2d-channel of height 1, we set

$$\gamma_{\eta_1}(0, T) = \min\{1 + \eta_1(y_1, t) \mid (y_1, t) \in [0, L] \times [0, T]\}.$$

The 'no contact condition' corresponds to

$$\gamma_{\eta_1}(0, T) > 0.$$

For $\gamma > 0$, $\mu > 0$ and $T > 0$, we define the ball

$$B(\gamma, \mu, T) = \{(\widehat{u}, \widehat{p}, \widehat{\eta}_1, \widehat{\eta}_2) \in E_T \mid$$
$$\|(\widehat{u}, \widehat{p}, \widehat{\eta}_1, \widehat{\eta}_2)\|_{E_T} \leq \mu, \ \gamma_{\widehat{\eta}_1}(0, T) \geq \gamma, \ (\widehat{u}, \widehat{\eta}_1, \widehat{\eta}_2)(0) = (u_0, \eta_1^0, \eta_2^0)\}.$$

When $\|(u_0, 0, \eta_2^0)\|_{E_0} = \|(u_0, \eta_1^0, \eta_2^0)\|_{E_0} \leq M$, we set

$$\gamma = 2\,\min\{1 + \eta_1^0(y_1) \mid (y_1) \in [0, L]\} = 2, \quad \mu = 2\,C_0\,M.$$

Combining regularity results and nonlinear estimates, we prove the following bound

$$\|\mathcal{N}(\widehat{u}, \widehat{p}, \widehat{\eta}_1, \widehat{\eta}_2)\|_{E_T} \leq C_0\,M + C_0\,C_{\rm nl}\,T^\alpha, \quad \alpha \in (0,1),$$

$$\text{for all } (\widehat{u}, \widehat{p}, \widehat{\eta}_1, \widehat{\eta}_2) \in B(\gamma, \mu, T), \quad \forall T \in (0,1),$$

the Lipschitz estimates

$$\|\mathcal{N}(\widehat{u}, \widehat{p}, \widehat{\eta}_1, \widehat{\eta}_2) - \mathcal{N}(\widetilde{u}, \widetilde{p}, \widetilde{\eta}_1, \widetilde{\eta}_2)\|_E$$

$$\leq C_0\,C_{\rm nl}\,T^\alpha\,\|(\widehat{u}, \widehat{p}, \widehat{\eta}_1, \widehat{\eta}_2) - (\widetilde{u}, \widetilde{p}, \widetilde{\eta}_1, \widetilde{\eta}_2)\|_E,$$

$$\forall (\widehat{u}, \widehat{p}, \widehat{\eta}_1, \widehat{\eta}_2) \in B(\gamma, \mu, T),\ \forall (\widetilde{u}, \widetilde{p}, \widetilde{\eta}_1, \widetilde{\eta}_2) \in B(\gamma, \mu, T).$$

and an estimate needed for the no contact condition

$$\|\eta_1\|_{L^\infty} = \|\eta_1 - \eta_1^0\|_{L^\infty} \leq C\,T^\alpha.$$

If $(u_0, \eta_1^0, \eta_2^0) \in E_0$, then there exists $T > 0$ such that the nonlinear FSI system admits a solution $(\widetilde{u}, \widetilde{p}, \eta_1, \eta_2)$ in E_T.

If $u_0 \in H^1_{\Gamma_i}(\Omega; \mathbb{R}^2)$, $\mathrm{div} u_0 = 0$, $\eta_2^0 \in H^1_0(\Gamma_s)$, $u_0 = \eta_2^0 \vec{e}_2$ on Γ_s, there exists $T > 0$ such that the FSI system admits a solution such that

$$\widetilde{u} \in L^2(0, T; H^2_\delta(\Omega; \mathbb{R}^2)) \cap H^1(0, T; L^2(\Omega; \mathbb{R}^2)) = H^{2,1}_\delta(Q_T; \mathbb{R}^2),$$

$$\widetilde{p} \in L^2(0, T; H^1_\delta(\Omega; \mathbb{R}^2)),$$

$$\eta_1 \in L^2(0, T; H^4(\Gamma_s)) \cap H^2(0, T; L^2(\Gamma_s)) = H^{4,2}(\Sigma_s^T),$$

$$\eta_2 \in L^2(0, T; H^2(\Gamma_s)) \cap H^1(0, T; L^2(\Gamma_s)) = H^{2,1}(\Sigma_s^T).$$

We rewriting the (non)homogeneous Differential algebraic system.
$F_f = 0,\ F_{\rm div} = 0,\ h = 0,\ H_s = 0,\ f = 0.$
Elimination of the pressure with the Leray projector

$$V^0_{n,\Gamma_d}(\Omega) = \Big\{ v \in L^2(\Omega;\mathbb{R}^2) \mid \ \text{div } v = 0,\ v\cdot n = 0 \text{ on } \Gamma_d \Big\},$$

$$L^2(\Omega;\mathbb{R}^2) = V^0_{n,\Gamma_d}(\Omega) \oplus \text{grad } H^1_{\Gamma_n}(\Omega),$$

$$P \ :\ L^2(\Omega) \longmapsto V^0_{n,\Gamma_d}(\Omega).$$

The pressure q satisfies

$$-\Delta q = 0 \quad \text{ in } Q_\infty, \quad q = 2\nu\,\varepsilon(v)n\cdot n \text{ on } \Sigma^\infty_n,$$

$$\frac{\partial q}{\partial n} = 2\nu\,\text{div}\varepsilon(v)\cdot n - v_t\cdot n = 2\nu\,\text{div}\varepsilon(v)\cdot n - \eta_{2,t}\chi_{\Gamma_s} \text{ on } \Sigma^\infty_d.$$

