ICTS - Bangalore

Recent advances on control theory of PDE systems

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Stabilization of fluid flows using ROM based on spectral projection
Numerical approximation of feedback gains based on ROM

Lecture 3

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Outline of the talk

Part I

- Issues for the Navier-Stokes equations

Lectures I and II

- Numerical approximation of Riccati based feedbacks for the Oseen system (Nonconforming approximation) - drawback.

Part III

- ROM based on spectral projections - Convergence rates for the approximation of Riccati based feedbacks based on these ROM.

Part IV

- Some errors estimates for the boundary control of the pseudo-compressible approximation.

Part I - Issues for the Navier-Stokes equations

- Ω is a bounded polyhedral domain in $\mathbb{R}^3$, not nec. convex.
- $(w_s, p_s) \in H^1(\Omega;\mathbb{R}^3) \times L^2_0(\Omega)$ is a stationary solution of the N.S.E:

$$(w_s \cdot \nabla)w_s - \nu\,\Delta w_s + \nabla p_s = f_s, \quad \operatorname{div} w_s = 0 \quad \text{in} \quad \Omega,$$
$$w_s = g_s \quad \text{on} \quad \Gamma = \partial\Omega.$$

- The controlled Navier-Stokes system

$$\frac{\partial w}{\partial t} + (w \cdot \nabla)w - \nu\,\Delta w + \nabla p = f_s \quad \text{in} \quad Q = \Omega \times (0,\infty),$$
$$\operatorname{div} u = 0 \quad \text{in} \quad Q, \quad w(0) = w_0 = w_s + z_0 \quad \text{in} \quad \Omega,$$
$$w = u_c + g_s \quad \text{on} \quad \Sigma_\infty = \Gamma \times (0,\infty),$$
$$u_c(x,t) = \sum_{i=1}^{N_c} f_i(t)\, g_i(x), \quad \int_\Gamma g_i \cdot n dx = 0,$$

Find a control f s. t. $\|w(t) - w_s\|_{L^2} \le C\, e^{-\omega t}\, \|z_0\|_{L^2}, \quad \omega > 0.$

The Oseen system

The nonlinear system satisfied by $(y, q) = (w, p) - (w_s, p_s)$ is

$$\frac{\partial y}{\partial t} + (w_s \cdot \nabla)y + (y \cdot \nabla)w_s + \kappa(y \cdot \nabla)y - \nu\,\Delta y + \nabla q = 0 \quad \text{in} \quad Q,$$

$$\operatorname{div} y = 0 \quad \text{in} \quad Q,$$

$$y = \textstyle\sum_{i=1}^{N_c} f_i\, g_i \quad \text{on} \quad \Sigma,$$

$$y(0) = z_0 \quad \text{in} \quad \Omega,$$

with $\kappa = 1$. The associated linearized system is obtained by setting $\kappa = 0$.

Main goal

Compute numerically a feedback gain $K = (K_1, \cdots, K_{N_c})$ so that

$$\frac{\partial y}{\partial t} + (w_s \cdot \nabla) y + (y \cdot \nabla) w_s + \kappa (y \cdot \nabla) y - \nu \, \Delta y + \nabla q = 0 \quad \text{in} \quad Q,$$

$$\operatorname{div} y = 0 \quad \text{in} \quad Q,$$

$$y(t) = \sum_{i=1}^{N_c} [K_i y(t)] \, g_i \quad \text{on} \quad \Sigma,$$

$$y(0) = z_0 \quad \text{in} \quad \Omega,$$

- the linearized closed-loop system (with $\kappa = 0$) is exponentially stable with a prescribed decay rate $\omega > 0$, in some space,
- and the nonlinear closed-loop system (with $\kappa = 1$) is locally exponentially stable in some space (under some smallness condition on z_0).

The method

- Consider the parabolic system (for $z = Py$, P is the Leray proj.)

$$z' = Az + Bf, \quad z(0) = z_0, \quad \text{in } Z \subset L^2(\Omega; \mathbb{R}^3).$$

- Assume that the pair (A, B) is stabilizable in Z.
- Choose a K such that $A + BK$ is exponentially stable in Z.
- Find a numerical approximation K_h of K by using

$$z_h' = A_h z_h + B_h f, \quad z_h(0) = z_{0,h}, \quad \text{in } Z_h.$$

We use a nonconforming approximation $Z_h \not\subset Z$ and

$$Z \subset H \quad \text{and} \quad Z_h \subset H.$$

$$P : H \longmapsto Z, \quad P_h : H \longmapsto Z_h.$$

- Prove error estimates for $\|KP - K_h P_h\|_{\mathcal{L}(H,U)}$.
- Deduce that $A + BK_h P_h$ is exponentially stable in Z.

Drawbacks and other strategies

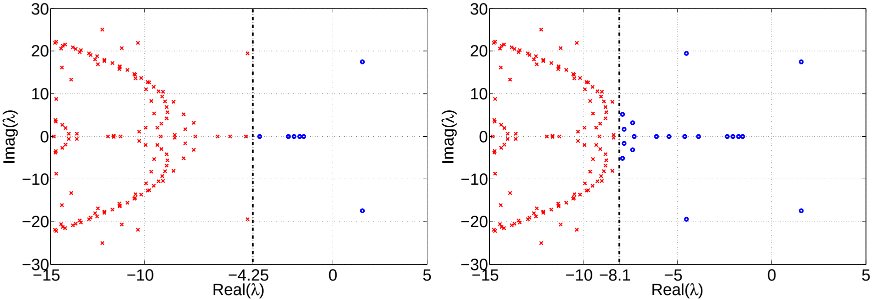
- The Riccati equations used to determine K_h are of very large dimension.

- We are going to prove convergence rates for reduced order models based on spectral projections.

- P. Benner, J. Heiland (2015), P. Benner et al. (2013, 2015). Efficient algorithms of large-scale Riccati equations for the stabilization of incompressible Navier-Stokes flows.

- For the other strategies based on POD, BPOD..., there is no convergence rates in terms of the discretization parameter h.

