ICTS - Bangalore

Recent advances on control theory of PDE systems

February 12-23, 2024

Numerical approximation of the Oseen system in polyhedral/polygonal domains

Lecture 2

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Outline of the lecture

- The Oseen system with NHBC. A, D, P and B.
- The approximate Oseen system with NHBC. A_h, D_h, P_h and B_h.
- To do list
- Approximation of A and D. Known/new $H^1 \times L_0^2$-estimates.
- New regularity results for the Oseen system with NHBC.
- Known/new $L^2 \times \mathcal{H}^{-1}$-estimates for regular data.
- Known/new $L^2 \times \mathcal{H}^{-1}$-estimates for irregular data.
- Additional results.
- Conclusion and perspectives

Issues for the Navier-Stokes equations

- Ω is a bounded polyhedral convex/non-convex domain in $\mathbb{R}^3$.
- $(w_s, \rho_s) \in H^1(\Omega; \mathbb{R}^3) \times L^2_0(\Omega)$ is a stationary solution of

$$(w_s \cdot \nabla) w_s - \nu \Delta w_s + \nabla \rho_s = f_s, \quad \operatorname{div} v_s = 0 \quad \text{in} \quad \Omega,$$
$$w_s = g_s \quad \text{on} \quad \Gamma = \partial\Omega.$$

- The controlled Navier-Stokes system

$$\frac{\partial w}{\partial t} + (w \cdot \nabla) w - \nu \Delta w + \nabla \rho = f_s, \quad \text{in} \quad Q = \Omega \times (0, \infty),$$
$$\operatorname{div} w = 0 \quad \text{in} \quad Q, \quad w = g_s + g \quad \text{on} \quad \Sigma = \Gamma \times (0, \infty),$$
$$w(0) = w_0 = w_s + y_0 \quad \text{in} \quad \Omega.$$

The control g with $\text{supp} g \subset \Gamma_c$ is either in a infinite dimensional space or of the form $g(x,t) = \sum_{i=1}^{N_c} f_i(t)\, g_i(x)$.

The Oseen system

The nonlinear system satisfied by $(y, p) = (w, \rho) - (w_s, \rho_s)$ is

$$\frac{\partial y}{\partial t} + (w_s \cdot \nabla) y + (y \cdot \nabla) w_s + \kappa (y \cdot \nabla) y - \nu \Delta y + \nabla p = 0 \quad \text{in} \quad Q,$$
$$\operatorname{div} y = 0 \quad \text{in} \quad Q,$$
$$y = g \quad \text{on} \quad \Sigma,$$
$$y(0) = y_0 \quad \text{in} \quad \Omega,$$

with $\kappa = 1$. The associated linearized system is obtained by setting $\kappa = 0$.

- The stationary Oseen system (for $\lambda_0 > 0$ large enough)

$$\lambda_0 y + (w_s \cdot \nabla) y + (y \cdot \nabla) w_s - \nu \Delta y + \nabla p = F \quad \text{in} \quad \Omega,$$

$$\operatorname{div} y = 0 \quad \text{in} \quad \Omega,$$

$$y = g \quad \text{on} \quad \Gamma = \partial\Omega.$$

The Leray projector P: $L^2(\Omega;\mathbb{R}^3) \longmapsto V^0_n(\Omega)$

$$V^0_n(\Omega) = \{y \in L^2(\Omega;\mathbb{R}^3) \mid \operatorname{div} y = 0,\ y \cdot n = 0 \text{ on } \Gamma\}.$$

$$L^2(\Omega;\mathbb{R}^3) = V^0_n(\Omega) \oplus \nabla H^1(\Omega).$$

Variational formulation of the PDE system

$$a(y,\zeta) = \int_\Omega \big(\nu\nabla y : \nabla\zeta + (w_s\cdot\nabla)y\cdot\zeta + (y\cdot\nabla)w_s\cdot\zeta\big)\,dx,$$

$$b(\zeta,q) = -\int_\Omega q\,\mathrm{div}\,\zeta\,dx,$$

For $\lambda_0 > 0$ large enough:

$$\lambda_0(y,\zeta)_\Omega + a(y,\zeta) + b(\zeta,p) = (F,\zeta)_\Omega \quad \text{for all } \zeta\in H_0^1(\Omega;\mathbb{R}^3),$$

$$b(y,\rho) = 0 \quad \text{for all } \rho\in L_0^2(\Omega),$$

$$y = g, \quad \text{on} \quad \Gamma.$$

$$\mathcal{D}(A) = \{y\in V_n^0(\Omega)\cap H_0^1(\Omega;\mathbb{R}^3) \mid |a(y,\zeta)| \le C\|\zeta\|_{L^2},\ \forall\zeta\in V_n^0(\Omega)\},$$

$$\big(-Ay,\zeta\big)_{L^2} = a(y,\zeta) \quad \text{for all } (y,\zeta)\in(\mathcal{D}(A))^2.$$

Coercivity condition

- We assume that $\lambda_0 > 0$ is large enough so that

$$\lambda_0(y, y) + a(y, y) \geq \frac{\nu}{2}\|y\|^2_{H^1(\Omega;\mathbb{R}^3)}.$$