Thus

$$q = -N_s(\eta_{2,t}) + N_v(v) = -N_s(\eta_{2,t}) + N_v(Pv) + N_v((I-P)v).$$

Equivalent formulation of the PDE system

$$M_a\, z' = \widehat{\mathcal{A}} z, \quad z(0) = z_0, \quad z = (Pv, \eta_1, \eta_2)^T,$$

$$(I - P)v(t) = (I - P)L(\eta_2(t)\, e_2),$$

L is a lifting operator of the Dirichlet boundary condition, A_0 is the Stokes operator with mixed B.C.

$$\widehat{\mathcal{A}} = \begin{pmatrix} A_0 & 0 & -A_0 PL \\ 0 & 0 & I \\ \gamma_s N_v & \alpha \Delta_s^2 & \delta \Delta_s + \gamma_s N_v \nabla N_s \end{pmatrix},$$

and

$$M_a = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I + \gamma_s N_s \end{pmatrix}.$$

The added mass operator, M_a, is symmetric.

We set

$$\mathcal{A} = M_a^{-1}\,\widehat{\mathcal{A}},$$

and, due to the corners

$$\mathcal{D}(\mathcal{A}) = \Big\{(Pv, \eta_1, \eta_2) \in H^{1/2+\alpha_0}(\Omega;\mathbb{R}^2) \times (H^4 \cap H_0^2)(\Gamma_s) \times H_0^2(\Gamma_s)$$

$$\mid Pv - PL(\eta_2\,\vec{e}_2) \in \mathcal{D}(A_0)\Big\}, \quad \text{with } \alpha_0 \in (0, 1/2),$$

where

$$\mathcal{D}(A_0) = \{v \in V^1_{\Gamma_s}(\Omega) \cap (H^{3/2+\alpha_0}(\Omega;\mathbb{R}^2)$$

$$\mid \mathrm{div}\sigma(v, N_v(v)) \in (L^2(\Omega))^d,\ \sigma(v, N_v(v))n|_{\Gamma_n} = 0\}.$$

Analyticity of $(\mathcal{A}, D(\mathcal{A}))$

The operator $(\mathcal{A}, D(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup on $Z = V^0_{n,\Gamma_d}(\Omega) \times H^2_0(\Gamma_s) \times L^2(\Gamma_s)$.

Analyticity of the Oseen operator + Analyticity for the damped elastic operator of the structure (Chen-Triggiani) + perturbation arguments

The resolvent of $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is compact in Z.

The regularity results for the nonhomongeneous system are based these results and on the characterization of $\mathcal{D}((-\mathcal{A})^{1/2})$.

Maximal in time solutions. For a given displacement $\eta(y_1, t)$ obeying $\eta(y_1, 0) = \eta_1^0$, define the transformation $X_{\eta_1^0,\eta}$ from $\Omega_{\eta_1^0}$ into $\Omega_{\eta(t)}$ by

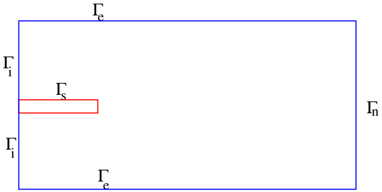
$$X_{\eta_1^0,\eta}(y, t) = X_\eta(X_{\eta_1^0}(y, t), t).$$

Now, we have to rewrite a fixed point argument for η_1^0 'arbitrary in some class of admissible solutions'. (Admissible: regularity and no contact.) Use a contradiction argument to prove the uniqueness of maximal in time solution.

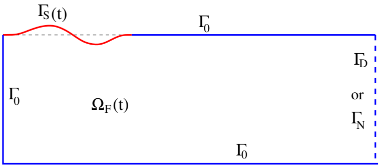
Difficulty. Prove regularity results in $\Omega_{\eta_1^0}$ with continuity constants independent of η_1^0, depending only of

$$\|\eta_1^0\|_{H^{3+\varepsilon_0}} \quad \text{and of} \quad \gamma_{\eta_1^0}.$$