The resolvent of A (resp. A_h) is compact in Z (resp. Z_h).



$$Z_u = \oplus_{j\in J_u} G_{\mathbb{R}}(\lambda_j), \quad Z = Z_u \oplus Z_s, \quad \dim Z_u = d_u < \infty.$$

Z_u and Z_s are invariant subspaces of A.

$$\mathrm{Re}\,\sigma(A|_{Z_u}) > -\omega_u \quad \text{and} \quad \mathrm{Re}\,\sigma(A|_{Z_s}) < -\omega,$$

$$\omega_u > \omega.$$

Spectral projections in Z

The projector $P_u \in \mathcal{L}(Z, Z_u)$ (and $P_u \in \mathcal{L}(H, Z_u)$) is defined by

$$P_u = \frac{1}{2i\pi} \int_{\Gamma_u} (\lambda I - A)^{-1} P \, d\lambda,$$

Γ_u is a union of Jordan curves, around $(\lambda_j)_{j \in J_u} \cup (\overline{\lambda_j})_{j \in J_u}$.

We split $z' = Az + Bu$ into two systems

$$A_u = A|_{Z_u}, \qquad A_s = A|_{Z_s}, \quad B_u = P_u B, \quad B_s = (I - P_u) B.$$

$$z = z_u + z_s, \quad z_u' = A_u z_u + B_u u, \quad z_s' = A_s z_s + B_s u.$$

Basic idea

If $K_u \in \mathcal{L}(Z_u, U)$ is a feedback such that

$$e^{t(A_u+\omega_u I+B_u K_u)} \text{ is exponentially stable on } Z_u,$$

then

$$\|e^{t(A+BK_uP_u)}\|_{\mathcal{L}(Z)} \le C\, e^{-\omega t}, \quad \forall t \ge 0.$$

Because

$$\omega_u > \omega \quad \text{and} \quad \mathrm{Re}\, \sigma(A|_{Z_s}) < -\omega.$$

Proof

$$z(t) = e^{t(A+BK_uP_u)}z_0, \quad z_u(t) = e^{t(A_u+B_uK_uP_u)}P_uz_0,$$

$$z_s(t) = e^{tA_s}P_sz_0 + \int_0^t e^{(t-\tau)A_s}B_sK_uz_u(\tau)\,d\tau.$$

We know that

$$\|e^{tA_s}\|_{\mathcal{L}(Z_s)} \le C\,e^{-\omega t}, \quad \forall t \ge 0,$$

and

$$\|e^{t(A_u+B_uK_u)}\|_{\mathcal{L}(Z_u)} \le C\,e^{-\omega_u t}, \quad \forall t \ge 0.$$

Thus

$$\|z_s(t)\|_Z \le C\,e^{-t\omega}\|P_sz_0\|_Z + C\int_0^t \frac{e^{-(t-\tau)\omega}}{(t-\tau)^\gamma}\|z_u(\tau)\|_Z\,d\tau.$$

$$\le C\,e^{-t\omega}\int_0^t \frac{e^{\tau(\omega-\omega_u)}}{(t-\tau)^\gamma}\|P_uz_0\|_Z\,d\tau.$$

$$z_h' = A_h z_h + B_h f, \quad z_h(0) = z_{0,h}, \quad \text{in } Z_h.$$

We use a nonconforming approximation $Z_h \not\subset Z$ and

$$Z \subset H \quad \text{and} \quad Z_h \subset H.$$

$$P \ : \ H \longmapsto Z, \quad P_h \ : \ H \longmapsto Z_h.$$

- Can we use spectral projections in Z_h ?

$$z_{h,u}' = A_{h,u} z_{h,u} + B_{h,u} f, \quad z_{h,u}(0) = P_{h,u} z_{0,h},$$

$$z_{h,s}' = A_{h,s} z_{h,s} + B_{h,s} f, \quad z_{h,s}(0) = P_{h,s} z_{0,h}.$$

- If yes, how to define $P_{h,u}$? Can we estimate $P_u - P_{h,u}$? Can we construct a feedback gain $K_{h,u} \in \mathcal{L}(Z_{h,u}, U)$ such that

$$\|e^{t(A+BK_{h,u}P_{h,u})}\|_{\mathcal{L}(Z)} \leq C\, e^{-\omega t}, \quad \forall t \geq 0 \ \ ?$$

Assumptions satisfied by (A_h, P_h, B_h)

We present the method for an abstract parabolic system.

- Uniform analyticity for A and A_h.
- $\|(\lambda_0 I - A)^{-1} P - (\lambda_0 I - A_h)^{-1} P_h\|_{\mathcal{L}(H)} \leq C\, h^s$.
- $\|(\lambda_0 I - A)^{-1} B - (\lambda_0 I - A_h)^{-1} B_h\|_{\mathcal{L}(U,H)} C\, h^r$.
- The last assumption on the uniform boundedness

$$\|e^{A_h t} B_h\|_{\mathcal{L}(V^0(\Gamma), Z_h)} \leq C\, \frac{e^{t\omega_0}}{t^{\overline{\gamma}}}, \quad \forall t \in (0, h^{r/(1-\gamma)}), \quad \forall h \in (0,1),$$

is not necessary.

With these conditions, we want to prove estimates between (A_u, P_u, B_u) and $(A_{h,u}, P_{h,u}, B_{h,u})$ similar to those for (A, P, B) and (A_h, P_h, B_h).

The approxim. param. h can be a meshsize or a penalty parameter.

The basic idea works for $K_{h,u}$

If $K_{h,u} \in \mathcal{L}(Z_{h,u}, U)$ is a family of feedbacks such that

$$\|e^{t(A_u + B_u K_{h,u} P_{h,u})}\|_{\mathcal{L}(Z_u)} \leq C\, e^{-\omega_u t}, \quad \forall t \geq 0, \quad \forall h \in (0, h_0),$$

then

$$\|e^{t(A + B K_{h,u} P_{h,u})}\|_{\mathcal{L}(Z)} \leq C\, e^{-\omega t}, \quad \forall t \geq 0.$$

$$z^h(t) = e^{t(A+BK_{h,u}P_{h,u})} z_0, \quad z_u^h(t) = e^{t(A_u+B_uK_{h,u}P_{h,u})} P_u z_0,$$

$$z_s^h(t) = e^{tA_s} P_s z_0 + \int_0^t e^{(t-\tau)A_s} B_s K_{h,u} P_{h,u} z_u^h(\tau)\, d\tau.$$

