ICTS - Bangalore

Recent advances on control theory of PDE systems

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Approximation of feedback gains for abstract parabolic systems

Lecture 1

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Outline of the four lectures

Lecture I

- Approximation of feedback gains for abstract parabolic systems

Lecture II

- Numerical approximation of the Oseen system in polyhedral domains - Approximation of feedback gains for the Oseen system

Lecture III

- Stabilization of fluid flows using ROM based on spectral projection - Numerical approximation of feedback gains based on ROM

Lecture IV

- Feedback stabilization of FSI problems

Outline of the talk

Part I

- Motivations – General framework – Assumptions

Part II

- Convergence rates for given feedbacks – New type gap theorem

Part III

- Convergence of Riccati based feedbacks

Part IV

- An application

Motivations - I

- The linear controlled system

$$z' = Az + Bu, \quad z(0) = z_0, \quad \text{in } Z.$$

- Assumptions.

* $(A, \mathcal{D}(A))$ is the inf. generator of an analytic semigroup on Z.

* The control operator $B \in \mathcal{L}(U, (\mathcal{D}(A^*))')$. But $(\lambda_0 I - A)^{-\gamma} B \in \mathcal{L}(U, Z)$ for some $\gamma \in (0, 1)$ and $\lambda_0 > 0$.

* The pair (A, B) is stabilizable in Z.

* The pair $(A, \mathcal{C})$ is detectable in Z, where $\mathcal{C} \in \mathcal{L}(Z, Y)$.

- Goal. We look for $F \in \mathcal{L}(Z, U)$ such that $(e^{t(A+BF)})_{t \geq 0}$ is exp. stable on Z.

Such a F can be obtained by solving a LQR problem.

Background

* $(A, \mathcal{D}(A))$ is the infinitesimal generator of an analytic semigroup on Z iff

The resolvent set $\quad \rho(A) \supset \{\omega_0\} + \mathbb{S}_{\pi/2+\delta}$.

$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(Z)} \leq \frac{C}{|\lambda - \omega_0|}, \quad \forall \lambda \in \{\omega_0\} + \mathbb{S}_{\pi/2+\delta},$

$\mathbb{S}_{\pi/2+\delta} = \{\lambda \in \mathbb{C} \mid |\arg(\lambda)| < \pi/2 + \delta\},\ 0 < \delta < \pi/2.$

Semigroup/Resolvent

$$e^{tA} = \frac{1}{2i\pi} \int_\Gamma e^{\lambda\, t} (\lambda I - A)^{-1}\, d\,\lambda.$$

Stability

$$\|e^{tA}\|_{\mathcal{L}(Z)} \leq C\, e^{t\omega_0}.$$

Motivations - II

- Approximate controlled system

$$z_\varepsilon' = A_\varepsilon z_\varepsilon + B_\varepsilon u, \quad z_\varepsilon(0) = z_{0,\varepsilon} \quad \text{in } Z_\varepsilon.$$

- The approximate controlled system: Finite Element approximation ($\varepsilon = h$, h is the meshsize) or a perturbation of the initial system (approximation by penalization, ε is the perturbation parameter).

- If $Z_\varepsilon \subset Z$, we have a conforming approximation. If $Z_\varepsilon \not\subset Z$, we have a nonconforming approximation.

We assume that $Z \subset H$, $Z_\varepsilon \subset H$, $P \in \mathcal{L}(H)$, and $P_\varepsilon \in \mathcal{L}(H)$ are projectors such that

$$PH = Z \quad \text{and} \quad P_\varepsilon H = Z_\varepsilon.$$

- Goal. Find an approximate feedback $F_\varepsilon \in \mathcal{L}(Z_\varepsilon, U)$ such that $(e^{t(A_\varepsilon + B_\varepsilon F_\varepsilon)})_{t\geq 0}$ is exp. stable on Z_ε and $(e^{t(A+BF_\varepsilon P_\varepsilon)})_{t\geq 0}$ is exp. stable on Z.

Known results for conforming approximations

If $Z_\varepsilon \subset Z$, $P_\varepsilon : Z \mapsto Z_\varepsilon$, we want to have $F_\varepsilon P_\varepsilon \to F$ as $\varepsilon \to 0$, and if possible $\|F_\varepsilon P_\varepsilon - F\|_{\mathcal{L}(Z,U)} \leq C\,\varepsilon^\alpha$, $\alpha > 0$.

- LQR problem for $(A, B, \mathcal{C})$: $\min \int_0^\infty (\|\mathcal{C}z(t)\|^2 + \|u(t)\|^2)dt$.

$u(t) = -B^*\Pi\, z(t)$, Π solves an A.R.E.

- K. Ito (1987). In order to prove that $\|F - F_h\|_{\mathcal{L}(Z,U)} \to 0$ as $h \to 0$, the uniform stabilizability of the pair $(A_h, B_h)_{h>0}$, with respect to h, is required. $B \in \mathcal{L}(U, Z)$.