Controlling a F–S system with a control in the structure equation



or



The fluid equation

$$u_t + (u\cdot\nabla)u - \operatorname{div}\sigma(u,p) = 0, \quad \operatorname{div} u = 0 \quad \text{in } \Omega_{\eta(t)},\ t>0,$$

$$u = \eta_2\vec{e}_2 \ \text{ on } \Gamma^s_{\eta(t)},\ t>0, \quad u = g_s \ \text{ on } \Sigma^\infty_0, \quad \Gamma_0 = \Gamma_d \setminus \Gamma_s,$$

$$\sigma(u,p)n = 0 \quad \text{on } \Sigma^\infty_n, \quad u(0) = u_0 \text{ in } \Omega,$$

$$\sigma(u,p) = \nu(\nabla u + \nabla u^T) - p\,I.$$

The structure equation

$$\eta_{1,t} = \eta_2 \quad \text{on } \Sigma^\infty_s,$$

$$\eta_{2,t} + \alpha\Delta^2_s\eta_1 - \delta\Delta_s\eta_2$$

$$= -\sigma(u,p)(-\eta_x\vec{e}_1 + \vec{e}_2)\cdot\vec{e}_2 - f_s + f\,\chi_{\Gamma_c} \quad \text{on } \Sigma^\infty_s,$$

$$\eta_1 = 0 \quad \text{and} \quad \frac{\partial\eta_1}{\partial n} = 0 \quad \text{on } \partial\Gamma_s\times(0,\infty),$$

$$\eta_1(0) = \eta^0_1 \quad \text{and} \quad \eta_2(0) = \eta^0_2 \quad \text{on } \Gamma_s.$$

The stationary solution $(u_s, p_s, 0, 0)$

$$(u_s \cdot \nabla)u_s - \operatorname{div} \sigma(u_s, p_s) = 0, \quad \operatorname{div} u_s = 0 \quad \text{in } \Omega,$$

$$u_s = \eta_2 \vec{e}_2 \ \text{ on } \Gamma_s, \quad u_s = g_s \ \text{ on } \Gamma_0, \quad \sigma(u_s, p_s)n = 0 \quad \text{on } \Gamma_n,$$

$$\eta_1 = \eta_2 \quad \text{on } \Gamma_s,$$
$$\alpha \Delta_s^2 \eta_1 - \delta \Delta_s \eta_2 = -\sigma(u_s, p_s)(-\eta_x \vec{e}_1 + \vec{e}_2) \cdot \vec{e}_2 - f_s \quad \text{on } \Gamma_s,$$
$$\eta_1 = 0 \quad \text{and} \quad \frac{\partial \eta_1}{\partial n} = 0 \quad \text{on } \partial\Gamma_s \times (0, \infty).$$

If $f_s = p_s$, then $\eta_1 = \eta_2 = 0$ and

$$(u_s \cdot \nabla)u_s - \operatorname{div} \sigma(u_s, p_s) = 0, \quad \operatorname{div} u_s = 0 \quad \text{in } \Omega,$$

$$u_s = 0 \ \text{ on } \Gamma_s, \quad u_s = g_s \ \text{ on } \Gamma_0, \quad \sigma(u_s, p_s)n = 0 \quad \text{on } \Gamma_n.$$

The system linearized around $(u_s, p_s, 0, 0)$

$$v_t - \operatorname{div}\sigma(v,q) + (u_s\cdot\nabla)v + (v\cdot\nabla)u_s - A_1\eta_1 - A_2\eta_2 = 0,$$

$$\operatorname{div} v = A_3\eta_1 \quad \text{in } Q_\infty,$$

$$v = \eta_2\, e_2\, \chi_{\Gamma_s} \quad \text{on } \Sigma_d^\infty, \quad \sigma(v,q)n = 0 \quad \text{on } \Sigma_n^\infty,$$

$$v(0) = v_0 = u_0 - u_s \text{ in } \Omega,$$

$$\eta_{1,t} = \eta_2 \quad \text{on } \Sigma_s^\infty,$$

$$\eta_{2,t} + \alpha\Delta_s^2\eta_1 - \delta\Delta_s\eta_2 - A_4\eta_1 = q + f \quad \text{on } \Sigma_s^\infty,$$

$$\eta_1 = 0 \quad \text{and} \quad \frac{\partial\eta_1}{\partial n} = 0 \quad \text{on } \partial\Gamma_s \times (0,\infty),$$

$$\eta_1(0) = \eta_1^0 \quad \text{and} \quad \eta_2(0) = \eta_2^0 \quad \text{on } \Gamma_s.$$

Expression of the pressure

$$
\begin{aligned}
-\Delta q &= A_3\eta_{1,t} + \mathrm{div}((u_s\cdot\nabla)v + (v\cdot\nabla)u_s - \nu\Delta(\mathrm{div} v)) \\
&\qquad -\mathrm{div}(A_1\eta_1) - \mathrm{div}(A_2\eta_2) \quad \text{in } \Omega,
\end{aligned}
$$

$$q = 2\nu\,\varepsilon(v)n\cdot n \qquad \text{on } \Gamma_n,$$

$$\frac{\partial q}{\partial n} = 2\nu\,\mathrm{div}\varepsilon(v)\cdot n - v_t\cdot n$$

$$= 2\nu\,\mathrm{div}\varepsilon(v)\cdot n - \eta_{2,t}, \quad \text{on } \Gamma_d.$$

Thus

$$q = -N_s(\eta_{2,t}) + N_d(A_3\,\eta_{1,t}) + N_v(v) + N(A_1\eta_1) + N(A_2\eta_2).$$

$$M_a z' = \widehat{\mathcal{A}} z + \mathcal{B} f, \quad z(0) = z_0,$$

$$(I - P)v(t) = (I - P)L(\eta_2(t)\, e_2, A_3\eta_1(t)),$$

$$z = (Pv, \eta_1, \eta_2)^T, \quad \mathcal{B} = (0\ 0\ I_{L^2(\Gamma_s)}\, \chi_{\Gamma_c})^T,$$