- Inf-Sup condition

$$\sup_{z \in H^1_0(\Omega;\mathbb{R}^3)} \frac{b(z, p)}{\|z\|_{H^1(\Omega;\mathbb{R}^3)}} \geq \beta\|p\|_{L^2_0(\Omega)} \quad \text{for all } p \in L^2_0(\Omega), \quad \beta > 0.$$

- We set

$$V^s(\Gamma) = \{g \in H^s(\Gamma; \mathbb{R}^3) \mid \int_\Gamma g \cdot n \, dx = 0\}, \quad s \geq 0.$$

Existence of variational solutions

For all $F \in H^{-1}(\Omega; \mathbb{R}^3)$, all $g \in V^{1/2}(\Gamma)$, the Oseen system admits a unique variational solution $(y, p) \in H^1(\Omega; \mathbb{R}^3) \times L^2_0(\Omega)$, and

$$\|y\|_{H^1(\Omega;\mathbb{R}^3)} + \|p\|_{L^2(\Omega)} \leq C\left(\|F\|_{H^{-1}(\Omega;\mathbb{R}^3)} + \|g\|_{V^{1/2}(\Gamma)}\right).$$

The control operator B

The Dirichlet operator D

$$Dg = y, \quad \text{where } y \text{ is solution to}$$

$$\lambda_0 y + (w_s \cdot \nabla) y + (y \cdot \nabla) w_s - \nu \Delta y + \nabla p = 0 \quad \text{in} \quad \Omega,$$

$$\operatorname{div} y = 0 \quad \text{in} \quad \Omega,$$

$$y = g \quad \text{on} \quad \Gamma,$$

when $g \in V^{1/2}(\Gamma)$. If $g \in V^0(\Gamma)$, D is defined by the transposition method.

The control operator B is defined by

$$(B \in \mathcal{L}(V^0(\Gamma), (\mathcal{D}(A^*))')): \; Bg = (\lambda_0 I - A) PD\, g.$$

and $(\lambda I - A)^{-\gamma} B \in \mathcal{L}(V^0(\Gamma), V_n^0(\Omega))$ for some $\gamma \in (0, 1)$.

Variational formulation versus Operator formulation

$(y, q) \in H^1(\Omega; \mathbb{R}^3) \times \in L^2(\Omega)$ is a solution to the variational Oseen system

if and only if

$$(\lambda_0 I - A)Py = Bg + PF,$$
$$(I - P)y = (I - P)Dg,$$
$$q = \cdots .$$

The solution (y, q) to the instationary controlled Oseen system satisfies

$$Py' = APy + Bg,$$
$$(I - P)y(t) = (I - P)Dg(t), \quad \forall t \geq 0,$$
$$q = \cdots .$$

Thus, we are in the functional setting of Lecture 1.

Explanation for Stokes - A_0

$$D_0\, g = y, \quad \text{where } y \text{ is solution to}$$

$$-\nu\,\Delta y + \nabla p = 0 \quad \text{in} \quad \Omega, \quad \operatorname{div} y = 0 \quad \text{in} \quad \Omega,$$

$$y = g \quad \text{on} \quad \Gamma.$$

For $\phi \in \mathcal{D}(A_0)$, $\nabla\psi = (I - P)\nu\Delta\phi$, that means

$A_0\phi = -\nabla\psi + \nu\Delta\phi$, and $B_0 = (-A_0)PD_0$, we have

$$B_0^*\phi = D_0^* P(-A_0^*)\phi = D_0^* F = -\nu\,\partial_n\phi + (\psi - c(\psi))\, n.$$

where $\phi \in \mathcal{D}(A_0)$, $-\nabla\psi + \nu\Delta\phi = F$.
Next

$$0 = \int_\Omega (-\nu\,\Delta y + \nabla p)\,\phi\, dx = \int_\Omega y\, A_0\phi\, dx + \int_\Gamma g\, B_0^*\phi,$$

which gives

$$A_0\, y + B_0\, g = 0.$$

Approximation spaces

Taylor-Hood finite element method or (P1-Bubble, P1) finite element method, with quasi-uniform families of triangulations.

- $X_h \subset H^1(\Omega;\mathbb{R}^3)$, $X_h^0 = X_h \cap H_0^1(\Omega;\mathbb{R}^3)$, velocity
- $M_h \subset L_0^2(\Omega)$, pressure
- $S_h = X_h|_\Gamma$, trace
- π_h is the orthogonal projection in $L^2(\Gamma;\mathbb{R}^3)$ onto S_h, • $Z_h = \{z \in X_h \mid b(q,z) = 0 \text{ for all } q \in M_h\}$,

and $Z_h^0 = Z_h \cap X_h^0$. velocity

- We have a nonconforming approximation: $Z_h \not\subset V_n^0(\Omega)$.

$$a_h(y,\zeta) = \int_\Omega \Big(\nu\nabla y : \nabla\zeta + (w_s^h\cdot\nabla)y\cdot\zeta + (y\cdot\nabla)w_s^h\cdot\zeta\Big)\, dx,$$

Approximate Oseen system

$$\lambda_0(y_h,\zeta) + a_h(y_h,\zeta) + b(\zeta,q_h) = (F,\zeta) \quad \text{for all } \zeta\in X_0^h,$$

$$b(y_h,\rho) = 0 \quad \text{for all } \rho\in M_h, \quad y_h = \pi_h g \quad \text{on} \quad \Gamma.$$