We know that

$$\|e^{tA_s}\|_{\mathcal{L}(Z_s)} \leq C\, e^{-\omega t}, \quad \forall t \geq 0,$$

and

$$\|e^{t(A_u+B_uK_{h,u}P_{h,u})}\|_{\mathcal{L}(Z_u)} \leq C\, e^{-\omega_u t}, \quad \forall t \geq 0.$$

Thus

$$\|z_s^h(t)\|_Z \leq C\, e^{-t\omega} \|P_s z_0\|_Z + C \int_0^t \frac{e^{-(t-\tau)\omega}}{(t-\tau)^\gamma} \|z_u^h(\tau)\|_Z\, d\tau.$$

$$\leq C\, e^{-t\omega} \int_0^t \frac{e^{\tau(\omega-\omega_u)}}{(t-\tau)^\gamma} \|P_u z_0\|_Z\, d\tau.$$

There exists $h_0 > 0$, such that $\Gamma_u \subset \rho(A_h), \ \forall h \in (0, h_0)$.

We prove that if, for $\lambda \in \{-\lambda_0\} + \Gamma_u$,

$$\|(\lambda_0 I_h - A_h)^{-1} P_h - (\lambda_0 I - A)^{-1} P\|_{\mathcal{L}(H)}$$
$$< \frac{1}{2[1+|\lambda| \max(\|P\|,\|P_h\|)]^2[1+\|(A-(\lambda_0+\lambda)I)^{-1}P\|_{\mathcal{L}(H,Z)}]},$$

then

$$\|(\mu I_h - A_h)^{-1}\|_{\mathcal{L}(H)} \le \sup_{\mu\in\Gamma_u}(1 + 2\|(\mu I - A)^{-1} P\|_{\mathcal{L}(H,Z)}),$$

with $\quad \mu = \lambda + \lambda_0.$

This result is proved in Theorem 1.3.

Spectral projections in Z_h - Approximation of P_u

Since there exists $h_0 > 0$, such that $\Gamma_u \subset \rho(A_h), \ \forall h \in (0, h_0)$, we set

$$P_{h,u} = \frac{1}{2i\pi} \int_{\Gamma_u} (\lambda I - A_h)^{-1} P_h \, d\lambda,$$

$$Z_{h,u} = P_{h,u} Z_h \quad \text{and} \quad Z_{h,s} = (I - P_{h,u}) Z_h,$$

$$A_{h,u} = A_h \big|_{Z_{h,u}}, \quad A_{h,s} = A_h \big|_{Z_{h,s}},$$

$$B_{h,u} = P_{h,u} \, B_h \quad B_{h,s} = (I - P_{h,u}) \, B_h.$$

We split $z_h' = A_h z_h + B_h u$ into two systems

$$z_h = z_{h,u} + z_{h,s}, \quad z_{h,u}' = A_{h,u} z_{h,u} + B_{h,u} u, \quad z_{h,s}' = A_{h,s} z_{h,s} + B_{h,s} u.$$

Estimate of $P_u - P_{h,u}$

For all $h \in (0, h_0)$, we have

$$\|P_u - P_{h,u}\|_{\mathcal{L}(H)} \leq C\, h^s = C\, h^{1+2\alpha_0} \quad \text{and} \quad \dim(Z_u) = \dim(Z_{h,u}).$$

$$P_u - P_{h,u} = \frac{1}{2i\pi} \int_{\Gamma_u} \left((\lambda I - A)^{-1} P - (\lambda I - A_h)^{-1} P_h \right) \, d\lambda.$$

An estimate of $\left((\lambda I - A)^{-1} P - (\lambda I - A_h)^{-1} P_h \right)$ is obtained with a resolvent identity and

$$\|(\lambda_0 I - A)^{-1} P - (\lambda_0 I - A_h)^{-1} P_h\|_{\mathcal{L}(H)} \leq C\, h^s.$$

If $\dim(Z_u) = d_u$ and $(e_1, \cdots, e_{d_u})$ is an orthonormal basis of Z_u, with the estimate of $P_u - P_{h,u}$, we prove that $(P_{h,u} e_1, \cdots, P_{h,u} e_{d_u})$ is a basis of $Z_{h,u}$ for all $h \in (0, h_0)$, for $h_0 > 0$ small enough.

Estimate of $(\lambda_0 I - A_u)^{-1} P_u - (\lambda_0 I - A_{h,u})^{-1} P_{h,u}$

For all $h \in (0, h_0)$, we have

$$\|(\lambda_0 I - A_u)^{-1} P_u - (\lambda_0 I - A_u)^{-1} P_{h,u}\|_{\mathcal{L}(H)} \leq C\, h^s$$

and

$$\|(\lambda_0 I - A_s)^{-1} P_s - (\lambda_0 I - A_s)^{-1} P_{h,s}\|_{\mathcal{L}(H)} \leq C\, h^s.$$

$$(\lambda_0 I - A_u)^{-1} P_u - (\lambda_0 I - A_u)^{-1} P_{h,u}$$
$$= (\lambda_0 I - A_u)^{-1} P (P_u - P_{h,u}) + ((\lambda_0 I - A)^{-1} P - (\lambda_0 I - A_h)^{-1} P_h) P_{h,u}.$$

For the second estimate, we notice that

$$(\lambda_0 I - A)^{-1} P = (\lambda_0 I - A_u)^{-1} P_u + (\lambda_0 I - A_s)^{-1} P_s$$

and

$$(\lambda_0 I - A_h)^{-1} P_h = (\lambda_0 I - A_{h,u})^{-1} P_{h,u} + (\lambda_0 I - A_{s,h})^{-1} P_{h,s}.$$

Uniform stability of $e^{t\,A_{s,h}}$

Remark. From

$$\|(\lambda_0 I - A_s)^{-1} P_s - (\lambda_0 I - A_s)^{-1} P_{h,s}\|_{\mathcal{L}(H)} \leq C\, h^s,$$

and

$$\|e^{tA_s}\|_{\mathcal{L}(Z_s)} \leq C\, e^{-t\omega}, \quad \omega > 0,$$

and the perturbation result in Lecture 1, we deduce

$$\|e^{t\,A_{s,h}}\|_{\mathcal{L}(Z_{h,s})} \leq C\, e^{-t\widetilde{\omega}}, \quad \widetilde{\omega} > 0, \quad \forall h \in (0, h_0).$$