L is the lifting operator of the divergence and Dirichlet boundary condition, and

$$\widehat{\mathcal{A}} = \begin{pmatrix} A & (PA_1 - APL(0, A_3)) & (PA_2 - APL(\cdot, 0)) \\ 0 & 0 & I \\ \gamma_s N_v & \alpha\Delta_s^2 + \cdots & \delta\Delta_s + \cdots \end{pmatrix},$$

and

$$M_a = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & \gamma_s N_d A_3 & I + \gamma_s N_s \end{pmatrix}.$$

The added mass operator, M_a, is no longer symmetric.

We set

$$\mathcal{A} = M_a^{-1}\,\widehat{\mathcal{A}},$$

and, due to the corners

$$D(\mathcal{A}) = \Big\{(Pv, \eta_1, \eta_2) \in H^{1/2+\alpha_0}(\Omega;\mathbb{R}^2) \times (H^4 \cap H_0^2)(\Gamma_s) \times H_0^2(\Gamma_s) \ \Big| \ Pv - PL(\eta_2\,\vec{e}_2, A_3\eta_1) \in D(A_0)\Big\}.$$

The adjoint FSI system

$$\lambda\phi - \operatorname{div}\sigma(\phi,\psi) - (u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi = F_f^*,$$

$$\operatorname{div}\phi = 0 \quad \text{in } \Omega,$$

$$\phi = \zeta_2\vec{e}_2 \ \text{ on } \ \Gamma_s, \qquad \phi = 0 \ \text{ on } \ \Gamma_0,$$

$$\sigma(\phi,\psi)n + u_s\cdot n\,\phi = 0 \ \text{ on } \ \Gamma_n,$$

$$\lambda\zeta_1 + \zeta_2 - (\alpha\Delta_s^2)^{-1}(-A_3^*\psi + A_1^*\phi + A_4^*\zeta_2) = G_s^* \quad \text{in } \Gamma_s,$$

$$\lambda\zeta_2 - \alpha\Delta_s^2\zeta_1 - \delta\Delta_s\zeta_2 - A_2^*\phi = \psi + H_s^* \ \text{ in } \Gamma_s,$$

$$\zeta_1 = 0 \quad \text{and} \quad \frac{\partial\zeta_1}{\partial n} = 0 \qquad \text{on } \partial\Gamma_s.$$

The adjoint operator for the FSI PDE

$$\mathcal{D}(\mathcal{A}^*) = \{M_a^*(P\phi, \zeta_1, \zeta_2) \in Z \mid$$

$$(P\phi, \zeta_1, \zeta_2) \in V_{n,\Gamma_0}^{1/2+\alpha_0}(\Omega) \times H^4(\Gamma_s) \cap H_0^2(\Gamma_s) \times H_0^2(\Gamma_s)$$

$$P(\phi - D\zeta_2) \in \mathcal{D}(A_0^*) = \mathcal{D}(A_0)\}.$$

Can we use a ROM based on spectral projections ?

The operator $(\mathcal{A}, D(\mathcal{A}))$ is the infinitesimal generator of an analytic semigroup on Z, and its resolvent is compact. The state is $z = (Pv, \eta_1, \eta_2)$.

Thus spectral projections can be used to define a R.O.M.

$$Z_u = \oplus_{j\in J_u} G_{\mathbb{R}}(\lambda_j) \quad \text{with} \quad Z = Z_u \oplus Z_s$$

$$Z_u^* = \oplus_{j\in J_u} G_{\mathbb{R}}^*(\lambda_j) \quad \text{with} \quad Z^* = Z = Z_u^* \oplus Z_s^*.$$

- For numerical issues we need a basis $\{e_1, \cdots, e_{d_u}\}$ of Z_u and a basis $\{\Phi_1, \cdots, \Phi_{d_u}\}$ of Z_u^* satisfying

$$(e_i, \Phi_j)_Z = \delta_{i,j}.$$

- These bases are used to determine the projector:

$$P_u F = \sum_{i=1}^{d_u} (F, \Phi_i)_Z \, e_i.$$

The direct eigenvalue problem for the FSI PDE

$$\lambda v - \operatorname{div} \sigma(v, q) + (u_s \cdot \nabla) v + (v \cdot \nabla) u_s - A_1 \eta_1 - A_2 \eta_2 = 0,$$

$$\operatorname{div} v = A_3 \eta_1 \quad \text{in } \Omega,$$

$$v = \eta_2 \, e_2 \quad \text{on } \Gamma_s, \quad v = 0 \quad \text{on } \Gamma_0,$$

$$\sigma(v, q) n = 0 \quad \text{on } \Gamma_n,$$

$$\lambda \eta_1 = \eta_2 \quad \text{on } \Gamma_s,$$

$$\lambda \eta_2 + \alpha \Delta_s^2 \eta_1 - \delta \Delta_s \eta_2 - A_4 \eta_1 = q \quad \text{on } \Gamma_s,$$

$$\eta_1 = 0 \quad \text{and} \quad \frac{\partial \eta_1}{\partial n} = 0 \quad \text{on } \partial \Gamma_s.$$

$$\lambda \in \mathbb{C}, \quad \mathcal{A}(Pv, \eta_1, \eta_2) = \lambda(Pv, \eta_1, \eta_2).$$

(Pv, η_1, η_2) is an eigenvector for $\mathcal{A}$ associated with λ,
$(I - P)v = \nabla N_s \eta_2 + \nabla N_{\mathrm{div}} A_3 \eta_1$, and $q = \cdots$

iff

(v, q, η_1, η_2) is an eigenvector associated with λ, for the direct PDE system.