Uniform coercivity : $\exists\lambda_0 > 0$, large enough, such that

$$\lambda_0\big(z,z\big)_{L^2(\Omega;\mathbb{R}^3)} + a(z,z) \ge \frac{\nu}{2}\|z\|^2_{H^1(\Omega;\mathbb{R}^3)}, \quad \forall z\in H^1(\Omega;\mathbb{R}^3),$$

and, for all $h\in(0,h_0)$,

$$\lambda_0\big(z,z\big)_{L^2(\Omega;\mathbb{R}^3)} + a_h(z,z) \ge \frac{\nu}{2}\|z\|^2_{H^1(\Omega;\mathbb{R}^3)}, \quad \forall z\in H^1(\Omega;\mathbb{R}^3),$$

and all w_s^h such that $\|w_s^h\|_{H^1} \le \|w_s\|_{H^1} + 1$.

Uniform Inf-Sup conditions

There exists $\beta > 0$ such that

- $$\sup_{z_h \in X_h^0} \frac{b(z_h, p_h)}{\|z_h\|_{H^1(\Omega;\mathbb{R}^3)}} \geq \beta \|p_h\|_{L_0^2(\Omega)} \quad \text{for all } p_h \in M_h, \text{ and all } h > 0,$$

(B.B.L. condition, Girault-Raviart, 2006, Bercovier-Pironneau, 1979) and

- $$\sup_{z_h \in X_h^0} \frac{\int_\Omega z_h \cdot \nabla p_h \, dx}{\|z_h\|_{L^2(\Omega;\mathbb{R}^3)}} \geq \beta \|\nabla p_h\|_{L^2(\Omega;\mathbb{R}^3)} \quad \text{for all } p_h \in M_h, \; h > 0.$$

(Bercovier-Pironneau, 1979)

For all $F \in H^{-1}(\Omega; \mathbb{R}^3)$, all $g \in V^{1/2}(\Gamma)$, and for all $h \in (0, h_0)$, the approximate Oseen system admits a unique variational solution $(y_h, p_h) \in X_h \times M_h$, and

$$\|y_h\|_{H^1(\Omega;\mathbb{R}^3)} + \|p_h\|_{L^2(\Omega)} \leq C\left(\|F\|_{H^{-1}(\Omega;\mathbb{R}^3)} + \|g\|_{V^{1/2}(\Gamma)}\right).$$

$y_h \in X_h$, $q_h \in M_h$ is a weak solution to the approximate variational Oseen system if and only if

$$(\lambda_0 I - A_h)P_h y_h + B_h g = P_h F,$$

$$(I - P_h)y_h = (I - P_h)D_h \pi_h g,$$

$$q_h = \cdots,$$

where $D_h \pi_h g = y_h$ when $F = 0$.

To do list

- Uniform analyticity for A and A_h.
- Estimate $(\lambda_0 I - A)^{-1}P - (\lambda_0 I - A_h)^{-1}P_h$.
- Estimate $(\lambda_0 I - A)^{-1}B - (\lambda_0 I - A_h)^{-1}B_h$.
- For convex domains $(\lambda_0 I - A_h)^{-\gamma}B_h$ is uniformly bounded in $\mathcal{L}(V^0(\Gamma), Z_h)$. For non-convex domain, we prove that, for some $\overline{\gamma} \in [\gamma, 1)$,

$$\|e^{A_h t}B_h\|_{\mathcal{L}(V^0(\Gamma),Z_h)} \leq C\,\frac{e^{t\omega_0}}{t^{\overline{\gamma}}}, \quad \forall t \in (0, h^{r/(1-\gamma)}), \quad \forall h \in (0, h_0).$$

- To study Oseen, we need estimates for $w_s - w_s^h$.

With these conditions, we prove the uniform stabilizability of the family $(A_h, B_h)_{0<h<h_0}$.

Next, we prove convergence rates for Riccati based feedbacks.

Uniform analyticity

Due to the coercivity of a and the uniform coercivity of a_h:

- There exists a sector

$$\mathbb{S}_{\pi/2+\delta} = \{\lambda \in \mathbb{C} \mid |\arg(\lambda)| < \pi/2 + \delta\},$$

$(\omega_0, \delta) \in (0, \lambda_0) \times]0, \pi/2[$ such that:

$$\{\omega_0\} + \mathbb{S}_{\pi/2+\delta} \subset \rho(A),$$

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(Z)} \leq \frac{C}{|\lambda - \omega_0|} \quad \text{for all } \lambda \in \{\omega_0\} + \mathbb{S}_{\pi/2+\delta},$$

and

$$\{\omega_0\} + \mathbb{S}_{\pi/2+\delta} \subset \rho(A_h),$$

$$\|(\lambda I - A_h)^{-1}\|_{\mathcal{L}(Z_h)} \leq \frac{C}{|\lambda - \omega_0|} \quad \text{for all } \lambda \in \{\omega_0\} + \mathbb{S}_{\pi/2+\delta}.$$