Estimate of $B_u - B_{h,u}$

- $B_u f = \sum_{i=1}^{N_c} f_i(\lambda_0 I - A) P_u D\, g_i$,
- $B_{h,u} f = \sum_{i=1}^{N_c} f_i(\lambda_0 I - A_h) P_{h,u} D_h\, g_i$.

$$\|(\lambda_0 I - A_u)^{-1} B_u - (\lambda_0 I - A_{h,u})^{-1} B_{h,u}\|_{\mathcal{L}(U,H)} \leq C\, h^s,$$

$$\|B_u - B_{h,u}\|_{\mathcal{L}(U,H)} \leq C\, h^s, \qquad \forall h \in (0, h_0).$$

For F.E. approximation $h^s = h^{1+2\alpha_0}$. This is much better than

$$\|(\lambda_0 I - A)^{-1} B - (\lambda_0 I - A_h)^{-1} B_h\|_{\mathcal{L}(U,H)} \leq C\, h^{\alpha_0}.$$

Uniform bound for $B_{h,u} \in \mathcal{L}(U, H)$

$$\sup_{h \in (0,h_0)} \|B_{h,u}\|_{\mathcal{L}(U,H)} < +\infty.$$

Uniform stabilizability of $(A_{h,u} + \omega_u P_{h,u}, B_{h,u})$

Assumption: Stabilizability and detectability conditions.

$(A_u + \omega_u P_u, B_u)$ is stabilizable in Z_u.

Either $(A_u + \omega_u P_u, \mathcal{C}|_{Z_u})$ is detectable or $\mathcal{C} = 0$.

We choose $\omega_u > \omega$ and $h_0 > 0$ such that

$$\mathrm{Re}\,\sigma(A_{h,u}) > -\omega_u \quad \text{and} \quad \mathrm{Re}\,\sigma(A_{h,s}) < -\omega, \quad \forall h \in (0, h_0).$$

$(A_{h,u} + \omega_u P_{h,u}, B_{h,u})$ is unif. stabilizable

$(A_{h,u} + \omega_u P_{h,u}, \mathcal{C}\big|_{Z_{h,u}})$ is unif. detectable.

The Riccati equation in Z_u

$$\Pi_u \in \mathcal{L}(Z_u, Z_u^*), \quad \Pi_u = \Pi_u^* \geq 0, \quad C_u = \mathcal{C}|_{Z_u},$$

$$\Pi_u(A_u + \omega_u P_u) + (A_u^* + \omega_u P_u^*)\Pi_u - \Pi_u B_u B_u^* \Pi_u + C_u^* C_u = 0,$$

$$A_u + \omega_u P_u - B_u B_u^* \Pi_u \quad \text{is exponentially stable in } Z_u.$$

If $K_u = -B_u^* \Pi_u$, then

$$\|e^{t(A+BK_u)}\|_{\mathcal{L}(H)} \leq Ce^{-t\,\omega}.$$

$$\Pi_{h,u} \in \mathcal{L}(Z_{h,u}, Z_{h,u}^*), \quad \Pi_{h,u} = P_{h,u}^*, \quad C_{h,u} = \mathcal{C}\big|_{Z_{h,u}},$$

$$\Pi_{h,u}(A_{h,u} + \omega_u P_{h,u}) + (A_{h,u}^* + \omega_u P_{h,u}^*)\Pi_{h,u} - \Pi_{h,u} B_{h,u} B_{h,u}^* \Pi_{h,u} + C_{h,u}^* C_{h,u} = 0,$$

$$A_{h,u} + \omega_u P_{h,u} - B_{h,u} B_{h,u}^* \Pi_{h,u} \quad \text{is exponentially stable in } Z_{h,u}.$$

The solutions Π_u and $\Pi_{h,u}$, and the feedbacks $K_u = -B_u^* \Pi_u P_u$ and $K_{h,u} = -B_{h,u}^* \Pi_{h,u} P_{h,u}$ obey

$$\|\Pi_u P_u - \Pi_{h,u} P_{h,u}\|_{\mathcal{L}(H)} \le C\, h^s,$$

and

$$\|K_u - K_{h,u}\|_{\mathcal{L}(H,U)} \le C\, h^s, \quad \forall h \in (0, h_0).$$

Convergence rates for the closed-loop systems

- $Py(t) = e^{(A+BK)t}y_0, \quad K = -B_u^*\Pi_u P_u$, f is the control.

  $(I-P)y(t) = \sum_{i=1}^{N_c} K_i y(t)(I-P)Dg_i$.
- $Py^h(t) = e^{(A+BK_h)t}y_0, \quad K_h = -B_{h,u}^*\Pi_{h,u}P_{h,u}$, f^h is the control.

  $(I-P)y^h(t) = \sum_{i=1}^{N_c} K_{h,i} y^h(t)(I-P)Dg_i$.
- $y_h(t) = e^{(A_h+B_hK_h)t}y_0, \quad K_h = -B_{h,u}^*\Pi_{h,u}P_{h,u}$, f_h is the control.

For all $h \in (0, h_0)$, we have

$$\|f_h(t) - f(t)\|_U \leq C\,\frac{e^{-\omega t}}{t}\,h^s\|y_0\|_H,$$

$$\|f_h - f\|_{L^p(0,\infty;U)} \leq C_p\,h^s\|y_0\|_H, \quad \forall p \in [1,\infty),$$

$$\|u_h - u^h\|_{L^p(0,\infty;U)} \leq C_p\,h^s\|y_0\|_H, \quad \forall p \in [1,\infty),$$

where $\omega > 0$ is the a priori prescribed decay rate.

• Convergence rates for the projections of the solutions of closed-loop systems

$$\|P_{h,u}y_h(t) - P_u y(t)\|_H \le C\,\frac{e^{-\omega t}}{t}\,h^s\|y_0\|_H,$$

$$\|P_{h,u}y_h - P_u y\|_{L^p(0,\infty;H)} \le C_p\,h^s|\ln(h)|\|y_0\|_H, \quad \forall p \in [1,\infty),$$

$$\|P_{h,u}y_h - P_u y^h\|_{L^p(0,\infty;H)} \le C_p\,h^s|\ln(h)|\|y_0\|_H, \quad \forall p \in [1,\infty).$$

• For the Oseen system, we could obtain convergence rates between y_h and y, on compact time intervals $[0, T]$, if we took a dynamic controller and if $y_0 \in V_n^0(\Omega) \cap (H_0^1(\Omega))^3$.