The adjoint eigenvalue problem for the FSI PDE

$$\lambda\phi - \mathrm{div}\,\sigma(\phi,\psi) - (u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi = 0,$$

$$\mathrm{div}\,\phi = 0 \quad \text{in } \Omega,$$

$$\phi = \zeta_2\vec{e}_2 \ \text{ on } \ \Gamma_s, \quad \phi = 0 \ \text{ on } \ \Gamma_0,$$

$$\sigma(\phi,\psi)n + u_s\cdot n\,\phi = 0 \ \text{ on } \ \Gamma_n,$$

$$\lambda\zeta_1 + \zeta_2 - (\alpha\Delta_s^2)^{-1}(-A_3^*\psi + A_1^*\phi + A_4^*\zeta_2) = 0 \quad \text{in } \Gamma_s,$$

$$\lambda\zeta_2 - \alpha\Delta_s^2\zeta_1 - \delta\Delta_s\zeta_2 - A_2^*\phi = \psi \ \text{ in } \Gamma_s,$$

$$\zeta_1 = 0 \quad \text{and} \quad \frac{\partial\zeta_1}{\partial n} = 0 \quad \text{on } \partial\Gamma_s.$$

The eigenvalue problem for $\mathcal{A}^*$

$$\lambda \in \mathbb{C}, \quad \mathcal{A}^* M_a^*(P\phi, \zeta_1, \zeta_2) = \lambda\, M_a^*(P\phi, \zeta_1, \zeta_2).$$

$M_a^*(P\phi, \zeta_1, \zeta_2)$ is an eigenvector for $\mathcal{A}^*$ associated with λ,
$(I - P)\phi = (I - P)L(\zeta_2 \vec{e}_2, 0) = \nabla N_s \zeta_2$, and $\psi = \cdots$

iff

$(\phi, \psi, \zeta_1, \zeta_2)$ is an eigenvector for the adjoint PDE system associated with λ.

Equivalence of the two bi-orthogonality conditions

The bi-orthogonality condition for eigenfunctions of the PDE systems is equivalent to the bi-orthogonality condition for eigenfunctions of $\mathcal{A}$ and $\mathcal{A}^*$.

$$((v_i, \eta_{1,i}, \eta_{2,i}), (\phi_j, \zeta_{1,j}, \zeta_{2,j}))_{L^2} = \delta_{i,j}$$

is equivalent to

$$((Pv_i, \eta_{1,i}, \eta_{2,i}), M_a^*(P\phi_j, \zeta_{1,j}, \zeta_{2,j}))_Z = \delta_{i,j}$$

We have to check the following unique continuation property.

If $(\lambda, \phi, \psi, \zeta_1, \zeta_2)$ is solution to the following eigenvalue problem

$$\lambda\phi - \operatorname{div}\sigma(\phi,\psi) - (u_s\cdot\nabla)\phi + (\nabla u_s)^T\phi = 0 \quad \text{and} \quad \operatorname{div}\phi = 0 \quad \text{in } \Omega,$$
$$\phi = \zeta_2\vec{e}_2 \ \text{ on } \ \Gamma_s, \quad \phi = 0 \ \text{ on } \ \Gamma_0, \quad \sigma(\phi,\psi)n + u_s\cdot n\,\phi = 0 \ \text{ on } \ \Gamma_n,$$
$$\lambda\zeta_1 + \zeta_2 - (\alpha\Delta_s^2)^{-1}(-A_3^*\psi + A_1^*\phi + A_4^*\zeta_2) = 0 \quad \text{in } \Gamma_s,$$
$$\lambda\zeta_2 - \alpha\Delta_s^2\zeta_1 - \delta\Delta_s\zeta_2 - A_2^*\phi = \psi \ \text{ in } \Gamma_s,$$
$$\zeta_1 = 0 \quad \text{and} \quad \frac{\partial\zeta_1}{\partial n} = 0 \quad \text{on } \partial\Gamma_s.$$

with $\operatorname{Re}\lambda \geq -\omega$ and

$$B^*(P\phi,\zeta_1,\zeta_2) = \zeta_2\,\chi_{\Gamma_c} = 0,$$

then

$$\phi = 0, \quad \psi = 0, \quad \zeta_1 = \zeta_2 = 0.$$

Stabilizability under additional conditions

• We assume that the spectrum of the linearized Stokes operator is disjoint from that of the damped beam operator.