Approximation of A

- $(\lambda_0 I - A)^{-1} P F = y$

where $(y, p) \in H^1_0(\Omega; \mathbb{R}^3) \times L^2_0(\Omega)$ is solution to

$$\lambda_0 \big(y, w\big) + a(y, w) + b(w, p) = \langle F, w \rangle_{H^{-1}, H^1_0} \quad \text{for all } w \in H^1_0(\Omega; \mathbb{R}^3),$$

$$b(y, \rho) = 0 \quad \text{for all } \rho \in L^2_0(\Omega).$$

- $(\lambda_0 I - A_h)^{-1} P_h F = y_h$

where $(y_h, p_h) \in Z^0_h \times M_h$ is solution to

$$\lambda_0 \big(y_h, w_h\big) + a_h(y_h, w_h) + b(w_h, p_h) = \big(F, w_h\big)_{L^2(\Omega; \mathbb{R}^3)} \quad \text{for all } w_h \in X^0_h,$$

$$b(y_h, \rho_h) = 0 \quad \text{for all } \rho_h \in M_h.$$

We have (Girault-Raviart, 2006, convex polyhedral domains)

$$\|(\lambda_0 I - A)^{-1} P - (\lambda_0 I - A_h)^{-1} P_h\|_{\mathcal{L}(H)} \leq C\, h^2.$$

Gunzburger-Hou-92, Girault-Raviart-86 and Girault-Raviart-06

If (y, p) is the solution to the Stokes system and (y_h, p_h) is the solution to the approximate Stokes system, we have

$$\|y - y_h\|_{H^1} + \|p - p_h\|_{L^2}$$

$$\leq C \left(\inf_{\zeta_h \in X_h} \|y - \zeta_h\|_{H^1} + \inf_{\rho_h \in M_h} \|p - \rho_h\|_{L^2} + \|g - \pi_h g\|_{V^{1/2}(\Gamma)}\right)$$

$$\leq C\, h^s (\|y\|_{H^{s+1}} + \|p\|_{\mathcal{H}^s}).$$

This last result come from

$$\|y - r_h y\|_{H^m} \leq C h^{s+1-m} \|y\|_{H^{s+1}}, \quad m = 0 \text{ or } 1,$$

where $r_h \in \mathcal{L}(H^\ell(\Omega); X_h)$, for $\ell > 1/2$, is an interploation operator.

L^2 estimates for the approximation of Stokes system

Gunzburger-Hou-92

For the L^2 estimate in a convex polyhedron, we have

$$\|y - y_h\|_{L^2}$$
$$\leq C \left(h\|y - y_h\|_{H^1} + h\|p - p_h\|_{L^2} + \sup_{v \in H^{1/2-\epsilon}(\Gamma)} \frac{(g - \pi_h g)_\Gamma}{\|v\|_{H^{1/2-\epsilon}}} \right)$$
$$\leq C\, h^{1-\epsilon} \left(\|y\|_{H^1} + \|p\|_{L^2} \right), \quad \forall \epsilon \in (0, 1/2).$$

Old/New ingredients

- To obtain $H^1 \times L^2$ estimates for Stokes/Oseen in non-convex domain, we need new regularity results for Stokes with non-homogeneous B.C.

- To obtain new regularity results for Stokes/Oseen with NHB.C., we use Dauge-89 for Stokes in non-convex domain with homogeneous B.C. and some sharp results for lifting NHB.C.

- To obtain $L^2 \times \mathcal{H}^{-1}$ estimates for Stokes/Oseen in convex/non-convex domain, in the case of regular B.C., we use a variant of the Aubin-Nitsche argument (with NHB.C., $\neq$ from G-H-92).

- To obtain $L^2 \times \mathcal{H}^{-1}$ estimates for Stokes/Oseen in convex/non-convex, in the case of irregular B.C., we use the transposition method and the approximation of the stress tensor of the adjoint state.

Old/New ingredients

- To prove the existence of (w_s, ρ_s) in $H^1 \times L^2$, we use Temam or Girault-Raviart. To prove regularity results for (w_s, ρ_s), we use our new regularity results for Stokes with non-homogeneous B.C.

- To prove estimates for $(w_s - w_s^h, \rho_s - \rho_s^h)$ in $H^1 \times L^2$, we use Girault-Raviart (or Casas-Mateos-R-07), and our new regularity results for Stokes with NHB.C.

Dauge-Sima-89

$\mathbb{E}_d = \mathbb{E}_3$ is the union of edges of Γ.

$$\mathcal{H}^\ell(\Omega) = H^\ell(\Omega) \cap L^2_0(\Omega), \quad \mathcal{H}^{-\ell}(\Omega) = (\mathcal{H}^\ell(\Omega))', \quad \ell \geq 0,$$

$$\mathcal{H}^{1+\ell}_{\mathbb{E}_d}(\Omega) = \{h \in \mathcal{H}^{1+\ell}(\Omega) \mid h|_{\mathbb{E}_d} = 0\}, \quad \text{for } \ell > 0,$$

$$\mathcal{H}^1_{00,\mathbb{E}_d}(\Omega) = \{h \in \mathcal{H}^1(\Omega) \mid (\mathrm{dist}(\cdot, \mathbb{E}_d))^{-1} h(\cdot) \in L^2(\Omega)\},$$

$$= [H^{1-\ell}(\Omega), \mathcal{H}^{1+\ell}_{\mathbb{E}_d}(\Omega)]_{1/2}, \quad \ell \in (0, 1/2).$$

$$\|h\|_{\mathcal{H}^1_{00,\mathbb{E}_d}(\Omega)} = \left(\|h\|^2_{\mathcal{H}^1(\Omega)} + \|(\mathrm{dist}(\cdot, \mathbb{E}_d))^{-1} h(\cdot)\|^2_{L^2(\Omega)} \right)^{1/2}.$$

$$y \in H^1_0(\Omega; \mathbb{R}^3), \quad -\nu\, \Delta y + \nabla q = F, \quad \mathrm{div}\, y = \mathrm{div}\, h \quad \text{in} \quad \Omega.$$

There exists $\alpha^* \in (0,1)$, depending on $\Omega \subset \mathbb{R}^d$, such that $\alpha^* \in (0,1/2)$ if Ω is non-convex, $\alpha^* \in (1/2,1)$ if Ω is convex, and for which the following regularity results are satisfied.