Applications of ROM based on spectral projections

- Numerical approximation of the Oseen system with a boundary control (lectures 2 and 3)
- Approximation of the Oseen system by the pseudo-compressibility method with internal control (lecture 1)
- Approximation of the Oseen system by the pseudo-compressibility method with a boundary control (see below)
- Numerical approximation of the Boussinesq system (doable)
- Stabilization of FSI systems (lecture 4) and their numerical approximation (under investigation lecture 4)

Ω convex or of class C^2.

The Oseen system without control

$$\frac{\partial y}{\partial t} + (w_s \cdot \nabla) y + (y \cdot \nabla) w_s - \nu \Delta y + \nabla p = F \quad \text{in} \quad Q,$$
$$\operatorname{div} y = 0 \quad \text{in} \quad Q, \quad y = 0 \quad \text{on} \quad \Sigma, \quad y(0) = y_0 \quad \text{in} \quad \Omega.$$

The Leray projector $P \in \mathcal{L}(H, Z)$, $H = (L^2(\Omega))^3$, $Z = V_n^0(\Omega)$.

$$V_n^0(\Omega) = \{ y \in L^2(\Omega; \mathbb{R}^3) \mid \operatorname{div} y = 0,\ y \cdot n = 0 \text{ on } \Gamma \}$$

The Oseen operator $(A, \mathcal{D}(A))$

$$Ay = P(\nu \Delta y - (w_s \cdot \nabla) y - (y \cdot \nabla) w_s),$$
$$\mathcal{D}(A) = V_n^0(\Omega) \cap (H_0^1(\Omega) \cap H^2(\Omega))^3.$$

The Oseen system

$$y' = Ay + PF, \quad y(0) = y_0.$$

Approximation by the pseudo-compressible model

- Pseudo-compressible approximation

$$\frac{\partial y_\varepsilon}{\partial t} - \nu\Delta y_\varepsilon + (y_\varepsilon \cdot \nabla) w_s^\varepsilon + (w_s^\varepsilon \cdot \nabla) y_\varepsilon + \nabla p_\varepsilon = F \ \text{ in } Q,$$

$$\operatorname{div} y_\varepsilon + \varepsilon p_\varepsilon = 0 \text{ in } Q, \qquad y_\varepsilon = 0 \ \text{ on } \Sigma, \qquad y_\varepsilon(0) = y_0 \ \text{ in } \Omega.$$

w_s^ε is an approximation of w_s.

- The equation for y_ε can be solved first

$$\frac{\partial y_\varepsilon}{\partial t} - \nu\Delta y_\varepsilon + (y_\varepsilon \cdot \nabla) w_s^\varepsilon + (w_s^\varepsilon \cdot \nabla) y_\varepsilon - \frac{1}{\varepsilon}\nabla \operatorname{div} y_\varepsilon = F \ \text{ in } Q.$$

Uniform coervivity condition

The stationary solution (w_s, ρ_s) belongs to $(H^1(\Omega))^3 \times L^2(\Omega)$.
For all w_s^ε satisfying the H^1-bound

$$\|w_s^\varepsilon\|_{(H^1(\Omega))^3} \leq \|w_s\|_{(H^1(\Omega))^3} + 1,$$

we set

$$a_\varepsilon(z,\zeta) = \int_\Omega (\nu \nabla z : \nabla \zeta + (w_s^\varepsilon \cdot \nabla) z \cdot \zeta + (z \cdot \nabla) w_s^\varepsilon \cdot \zeta)\ dx,$$

for all $z \in (H^1(\Omega))^3$, $\zeta \in (H^1(\Omega))^3$.

We can choose $\lambda_0 > 0$ such that

$$\lambda_0 \|z\|^2_{(L^2(\Omega))^3} + a_\varepsilon(z,z) \geq \frac{\nu}{2} \|z\|^2_{(H^1(\Omega))^3},$$

for all $z \in (H^1(\Omega))^3$ and all w_s^ε satisfying the H^1-bound.

We assume that $\|w_s^\varepsilon - w_s\|_{(H^1(\Omega))^3} \leq C_s\,\varepsilon, \quad \forall \varepsilon \in (0,1).$
We set $\varepsilon_0 = 1/C_s$. The pseudo-compressible Oseen operator A_ε is

$$\mathcal{D}(A_\varepsilon) = (H^2(\Omega) \cap H_0^1(\Omega))^3,$$
$$A_\varepsilon y = \nu\Delta y - (y\cdot\nabla)w_s^\varepsilon - (w_s^\varepsilon\cdot\nabla)y + \tfrac{1}{\varepsilon}\nabla(\mathrm{div}\, y).$$

The pseudo-compressible system can be rewritten in the form

$$y_\varepsilon' = A_\varepsilon y_\varepsilon + F, \quad y_\varepsilon(0) = y_0.$$

For all $\varepsilon \in (0,\varepsilon_0)$, the operator $(A_\varepsilon, \mathcal{D}(A_\varepsilon))$ is the infinitesimal generator of an analytic semigroup on $(L^2(\Omega))^3$. We have

$$\{\omega_0\} + \mathbb{S}_{\pi/2+\delta} \subset \rho(A_\varepsilon),$$
$$\|(\lambda I - A_\varepsilon)^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \leq \tfrac{C}{|\lambda-\omega_0|} \quad \text{for all } \lambda \in \{\omega_0\} + \mathbb{S}_{\pi/2+\delta},$$

for all $\varepsilon \in (0,\varepsilon_0)$.