• We can choose a control space of finite dimension so that the unique continuation property reduces to the unique continuation property for the Oseen operator.

• Under that condition and if $u_s = 0$, we show that there exists a control space of finite dimension for which the system is stabilizable (Osses-Puel, 09).

• If $u_s \neq 0$ and if $\|u_s\|_{H^1}$ is small enough, $(\mathcal{A}_{u_s}, \mathcal{B}_{u_s})$ is a perturbation of $(\mathcal{A}_0, \mathcal{B}_0)$ that satisfies the assumptions of Theorem 1.1, with $\varepsilon = \|u_s\|_{H^1}$. Thus $(\mathcal{A}_{u_s}, \mathcal{B}_{u_s})$ is satbilizable in Z for $\|u_s\|_{H^1}$ is small enough.

A sufficient condition for stabilizability is that

$$\int_0^\infty \|\mathcal{B}_u^*\, e^{-t(\mathcal{A}_u^*+\omega P_u^*)}\Phi\|^2_{L^2(\Gamma_s)}\, dt \geq \beta\|\Phi\|^2_Z. \quad \forall \Phi \in Z_u.$$

We notice that

$$e^{-t(\mathcal{A}_u+\omega P_u)}\mathcal{B}_u\,\mathcal{B}_u^*\, e^{-t(\mathcal{A}_u^*+\omega P_u^*)} = (X_{\omega,u})^{-1}$$

where $X_{\omega,u}$ is the solution to the Riccati equation

$$X_{\omega,u} \in \mathcal{L}(Z_u, Z_u^*), \quad X_{\omega,u} > 0,$$

$$X_{\omega,u}(\mathcal{A}_u + \omega P_u) + (\mathcal{A}_u^* + \omega P_u^*)X_{\omega,u} - X_{\omega,u}\mathcal{B}_u\,\mathcal{B}_u^* X_{\omega,u} = 0.$$

We can determine a numerical approximation $X^h_{\omega,u}$ of $X_{\omega,u}$. If that approximate solution satisfies

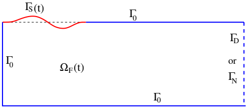
$$(X^h_{\omega,u})^{-1} \geq \beta\, I, \quad \text{for some } \beta > 0,$$

then $(\mathcal{A}_h, \mathcal{B}_h)$ is stabilizable.

Local stabilization of the closed loop nonlinear F–S system

If $u_0 \in H^1_{\Gamma_i}(\Omega;\mathbb{R}^2)$, $\eta_1^0 \in H^3(\Gamma_s)\cap H^2_0(\Gamma_s)$, $\eta_2^0 \in H^1_0(\Gamma_s)$, $u_0 = \eta_2^0\,\vec{e}_2$ on Γ_s, and $\mathrm{div} u_0 = \mathcal{G}(u_0,\eta_1)$, and if $(u_0 - u_s, p_0, p_s, \eta_1^0, \eta_2^0)$ is small enough in $H^1_{\Gamma_i}(\Omega;\mathbb{R}^2) \times H^3(\Gamma_s)\cap H^2_0(\Gamma_s) \times H^1_0(\Gamma_s)$, then the closed loop nonlinear system admits a solution decaying exponentially to the stationary solution in $H^{2,1}_\delta(Q_\infty;\mathbb{R}^2) \times H^{4,2}(\Sigma_s^\infty) \times H^{2,1}(\Sigma_s^\infty)$.

Based on a work in progress with T. Gudi.



The linearized system around $(0,0,0)$

$$\lambda u - \operatorname{div}\sigma(u,p) = F_f, \quad \operatorname{div} u = 0 \quad \text{in } \Omega,$$

$$u = \eta_2\, e_2\, \chi_{\Gamma_s} \quad \text{on } \Gamma_d, \quad \sigma(u,p)n = 0 \quad \text{on } \Gamma_n,$$

$$\lambda\eta_1 - \eta_2 = G_s \quad \text{on } \Gamma_s,$$

$$\lambda\eta_2 + \alpha\Delta_s^2\eta_1 - \delta\Delta_s\eta_2 = -\sigma(u,p)n\cdot n + H_s \quad \text{on } \Gamma_s,$$

$$\eta_1 = 0 \quad \text{and} \quad \frac{\partial\eta_1}{\partial n} = 0 \quad \text{on } \partial\Gamma_s.$$

The goal is to estimate $\|(\lambda I - \mathcal{A})^{-1}P - (\lambda I - \mathcal{A}_h)^{-1}P_h\|_{L^2\times H_0^2\times L^2}$ for $\lambda > 0$ large enough.

Bilinear forms of the FSI system

$$\sigma(u,p) = 2\nu\,\varepsilon(u) - p\,I,$$

$$a_f(u,v) = \int_\Omega 2\nu\,\varepsilon(u) : \varepsilon(v)\,dx\,dy,$$

$$b(v,p) = -\int_\Omega p\,\mathrm{div}\,v\,dx\,dy,$$

$$a_s(\eta,\xi) = \int_{\Gamma_s} \alpha\,\eta_{xx}\,\xi_{xx}\,dx,$$

$$a_d(\eta,\xi) = \int_{\Gamma_s} \delta\,\eta_x\,\xi_x\,dx.$$

We define $\mathbb{V} = \{v \in H^1_{\Gamma_d \setminus \Gamma_s}(\Omega; \mathbb{R}^2) \mid v_1|_{\Gamma_s} = 0\}$.