(i) If Ω is non-convex, for all $\alpha_0 \in (0,\alpha^*)$, we have

$$\|y\|_{H^{3/2+\alpha_0}} + \|q\|_{\mathcal{H}^{1/2+\alpha_0}} \leq C\,(\|F\|_{H^{-1/2+\alpha_0}(\Omega;\mathbb{R}^3)} + \|h\|_{\mathcal{H}^{1/2+\alpha_0}(\Omega)}).$$

(ii) If Ω is convex, for all $\alpha_0 \in (1/2,\alpha^*)$, we have

$$\|y\|_{H^{3/2+\alpha_0}} + \|q\|_{\mathcal{H}^{1/2+\alpha_0}} \leq C\,(\|F\|_{H^{-1/2+\alpha_0}(\Omega;\mathbb{R}^3)} + \|h\|_{\mathcal{H}^{1/2+\alpha_0}_{\mathbb{E}_d}(\Omega)}),$$

$$\forall \alpha_0 \in (0,1/2),$$

$$\|y\|_{H^2} + \|q\|_{\mathcal{H}^1} \leq C\,(\|F\|_{L^2(\Omega;\mathbb{R}^3)} + \|h\|_{\mathcal{H}^1_{00,\mathbb{E}_d}(\Omega)}).$$

$$H^{\ell-1/2}(\Gamma) = \gamma_0 H^{\ell}(\Omega), \quad \forall \ell \in (1/2, 5/2).$$

There exists $L \in \mathcal{L}(H^{\ell-1/2}(\Gamma), H^{\ell}(\Omega))$ such that

$$\gamma_0 L g = g, \quad \forall g \in H^{\ell-1/2}(\Gamma).$$

If $g \in H^{\ell-1/2}(\Gamma)$, then $g|_{\Gamma_i} \in H^{\ell-1/2}(\Gamma_i)$ for all face Γ_i of Γ.

If $1/2 < \ell < 1$, no additional condition.

If $\ell = 1$, an integral condition is needed at the edges.

If $1 < \ell < 3/2$, equalities of the traces at the edges

$$\gamma_{i,j}(z|_{\Gamma_i}) = \gamma_{j,i}(z|_{\Gamma_j}).$$

If $\ell = 3/2$, equalities of the traces at the edges.

If $3/2 < \ell < 5/2$, equalities of the traces at the edges and the corners.

Traces of divergence free functions

$$V^\ell(\Gamma) = \{g \in H^\ell(\Gamma;\mathbb{R}^3) \mid \int_\Gamma g \cdot n\,dx = 0\}, \quad \ell \geq 0,$$

$$\dot{V}^{3/2+\ell}(\Gamma) = \{g \in V^{3/2+\ell}(\Gamma) \mid \operatorname{div} Lg = 0 \quad \text{on } \mathbb{E}_3\}$$

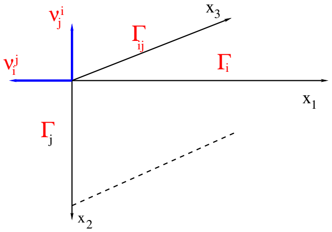
The condition $\operatorname{div} Lg = 0 \quad \text{on } \mathbb{E}_3$ is independent of L.

$$\operatorname{div} Lg = 0 \quad \text{on } \Gamma_{i,j} \;\Leftrightarrow\; \sin(\omega_{i,j})\partial_{\sigma_i^j} g_{\sigma_i^j} + \partial_{\nu_i^j} g^i_{n_j} + \partial_{\nu_j^i} g^j_{n_i} = 0,$$

$$\dot{V}^{3/2}(\Gamma) = [V^{3/2-\ell}(\Gamma), \dot{V}^{3/2+\ell}(\Gamma)]_{1/2}, \quad \ell \in (0,1/2)$$

$$= \{g \in V^{3/2}(\Gamma) \mid \operatorname{div} Lg \in \mathcal{H}^1_{00,\mathbb{E}_d}(\Omega)\}.$$

div $Lg = 0$ on $\Gamma_{i,j}$



$\omega_{i,j}$ is the angle between Γ_i and Γ_j,

$(n_i, \nu_i^j, \sigma_i^j)$ is a direct orthonormal basis,

n_i is the normal to Γ_i, exterior to Ω,

ν_i^j is the normal to $\Gamma_{i,j}$, parallel to Γ_i, exterior to Γ_i,

$(g_{n_i}, g_{\nu_i^j}, g_{\sigma_i^j})$ is the coordinate vector of g in the basis $(n_i, \nu_i^j, \sigma_i^j)$,

$g_{n_i}^j$ is the restriction of g_{n_i} to Γ_j.

$$\sin(\omega_{i,j})\partial_{\sigma_i^j} g_{\sigma_i^j} + \partial_{\nu_i^j} g^i_{n_j} + \partial_{\nu_j^i} g^j_{n_i} = 0.$$

is a condition which is expressed only in terms of g on Γ, while

$$\operatorname{div} Lg = 0 \quad \text{on } \Gamma_{i,j},$$

is a condition on Lg defined on Ω.