Estimate of $(\lambda_0 I - A)^{-1}P - (\lambda_0 I - A_\varepsilon)^{-1}$

- The following bounds hold, uniformly in $\varepsilon \in (0, \varepsilon_0)$:

$$\|y\|_{(H^2(\Omega))^3} + \frac{1}{\varepsilon}\|\mathrm{div}\, y\|_{H^1(\Omega)} \leq C\|(\lambda_0 I - A_\varepsilon)y\|_{(L^2(\Omega))^3}, \ \forall z \in \mathcal{D}(A_\varepsilon),$$

$$\|\phi\|_{(H^2(\Omega))^3} + \frac{1}{\varepsilon}\|\mathrm{div}\, \phi\|_{H^1(\Omega)} \leq C\|(\lambda_0 I - A_\varepsilon^*)\phi\|_{(L^2(\Omega))^3}, \ \forall \phi \in \mathcal{D}(A_\varepsilon^*).$$

proved by rewriting the divergence eq. as for the imcompressible case.

- We have to prove

$$\|(\lambda_0 I - A)^{-1}P - (\lambda_0 I - A_\varepsilon)^{-1}\|_{\mathcal{L}((L^2(\Omega))^3)} \leq C\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

- $y = (\lambda_0 I - A)^{-1} PF$ is solution of

$$\lambda_0 y - \nu\Delta y + (y\cdot\nabla)w_s + (w_s\cdot\nabla)y + \nabla q = F \ \text{ in } \Omega,$$
$$\operatorname{div} y = 0 \text{ in } \Omega, \quad y = 0 \ \text{ on } \Gamma.$$

- $y^\varepsilon = (\lambda_0 I - A_{w_s^\varepsilon})^{-1} PF$ is solution of

$$\lambda_0 y - \nu\Delta y + (y\cdot\nabla)w_s^\varepsilon + (w_s^\varepsilon\cdot\nabla)y + \nabla q^\varepsilon = F \ \text{ in } \Omega,$$
$$\operatorname{div} y = 0 \text{ in } \Omega, \quad y = 0 \ \text{ on } \Gamma.$$

- $y_\varepsilon = (\lambda_0 I - A_\varepsilon)^{-1} F$ is solution of

$$\lambda_0 y_\varepsilon - \nu\Delta y_\varepsilon + (y_\varepsilon\cdot\nabla)w_s^\varepsilon + (w_s^\varepsilon\cdot\nabla)y_\varepsilon + \nabla q_\varepsilon = F \ \text{ in } \Omega,$$
$$\operatorname{div} y_\varepsilon + \varepsilon q_\varepsilon = 0 \text{ in } \Omega, \quad y_\varepsilon = 0 \ \text{ on } \Gamma.$$

$\|y - y^\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)}$ can be estimated with regularity results for the Oseen system and with the estimate on $\|w_s - w_s^\varepsilon\|_{H^1(\Omega;\mathbb{R}^3)}$.

Estimate of $\|y_\varepsilon - y^\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)}$

The differences $z_\varepsilon = y_\varepsilon - y^\varepsilon$ and $p_\varepsilon = q_\varepsilon - q$ obey

$$\lambda_0 z_\varepsilon - \nu\Delta z_\varepsilon + (z_\varepsilon \cdot \nabla) w_s^\varepsilon + (w_s^\varepsilon \cdot \nabla) z_\varepsilon + \nabla p_\varepsilon = 0 \ \text{ in } \Omega,$$
$$\operatorname{div} z_\varepsilon + \varepsilon p_\varepsilon = -\varepsilon q \text{ in } \Omega, \quad z_\varepsilon = 0 \ \text{ on } \Gamma.$$

With the adjoint system

$$\lambda_0 \Phi_\varepsilon - \nu\Delta\Phi_\varepsilon + (\nabla w_s^\varepsilon)^T \Phi_\varepsilon - (w_s^\varepsilon \cdot \nabla)\Phi_\varepsilon + \nabla\psi_\varepsilon - \operatorname{div}(w_s^\varepsilon)\Phi_\varepsilon$$
$$= y_\varepsilon - y^\varepsilon \ \text{ in } \Omega,$$
$$\operatorname{div} \Phi_\varepsilon + \varepsilon\psi_\varepsilon = 0 \text{ in } \Omega, \quad \Phi_\varepsilon = 0 \ \text{ on } \Gamma,$$

we obtain

$$\int_\Omega |y_\varepsilon - y^\varepsilon|^2 \mathrm{d}x = \varepsilon \int_\Omega q\psi_\varepsilon \mathrm{d}x$$
$$\leq \varepsilon \|q\|_{L^2(\Omega)} \|\psi_\varepsilon\|_{L^2(\Omega)} \leq C\,\varepsilon \|y_\varepsilon - y^\varepsilon\|_{L^2(\Omega)}.$$

The Oseen system with a boundary control and the penalty method

$$\frac{\partial y}{\partial t} + (w_s \cdot \nabla) y + (y \cdot \nabla) w_s - \nu \, \Delta y + \nabla p = 0 \quad \text{in} \quad Q,$$
$$\operatorname{div} y = 0 \quad \text{in} \quad Q, \quad y(x,t) = \sum_{i=1}^{N_c} f_i(t)\, g_i(x) \quad \text{on} \quad \Sigma,$$
$$y(0) = y_0 \quad \text{in} \quad \Omega,$$

with the control space $U = \mathbb{R}^{N_c}$.

Assumption. $g_i \in H^{3/2}(\Gamma)$, $\int_\Gamma g_i \cdot n \, dx = 0$. $(A + \omega_u I, B)$ is exponentially stabilizable, with $\omega_u > \omega > 0$.