Find $u \in \mathbb{V},\ p \in L^2(\Omega),\ \theta \in H^{-1/2}(\Gamma_s; \mathbb{R}^2)$ such that

$$\lambda(u, v)_\Omega + a_f(u, v) + b(v, p) - \langle v, \theta \rangle_{\Gamma_s} = (F_f, v), \quad \forall v \in \mathbb{V},$$

$$b(u, q) = 0, \quad \forall q \in L^2(\Omega),$$

$$\langle u, \mu \rangle_{\Gamma_s} = \langle \eta_2\, n, \mu \rangle_{\Gamma_s}, \quad \forall \mu \in H^{-1/2}(\Gamma_s; \mathbb{R}^2),$$

$$(\lambda \eta_1 - \eta_2, \zeta) = (G_s, \zeta), \quad \forall \zeta \in H^2_0(\Gamma_s),$$

$$(\lambda \eta_2, \xi)_{\Gamma_s} + a_s(\eta_1, \xi) + a_d(\eta_2, \xi) = (-\theta \cdot n + H_s, \xi)_{\Gamma_s}, \quad \forall \xi \in H^2_0(\Gamma_s).$$

When $(u, p) \in H^{3/2+\alpha_0}(\Omega; \mathbb{R}^2) \times H^{1/2+\alpha_0}(\Omega)$ with $\alpha_0 \in (0, 1/2)$, we have

$$\theta = \sigma(u, p)n \in H^{\varepsilon_0}(\Gamma_s; \mathbb{R}^2).$$

Elimination of η_2

We replace η_2 by $\lambda\eta_1 - G_s$ in the last equation, and we set $\eta_1 = \eta$, we obtain

$$\lambda(u,v)_\Omega + a_f(u,v) + b(v,p) - \langle v, \theta\rangle_{\Gamma_s} = (F_f, v)_\Omega, \quad \forall v \in \mathbb{V},$$

$$b(u,q) = 0 \quad \forall q \in L^2(\Omega),$$

$$\langle u, \mu\rangle_{\Gamma_s} = \langle(\lambda\eta - G_s)n, \mu\rangle_{\Gamma_s}, \quad \forall \mu \in H^{-1/2}(\Gamma_s;\mathbb{R}^2),$$

$$\lambda^2(\eta,\xi)_{\Gamma_s} + a_s(\eta,\xi) + \lambda a_d(\eta,\xi)$$

$$= (-\theta\cdot n + H_s + G_s, \xi)_{\Gamma_s} + a_d(G_s,\xi), \quad \forall \xi \in H_0^2(\Gamma_s).$$

Monolythic formulation

If $u \in H^{3/2+\alpha_0}(\Omega)$ and $p \in H^{1/2+\alpha_0}(\Omega)$, we have

$$\lambda(u,v)_\Omega + a_f(u,v) + b(v,p) + \lambda^2(\eta,\xi)_{\Gamma_s} + a_s(\eta,\xi) + \lambda a_d(\eta,\xi)$$

$$= (F_f, v)_\Omega + (H_s + G_s, \xi) + a_d(G_s,\xi),$$

$$\forall v \in \mathbb{V}, \quad \forall \xi \in H_0^2(\Gamma_s) \quad \text{such that } v \cdot n = \xi,$$

$$b(u,q) = 0, \quad \forall q \in L^2(\Omega),$$

$$\langle u, \mu \rangle_{\Gamma_s} = \langle (\lambda\eta - G_s)n, \mu \rangle_{\Gamma_s}, \quad \forall \mu \in H^{-1/2}(\Gamma_s; \mathbb{R}^2).$$

We define a triangulation $\mathcal{T}$. We set

$$X_h = \{v_h \in \mathbb{V} \cap (C(\overline{\Omega}))^2 \mid v_h|_K \in \mathbb{P}_2, \forall K \in \mathcal{T}\}, \quad X_h^0 = X_h \cap (H_0^1(\Omega))^2,$$

$$M_h = \{p_h \in C(\overline{\Omega}) \mid p_h|_K \in \mathbb{P}_1, \forall K \in \mathcal{T}\},$$

$$S_h = \{\xi \in H_0^2(\Gamma_s) \mid \xi|_e \in \mathbb{P}_3, \forall e \in \mathcal{T} \cap \Gamma_s\}, \quad \text{cubic Hermite pol.}$$

$$Z_h = \{v_h \in X_h \mid b(v_h, q_h) = 0 \ \forall q_h \in M_h\}, \quad Z_h^0 = Z_h \cap (H_0^1(\Omega))^2,$$

$$X_{\Gamma_s}^h = \{v_h|_{\Gamma_s} \mid v_h \in X_h\}.$$

P_h is the $L^2(\Gamma_s)$-projection operator onto $X_{\Gamma_s}^h$.