In the case of the above figure the condition is

$$\partial_{x_3} g_3 + \partial_{x_1} g^i_1 + \partial_{x_2} g^j_2 = 0 \quad \text{on } \Gamma_{i,j}.$$

$$-\nu\,\Delta y + \nabla q = F, \quad \operatorname{div} y = 0 \quad \text{in} \quad \Omega, \quad y = g \quad \text{on} \quad \Gamma.$$

(i) If Ω is non-convex, for all $\alpha_0 \in (0, \alpha^*)$, we have

$$\|y\|_{H^{3/2+\alpha_0}} + \|p\|_{\mathcal{H}^{1/2+\alpha_0}} \leq C_{\alpha_0}(\|F\|_{H^{\alpha_0-1/2}} + \|g\|_{V^{1+\alpha_0}}).$$

(ii) If Ω is convex, we have

$$\|y\|_{H^2} + \|p\|_{\mathcal{H}^1} \leq C\,(\|F\|_{L^2} + \|g\|_{\dot{V}^{3/2}}).$$

Moreover

$$\dot{V}^{3/2}(\Gamma) = \gamma_0\left(\{v \in H^2(\Omega) \mid \operatorname{div} v = 0\}\right) = \gamma_0(V^2(\Omega)).$$

If Ω is non-convex, we have

$$\|y - y_h\|_{H^1} + \|p - p_h\|_{L^2} \leq C_{\alpha_0}\, h^{1/2+\alpha_0}(\|F\|_{H^{\alpha_0-1/2}} + \|g\|_{V^{1+\alpha_0}}),$$
$\forall \alpha_0 \in (0, \alpha^*)$.

If Ω is convex, we have

$$\|y - y_h\|_{H^1} + \|p - p_h\|_{L^2} \leq C\, h\,(\|F\|_{L^2} + \|g\|_{\dot{V}^{3/2}}).$$

We mainly use G-H + new regularity results.

If Ω is non-convex, for all $\alpha_0 \in (0, \alpha^*)$, we have

$$\|y - y_h\|_{L^2} + \|p - p_h\|_{\mathcal{H}^{-1}} \leq C_{\alpha_0}\, h^{1+2\alpha_0}\big(\|F\|_{H^{\alpha_0-1/2}} + \|g\|_{V^{1+\alpha_0}}\big),$$

If Ω is convex, for all $\alpha_0 \in (1/2, \alpha^*)$, we have

$$\|y - y_h\|_{L^2} + \|p - p_h\|_{(\mathcal{H}^1_{00,\mathbb{E}_d})'} \leq C_{\alpha_0}\, h^{1+2\alpha_0}\big(\|F\|_{H^{\alpha_0-1/2}} + \|g\|_{\dot{V}^{1+\alpha_0}}\big),$$

If Ω is convex, for all $\epsilon \in (0, 1/2)$, we have

$$\|y - y_h\|_{H^\epsilon} + \|p - p_h\|_{\mathcal{H}^{-1+\epsilon}} \leq C\, h^{2-\epsilon}\,\big(\|F\|_{L^2} + \|g\|_{\dot{V}^{3/2}}\big).$$

For all $g \in V^{3/2}$, compactly supported in $\Gamma \setminus \mathbb{E}_3$, we have

$$\|y - y_h\|_{L^2} + \|p - p_h\|_{(\mathcal{H}^1_{00,\mathbb{E}_d})'} \leq C_{\bar\delta}\, h^2\big(\|F\|_{L^2} + \|g\|_{V^{3/2}}\big),$$

where $\bar\delta = \operatorname{dist}(\operatorname{supp} g, \mathbb{E}_3)$.

We choose $F = 0$.

If Ω is non-convex, we have

$$\|y - y_h\|_{L^2} + \|p - p_h\|_{\mathcal{H}^{-1}} \leq C_{\alpha_0}\, h^{\alpha_0}\|g\|_{V^0}, \quad \forall \alpha_0 \in (0, \alpha^*).$$

If Ω is convex, we have

$$\|y - y_h\|_{H^\epsilon} + \|p - p_h\|_{\mathcal{H}^{-1+\epsilon}} \leq C_\epsilon\, h^{1/2-\epsilon}\|g\|_{V^0}, \quad \forall \epsilon \in (0, 1/2).$$

If Ω is convex, we have

$$\|y - y_h\|_{L^2} + \|p - p_h\|_{(\mathcal{H}_0^1)'} \leq C\, h^{1/2}\|g\|_{V^0},$$

$\forall g \in L^2(\Gamma)$, such that $g \cdot n = 0$, and

$$\|y - y_h\|_{L^2} + \|p - p_h\|_{(\mathcal{H}_0^1)'} \leq C_{\bar\delta}\, h^{1/2}\|g\|_{V^0},$$

$\forall g \in L^2(\Gamma)$, such that $0 < \bar\delta \leq \mathrm{dist}(\mathrm{supp}\, g, \mathbb{E}_3)$.