Approximation by the pseudo-compressible model

$$
\begin{aligned}
&\frac{\partial y_\varepsilon}{\partial t} - \nu\Delta y_\varepsilon + (y_\varepsilon\cdot\nabla)w_s^\varepsilon + (w_s^\varepsilon\cdot\nabla)y_\varepsilon + \nabla p_\varepsilon = 0 \ \text{ in } Q,\\
&\operatorname{div} y_\varepsilon + \varepsilon p_\varepsilon = 0 \text{ in } Q,\\
&y_\varepsilon = \textstyle\sum_{i=1}^{N_c} f_i(t)\, g_i \ \text{ on } \Sigma,\\
&y_\varepsilon(0) = y_0 \ \text{ in } \Omega.
\end{aligned}
$$

- $v = Dg$ is solution of

$$
\begin{aligned}
&\lambda_0 v - \nu\Delta v + (v\cdot\nabla)w_s + (w_s\cdot\nabla)v + \nabla q = 0 \ \text{ in } \Omega,\\
&\operatorname{div} v = 0 \text{ in } \Omega, \quad v = g \ \text{ on } \Gamma.
\end{aligned}
$$

- $v_\varepsilon = D_\varepsilon g$ is solution of

$$
\begin{aligned}
&\lambda_0 v_\varepsilon - \nu\Delta v_\varepsilon + (v_\varepsilon\cdot\nabla)w_s^\varepsilon + (w_s^\varepsilon\cdot\nabla)v_\varepsilon + \nabla q_\varepsilon = 0 \ \text{ in } \Omega,\\
&\operatorname{div} v_\varepsilon + \varepsilon q_\varepsilon = 0 \text{ in } \Omega, \quad v_\varepsilon = g \ \text{ on } \Gamma.
\end{aligned}
$$

Difficulty

The Oseen system is a differential algebraic system of the form

$$Py'(t) = APy(t) + Bf, \quad Bf = (\lambda_0 I - A)\sum_{i=1}^{N_c} f_i\, P\, Dg_i,$$
$$(I-P)y(t) = (I-P)\sum_{i=1}^{N_c} f_i(t)\, Dg_i,$$

while the pseudo-compressible Oseen system is of the form

$$y_\varepsilon'(t) = A_\varepsilon y_\varepsilon(t) + B_\varepsilon f, \quad B_\varepsilon f = (\lambda_0 I - A_\varepsilon)\sum_{i=1}^{N_c} f_i\, D_\varepsilon g_i.$$

We have good approximation properties for $A - A_\varepsilon$, for $D - D_\varepsilon$

$$\|Dg - D_\varepsilon g\|_{(L^2(\Omega))^3} \le C\,\varepsilon\,\|g\|_{(H^{1/2}(\Gamma))^3}$$

but not for $P - I$, and thus not for $B - B_\varepsilon$.

Possible remedy

- Compute a feedback for a ROM based on a spectral projection.

Spectral projections in Z

The projector $P_u \in \mathcal{L}(Z, Z_u)$ (and $P_u \in \mathcal{L}(H, Z_u)$) is defined by

$$P_u = \frac{1}{2i\pi} \int_{\Gamma_u} (\lambda I - A)^{-1} P \, d\lambda,$$

Γ_u is a union of Jordan curves, around $(\lambda_j)_{j \in J_u} \cup (\overline{\lambda_j})_{j \in J_u}$.

We split $z' = Az + Bu$ into two systems

$$A_u = A|_{Z_u}, \qquad A_s = A|_{Z_s}, \quad B_u = P_u B, \quad B_s = (I - P_u) B.$$

$$z = z_u + z_s, \quad z_u' = A_u z_u + B_u f, \quad z_s' = A_s z_s + B_s f.$$

There exists $\varepsilon_0 > 0$, such that $\Gamma_u \subset \rho(A_\varepsilon), \ \forall \varepsilon \in (0, \varepsilon_0)$.

We set

$$P_{\varepsilon,u} = \frac{1}{2i\pi} \int_{\Gamma_u} (\lambda I - A_\varepsilon)^{-1} P_\varepsilon \, d\lambda,$$

$$Z_{\varepsilon,u} = P_{\varepsilon,u} Z_\varepsilon \quad \text{and} \quad Z_{\varepsilon,s} = (I - P_{\varepsilon,u}) Z_\varepsilon,$$

$$A_{\varepsilon,u} = A_\varepsilon \big|_{Z_{\varepsilon,u}}, \quad A_{\varepsilon,s} = A_\varepsilon \big|_{Z_{\varepsilon,s}},$$

$$B_{\varepsilon,u} = P_{\varepsilon,u} B_\varepsilon \quad B_{\varepsilon,s} = (I - P_{\varepsilon,u}) B_\varepsilon.$$

We split $z_\varepsilon' = A_\varepsilon z_\varepsilon + B_\varepsilon f$ into two systems

$$z_\varepsilon = z_{\varepsilon,u} + z_{\varepsilon,s}, \quad z_{\varepsilon,u}' = A_{\varepsilon,u} z_{\varepsilon,u} + B_{\varepsilon,u} f, \quad z_{\varepsilon,s}' = A_{\varepsilon,s} z_{\varepsilon,s} + B_{\varepsilon,s} f.$$

We choose $\varepsilon_0 > 0$, and $\exists C > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$:

$$\|P_u - P_{\varepsilon,u}\|_{\mathcal{L}(H)} \leq C\,\varepsilon \quad \text{and} \quad \dim(Z_u) = \dim(Z_{\varepsilon,u}).$$

Estimates in Z_u and $Z_{\varepsilon,u}$

- $B_u f = \sum_{i=1}^{N_c} f_i (\lambda_0 I - A) P_u D\, g_i$,
- $B_{\varepsilon,u} f = \sum_{i=1}^{N_c} f_i (\lambda_0 I - A_\varepsilon) P_{\varepsilon,u} D_\varepsilon\, g_i$.

$$\|(\lambda_0 I - A_u)^{-1} P_u - (\lambda_0 I - A_{\varepsilon,u})^{-1} P_{\varepsilon,u}\|_{\mathcal{L}(H)} \le C\,\varepsilon,$$

$$\|(\lambda_0 I - A_u)^{-1} B_u - (\lambda_0 I - A_{\varepsilon,u})^{-1} B_{\varepsilon,u}\|_{\mathcal{L}(U,H)} \le C\,\varepsilon,$$

$$\|B_u - B_{\varepsilon,u}\|_{\mathcal{L}(U,H)} \le C\,\varepsilon, \qquad \forall \varepsilon \in (0, \varepsilon_0).$$

Uniform bound for $B_{\varepsilon,u} \in \mathcal{L}(U,H)$

$$\sup_{\varepsilon \in (0,\varepsilon_0)} \|B_{\varepsilon,u}\|_{\mathcal{L}(U,H)} < +\infty.$$

Uniform stabilizability of $(A_{\varepsilon,u} + \omega_u P_{\varepsilon,u}, B_{\varepsilon,u})$

Assumption: Stabilizability and detectability conditions.