Approximate variational formulation

Find $u_h \in X_h,\ p_h \in M_h,\ \theta_h \in X^h_{\Gamma_s},\ \eta_h \in S_h$ such that

$$\lambda(u_h, v_h)_\Omega + a_f(u_h, v_h) + b(v_h, p_h) - \langle v_h, \theta_h\rangle_{\Gamma_s} = (F_f, v_h)_\Omega,\ \forall v_h \in X_h,$$

$$b(u_h, q_h) = 0, \quad \forall q_h \in M_h,$$

$$\langle u_h, \mu_h\rangle_{\Gamma_s} = \langle(\lambda P_h\eta_h - G_s)n, \mu_h\rangle_{\Gamma_s}, \quad \forall \mu_h \in X^h_{\Gamma_s},$$

$$\lambda^2(\eta_h, \xi_h)_{\Gamma_s} + a_s(\eta_h, \xi_h) + \lambda a_d(\eta_h, \xi_h)$$
$$= (-\theta_h \cdot n + H_s + G_s, \xi_h)_{\Gamma_s} + a_d(G_s, \xi_h), \quad \forall \xi_h \in S_h.$$

Monolythic formulation for the approximate problem

$$\lambda(u_h, v_h)_\Omega + a_f(u_h, v_h) + b(v_h, p_h) + \lambda^2(\eta_h, \xi_h)_{\Gamma_s} + a_s(\eta_h, \xi_h)$$
$$+\lambda a_d(\eta_h, \xi_h)$$
$$= (F_f, v_h)_\Omega + (H_s + G_s, \xi_h)_{\Gamma_s} + a_d(G_s, \xi_h),$$
$$\forall v_h \in X_h \quad \forall \xi_h \in S_h \quad \text{such that } v_h|_{\Gamma_s} \cdot n = P_h \xi_h,$$
$$b(u_h, q_h) = 0, \quad \forall q \in M_h,$$
$$\langle u_h, \mu_h \rangle_{\Gamma_s} = \langle (\lambda P_h \eta_h - G_s) n, \mu_h \rangle_{\Gamma_s}, \quad \forall \mu_h \in X^h_{\Gamma_s},$$

From now on (to simplify), we assume that $G_s = 0$.

Construction of special test functions

π_h is the orthogonal projector in $L^2(\Gamma_s)$ onto S_h.

For the function $u \in H^{3/2+\alpha_0}(\Omega; \mathbb{R}^2) \cap \mathbb{V}$ such that (u, p, η_1, η_2) is the solution to the exact FSI problem, satisfying $u = \lambda\eta\, n$ on Γ_s, we denote by $(\Pi_h u, \widetilde{p}_h) \in Z_h \times M_h$ the solution to the following Stokes equation

$$\lambda(\Pi_h u, v_h)_\Omega + a_f(\Pi_h u, v_h) + b(v_h, \widetilde{p}_h) = (F_f, v_h)_\Omega, \quad \forall v_h \in X_h^0,$$

$$b(\Pi_h u, q_h) = 0, \quad \forall q_h \in M_h,$$

$$\Pi_h u = \lambda P_h \pi_h \eta\, n, \quad \text{on } \Gamma_s.$$

We notice that

$$(\Pi_h u - u_h) \cdot n = \lambda P_h(\pi_h \eta - \eta_h).$$

Special test functions in the exact problem

We replace v by $\Pi_h u - u_h$ and ξ by $\lambda(\pi_h\eta - \eta_h)$ in the variational formulation of the exact problem, we obtain

$$\lambda(u, \Pi_h u - u_h)_\Omega + a_f(u, \Pi_h u - u_h) + b(\Pi_h u - u_h, p)$$

$$+(\lambda^2\eta, \lambda(\pi_h\eta - \eta_h))_{\Gamma_s} + a_s(\eta, \lambda(\pi_h\eta - \eta_h)) + \lambda a_d(\eta, \lambda(\pi_h\eta - \eta_h))$$

$$= (F_f, \Pi_h u - u_h)_\Omega + (H_s + G_s, \lambda(\pi_h\eta - \eta_h))_{\Gamma_s} + a_d(G_s, \lambda(\pi_h\eta - \eta_h)).$$

We next write the difference of the exact and the approximate variational equations.

We set

$$e_u = u - u_h, \quad e_\eta = \eta - \eta_h, \quad \varepsilon_u = u - \Pi_h u, \quad \varepsilon_\eta = \eta - \pi_h \eta, \quad \varepsilon_p = p - r_h p.$$

Using estimates of ε_u, ε_p, and ε_η, we obtain

$$\lambda \|e_u\|^2_{L^2(\Omega)} + \|e_u\|^2_1 + \lambda^3 \|e_\eta\|^2_{L^2(\Gamma_s)} + \lambda\alpha \|\Delta e_\eta\|^2_{L^2(\Gamma_s)} + \lambda^2 \delta \|\nabla e_\eta\|^2_{L^2(\Gamma_s)}$$
$$\leq C\, h^{1+2\alpha_0} \left(\|F_s\|^2_{L^2(\Omega;\Omega^2)} + \|H_s\|^2_{L^2(\Gamma_s)} \right).$$

The approximation error e_p on the pressure can be found a posteriori with the fluid equation.

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