Aubin-Nitsche argument for regular data

(y, p) the solution to Stokes with (F, g) as RHS.

(y_h, p_h) the solution to approx-Stokes with $(F, \pi_h g)$ as RHS.

(ϕ, ψ) the solution to (adjoint)-Stokes with (ξ, f) as RHS with H.B.C. ($\mathrm{div}\phi = f$).

$$
\begin{aligned}
&(\xi, y - y_h)_\Omega + \langle f, p - p_h \rangle \\
&= a(y - y_h, \phi - r_h\phi) + b(\phi - r_h\phi, p - p_h) \\
&\qquad + b(y - y_h, \psi - s_h\psi_h) - \langle \mathbf{t}, g - \pi_h g \rangle,
\end{aligned}
$$

where $\mathbf{t} = \nu\, \partial_n \phi - \psi$.

Transposition argument for irregular data

$(\phi, \psi, \mathbf{t})$ the solution to (adjoint)-Stokes with (ξ, f) as RHS with H.B.C. $(\text{div}\phi = f)$.

$(\phi_h, \psi_h, \mathbf{t}_h)$ the solution to approx. (adjoint)-Stokes with (ξ, f) as RHS with H.B.C. $(\text{div}\phi = f)$.

$$(\xi, y - y_h)_\Omega + (f, p - p_h)_\Omega = (\mathbf{t} - \mathbf{t}_h, g)_\Gamma.$$

We need to estimate $\mathbf{t} - \mathbf{t}_h$.

Equations for $\mathbf{t}$ and $\mathbf{t}_h$

$(\phi, \psi, \mathbf{t})$ is the solution to

$$
\begin{aligned}
&a(\zeta, \phi) + b(\zeta, \psi) - \langle \mathbf{t}, \zeta \rangle = (\xi, \zeta)_\Omega, \quad \forall \zeta \in (H^1(\Omega))^3,\\
&b(\phi, \rho) = (f, \rho)_\Omega, \quad \forall \rho \in L^2_0(\Omega),\\
&\langle \boldsymbol{\lambda}, \phi \rangle = 0, \quad \forall \boldsymbol{\lambda} \in (H^{-1/2}(\Gamma))^3.
\end{aligned}
$$

$(\phi_h, \psi_h, \mathbf{t}_h)$ is the solution to

$$
\begin{aligned}
&a(\zeta_h, \phi_h) + b(\zeta_h, \psi_h) - \langle \mathbf{t}_h, \zeta_h \rangle = (\xi, \zeta_h)_\Omega, \quad \forall \zeta \in X_h,\\
&b(\phi_h, \rho_h) = (f, \rho_h)_\Omega, \quad \forall \rho_h \in M_h,\\
&\langle \boldsymbol{\lambda}_h, \phi_h \rangle = 0, \quad \forall \boldsymbol{\lambda}_h \in S_h.
\end{aligned}
$$

If Ω is non-convex, for all $\alpha_0 \in (0, \alpha^*)$, we have

$$\|\pi_h \mathbf{t} - \mathbf{t}_h\|_{L^2(\Gamma;\mathbb{R}^3)} \leq C_{\alpha_0}\, h^{\alpha_0}(\|\xi\|_{H^{-1/2\alpha_0}} + \|f\|_{\mathcal{H}^{1/2\alpha_0}}),$$

and

$$\|\mathbf{t} - \pi_h \mathbf{t}\|_{L^2(\Gamma;\mathbb{R}^3)} \leq C_{\alpha_0}\, h^{\alpha_0}(\|\xi\|_{H^{-1/2\alpha_0}} + \|f\|_{\mathcal{H}^{1/2\alpha_0}}),$$

If Ω is convex, we have

$$\|\pi_h \mathbf{t} - \mathbf{t}_h\|_{L^2(\Gamma;\mathbb{R}^3)} \leq C\, h^{1/2}(\|\xi\|_{L^2} + \|f\|_{\mathcal{H}^1_{00,\mathbb{E}_d}}),$$

and

$$\|\mathbf{t} - \pi_h \mathbf{t}\|_{L^2(\Gamma;\mathbb{R}^3)} \leq C\, h^{1/2-\epsilon}(\|\xi\|_{L^2} + \|f\|_{\mathcal{H}^1_{00,\mathbb{E}_d}}), \quad \forall \epsilon \in (0, 1/2).$$

Approximation of B - I

We have to estimate $\|(\lambda_0 I - A)^{-1} B - (\lambda_0 I - A_h)^{-1} B_h\|_{\mathcal{L}(U,L^2)}$.

If $U = V^0(\Gamma)$, $\;Bg = (\lambda_0 I - A)PD\, g$,

$B_h g = (\lambda_0 I - A_h) P_h D_h\, \pi_h g$, and we have to estimate

$$\|PD\, g - P_h D_h\, \pi_h g\|_{V_n^0(\Omega)}$$
$$\leq \|P(D\, g - D_h\, \pi_h g)\|_{V_n^0(\Omega)} + \|(P - P_h) D\, g\|_{V_n^0(\Omega)}.$$

If $U = \mathbb{R}^{N_c}$, $\;Bf = \sum_{i=1}^{N_c} f_i (\lambda_0 I - A) PD\, g_i$,

$B_h f = \sum_{i=1}^{N_c} f_i (\lambda_0 I - A_h) P_h D_h\, \pi_h g_i$.