$$(A_u + \omega_u P_u, B_u) \quad \text{is stabilizable in } Z_u.$$

$$\text{Either } (A_u + \omega_u P_u, \mathcal{C}|_{Z_u}) \text{ is detectable or } \mathcal{C} = 0.$$

We choose $\varepsilon_0 > 0$ such that

$$\mathrm{Re}\,\sigma(A_{\varepsilon,u}) > -\omega_u \quad \text{and} \quad \mathrm{Re}\,\sigma(A_{\varepsilon,s}) < -\omega, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

$$(A_{\varepsilon,u} + \omega_u P_{\varepsilon,u}, B_{\varepsilon,u}) \text{ is unif. stabilizable}$$

$$(A_{\varepsilon,u} + \omega_u P_{\varepsilon,u}, \mathcal{C}\big|_{Z_{\varepsilon,u}}) \text{ is unif. detectable.}$$

$$\Pi_u \in \mathcal{L}(Z_u, Z_u^*), \quad \Pi_u = \Pi_u^* \geq 0, \quad C_u = \mathcal{C}|_{Z_u},$$

$$\Pi_u(A_u + \omega_u P_u) + (A_u^* + \omega_u P_u^*)\Pi_u - \Pi_u B_u B_u^* \Pi_u + C_u^* C_u = 0,$$

$$A_u + \omega_u P_u - B_u B_u^* \Pi_u \quad \text{is exponentially stable in } Z_u.$$

If $K_u = -B_u^* \Pi_u$, then

$$\|e^{t(A+BK_u)}\|_{\mathcal{L}(H)} \leq Ce^{-t\omega}.$$

$$\Pi_{\varepsilon,u} \in \mathcal{L}(Z_{\varepsilon,u}, Z^*_{\varepsilon,u}), \quad \Pi_{\varepsilon,u} = P^*_{\varepsilon,u}, \quad C_{\varepsilon,u} = \mathcal{C}\big|_{Z_{\varepsilon,u}},$$

$$\Pi_{\varepsilon,u}(A_{\varepsilon,u} + \omega_u P_{\varepsilon,u}) + (A^*_{\varepsilon,u} + \omega_u P^*_{\varepsilon,u})\Pi_{\varepsilon,u} - \Pi_{\varepsilon,u} B_{\varepsilon,u} B^*_{\varepsilon,u}\Pi_{\varepsilon,u} + C^*_{\varepsilon,u} C_{\varepsilon,u} = 0,$$

$$A_{\varepsilon,u} + \omega_u P_{\varepsilon,u} - B_{\varepsilon,u} B^*_{\varepsilon,u}\Pi_{\varepsilon,u} \quad \text{is exponentially stable in } Z_{\varepsilon,u}.$$

The solutions Π_u and $\Pi_{\varepsilon,u}$, and the feedbacks $K_u = -B^*_u \Pi_u P_u$ and $K_{\varepsilon,u} = -B^*_{\varepsilon,u}\Pi_{\varepsilon,u}P_{\varepsilon,u}$ obey

$$\|\Pi_u P_u - \Pi_{\varepsilon,u}P_{\varepsilon,u}\|_{\mathcal{L}(H)} \leq C\,\varepsilon,$$

and

$$\|K_u - K_{\varepsilon,u}\|_{\mathcal{L}(H,U)} \leq C\,\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

Convergence rates for the closed-loop systems

- $Py(t) = e^{(A+BK)t}y_0, \quad K = -B_u^*\Pi_u P_u$, f is the control.

  $(I-P)y(t) = \sum_{i=1}^{N_c} K_i y(t)(I-P)Dg_i$.
- $Py^\varepsilon(t) = e^{(A+BK_\varepsilon)t}y_0, \quad K_\varepsilon = -B_{\varepsilon,u}^*\Pi_{\varepsilon,u}P_{\varepsilon,u}$, f^ε is the control.

  $(I-P)y^\varepsilon(t) = \sum_{i=1}^{N_c} K_{\varepsilon,i} y^\varepsilon(t)(I-P)Dg_i$.
- $y_\varepsilon(t) = e^{(A_\varepsilon+B_\varepsilon K_\varepsilon)t}y_0, \quad K_\varepsilon = -B_{\varepsilon,u}^*\Pi_{\varepsilon,u}P_{\varepsilon,u}$, f_ε is the control.

For all $\varepsilon \in (0, \varepsilon_0)$, we have

$$\|f_\varepsilon(t) - f(t)\|_U \leq C\,\frac{e^{(-\omega+\varrho\varepsilon)t}}{t}\,\varepsilon\|y_0\|_H,$$

$$\|f_\varepsilon - f\|_{L^p(0,\infty;U)} \leq C_p\,\varepsilon^{1/p}\|y_0\|_H, \quad \forall p \in (1,\infty),$$

$$\|f_\varepsilon - f^\varepsilon\|_{L^p(0,\infty;U)} \leq C_p\,\varepsilon^{1/p}\|y_0\|_H, \quad \forall p \in (1,\infty),$$

where $\omega > 0$ is the a priori prescribed decay rate.

- Convergence rates for the projections of the solutions of closed-loop systems

$$\|P_{\varepsilon,u}y_\varepsilon(t) - P_u y(t)\|_H \leq C\,\frac{e^{(-\omega+\varrho\varepsilon)t}}{t}\,\varepsilon\|y_0\|_H,$$

$$\|P_{\varepsilon,u}y_\varepsilon - P_u y\|_{L^p(0,\infty;H)} \leq C_p\,\varepsilon^{1/p}\|y_0\|_H, \quad \forall p \in (1,\infty),$$

$$\|P_{\varepsilon,u}y_\varepsilon - P_u y^\varepsilon\|_{L^p(0,\infty;H)} \leq C_p\,\varepsilon^{1/p}\|y_0\|_H, \quad \forall p \in (1,\infty).$$

- We could obtain convergence rates between y_ε and y, on compact time intervals $[0, T]$, if we took a dynamic controller and if $y_0 \in V^0_n(\Omega) \cap (H^1_0(\Omega))^3$.

- This framework can be used for many other parabolic systems: Oseen with mixed B.C., Boussinesq, Fluid-structure interaction system, penalty methods like the pseudo-compressibility method of the Robin penalized B.C. to approximate Dirichlet B.C.

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Thank you for your attention