We have (in convex domain)

$$\|(D - D_h\, \pi_h) g\|_{L^2(\Omega;\mathbb{R}^3)} \leq C_{\overline{\delta}}\, h^{1/2}\, \|g\|_{V^0(\Gamma)},$$

if $\mathrm{dist}(\mathrm{supp} g, \mathbb{E}_3) \geq \overline{\delta} > 0$.

Approximation of P

We use the Inf-Sup condition from Bercovier-Pironneau.

- $PF = z$ where $(z, q) \in V_n^0(\Omega) \times (H^1(\Omega) \cap L_0^2(\Omega))$ is sol. to

$$(z, w)_{L^2(\Omega;\mathbb{R}^3)} + \int_\Omega \nabla q \cdot w \, dx = \big(F, w\big)_{L^2(\Omega;\mathbb{R}^3)} \quad \text{for all } w \in H_0^1(\Omega;\mathbb{R}^3),$$

$$\int_\Omega \nabla \rho \cdot z \, dx = 0 \quad \text{for all } \rho \in (H^1(\Omega) \cap L_0^2(\Omega)).$$

- $P_h F = z_h$ where $(z_h, q_h) \in X_h^0 \times M_h$ is solution to

$$(z_h, w_h)_{L^2(\Omega;\mathbb{R}^3)} + \int_\Omega \nabla q_h \cdot w_h \, dx = \big(F, w_h\big)_{L^2(\Omega;\mathbb{R}^3)} \quad \text{for all } w_h \in X_h^0,$$

$$\int_\Omega \nabla \rho_h \cdot z_h \, dx = 0 \quad \text{for all } \rho_h \in M_h.$$

We have (Badra-R, 23)

$$\|P - P_h\|_{\mathcal{L}(H^\ell(\Omega;\mathbb{R}^3), L^2(\Omega;\mathbb{R}^3))} \le C\, h^\ell, \quad \forall \ell \in [0, 1/2).$$

Approximation of B - II

- $Bg = (\lambda_0 I - A)PD\, g,\ U = V^0(\Gamma),\ H = L^2(\Omega; \mathbb{R}^3)$
- $B_h f = (\lambda_0 I - A_h)P_h D_h\, \pi_h g,$

We have

If Ω is non-convex, we have

$$\|(\lambda_0 I - A)^{-1}B - (\lambda_0 I - A_h)^{-1}B_h\|_{\mathcal{L}(U,H)} \le C_{\alpha_0}\, h^{\alpha_0}, \quad \forall \alpha_0 \in (0, \alpha^*).$$

If Ω is convex, we have

$$\|(\lambda_0 I - A)^{-1}B - (\lambda_0 I - A_h)^{-1}B_h\|_{\mathcal{L}(U,H)} \le C_\epsilon\, h^{1/2-\epsilon}, \quad \forall \epsilon \in (0, 1/2).$$

Estimate of $\|e^{A_h t} B_h\|_{\mathcal{L}(V^0(\Gamma), Z_h)}$

$$y \longmapsto \|(\lambda_0 - A_h)^{1/2} y\|_{L^2}$$

is a norm on Z_h equivalent to the H^1-norm.

$$\|e^{A_h t} B_h\|_{\mathcal{L}(V^0(\Gamma), Z_h)} \leq C\, h^r \frac{e^{\omega_0 t}}{t}, \quad \forall t \in (0, h^{r/(1-\gamma)}),$$

and

$$\|e^{A_h t} B_h\|_{\mathcal{L}(V^0(\Gamma), Z_h)} \leq C\, h^{-1} \frac{e^{\omega_0 t}}{t^{1/2}}.$$

$$\|e^{A_h t} B_h\|_{\mathcal{L}(V^0(\Gamma), Z_h)} \leq C \frac{e^{\omega_0 t}}{t^{(r+2)/(2r+2)}}, \quad \forall t \in (0, h^{r/(1-\gamma)}).$$

Estimates of $w_s - w_s^h$

Assumptions $g_s \in V_{cc}^{1+\alpha_0}(\Gamma)$ if Ω is non-convex and $g_s \in V_{cc}^{3/2}(\Gamma) \cap \dot{V}^{3/2}(\Gamma)$ if Ω is convex.

In addition (w_s, ρ_s) is a nonsingular solution of the NSE.

If Ω is non-convex, for all $\alpha_0 \in (0, \alpha^*)$, we have

$$h^{1/2+\alpha_0}\|w - w_s^h\|_{H^1} + \|w - w_s^h\|_{L^2} + h^{1/2+\alpha_0}\|\rho_s - \rho_s^h\|_{L_0^2}$$

$$\leq C(\alpha_0, w_s)\, h^{1+2\alpha_0},$$

If Ω is convex, we have

$$\|w - w_s^h\|_{H^1} + \|\rho_s - \rho_s^h\|_{L_0^2} \leq C(w_s)\, h.$$

If Ω is convex, and if $g_s \in V_{cc}^{1+\alpha_0}(\Gamma) \cap \dot{V}^{1+\alpha_0}(\Gamma)$ for some $\alpha_0 \in (1/2, \alpha^*)$, we have

$$\|w - w_s^h\|_{L^2} \leq C(w_s)\, h^2.$$

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Thank you for your attention