- H. T. Banks, K. Kunisch, 1984. The uniform stabilizability is satisfied for parabolic equations. $B \in \mathcal{L}(U, Z)$.

- Extension to the case of unbounded control operators. $(\lambda_0 I - A)^{-\gamma} B \in \mathcal{L}(U, Z)$, $0 < \gamma < 1$. I. Lasiecka (1992). R. Triggiani (1994). Series of works, monography (2000).

- In all these results, $Z_h \subset Z$ (conforming approximation).

Motivations

• For the approximation by a FEM of the Navier-Stokes equations in $\Omega \subset \mathbb{R}^d$, we have

$$H = (L^2(\Omega))^d,$$

$$Z = V_n^0(\Omega) = \{z \in (L^2(\Omega))^d \mid \operatorname{div} z = 0,\ z \cdot n = 0 \text{ on } \Gamma\},$$

$$Z_h = \{z_h \in X_h \mid \int_\Omega \operatorname{div} z_h \, q_h \, dx = 0,\ \forall q_h \in M_h\},$$

where $X_h \subset (H^1(\Omega))^d$ is a F.E. space for the velocity, and $M_h \subset L_0^2(\Omega)$ is a F.E. space for the pressure.

• For the pseudo-compressibility (or penalty) method $\operatorname{div} z_\varepsilon + \varepsilon p_\varepsilon = 0$, we have

$$Z = V_n^0(\Omega) \quad \text{and} \quad Z_\varepsilon = (L^2(\Omega))^d = H.$$

We are in the case of nonconforming approximation: $Z_\varepsilon \not\subset Z$, but $Z \subset H$ and $Z_\varepsilon \subset H$.

Part II - A general framework - Assumptions I

(A, B, P) and $(A_\varepsilon, B_\varepsilon, P_\varepsilon)$ given. We assume (to check for each application)

- Projectors $P \; : \; H \mapsto Z$ and $P_\varepsilon \; : \; H \mapsto Z_\varepsilon$
- Uniform analytic estimates. $\mathcal{D}(A)$ dense in Z, $\mathcal{D}(A_\varepsilon)$ dense in Z_ε

The resolvent set $\quad \rho(A_\varepsilon) \supset \{\omega_0\} + \mathbb{S}_{\pi/2+\delta}$.

$$\|(\lambda I - A_\varepsilon)^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \leq \frac{C}{|\lambda-\omega_0|}, \quad \forall \lambda \in \{\omega_0\} + \mathbb{S}_{\pi/2+\delta}, \ \forall \varepsilon \in (0,1),$$

$$\mathbb{S}_{\pi/2+\delta} = \{\lambda \in \mathbb{C} \mid |\arg(\lambda)| < \pi/2 + \delta\}, \ 0 < \delta < \pi/2.$$

- Approximation assumption for A:

$$\sup_{\varepsilon \in (0,1)} \|P_\varepsilon\|_{\mathcal{L}(H)} < +\infty,$$

$$\|(\lambda_0 I - A)^{-1} P - (\lambda_0 I - A_\varepsilon)^{-1} P_\varepsilon\|_{\mathcal{L}(H)} \leq C\varepsilon^s, \quad \text{with } s > 0,$$

with $\lambda_0 > \omega_0$.

A general framework - Assumptions II

- Approximation assumption for B:

$$(\lambda_0 I - A)^{-\gamma} B \in \mathcal{L}(U, Z) \quad \text{for some } \gamma \in [0,1).$$

$$\|(\lambda_0 I - A)^{-1} B - (\lambda_0 I - A_\varepsilon)^{-1} B_\varepsilon\|_{\mathcal{L}(U,H)} \leq C\varepsilon^r,$$

with $0 \leq r \leq s(\gamma - 1)$.

- A uniform boundedness condition - weaker than:

$$\sup_{\varepsilon\in(0,1)} \|(\lambda_0 I - A_\varepsilon)^{-\gamma} B_\varepsilon\|_{\mathcal{L}(U,H)} < \infty.$$

For all $\varepsilon \in (0,1)$, $(\lambda_0 I - A_\varepsilon)^{-\gamma} B_\varepsilon \in \mathcal{L}(U,H)$ and, for all $\varepsilon \in (0,1)$, the following uniform bound holds

$$\|e^{tA_\varepsilon} B_\varepsilon\|_{\mathcal{L}(U,H)} \leq C\,\frac{e^{\omega_0 t}}{t^{\overline{\gamma}}}, \quad \forall t \in (0, \varepsilon^{r/(1-\gamma)}),\ \overline{\gamma} \in [\gamma, 1).$$

- Stabilizability

The pair (A, B) is stabilizable in Z.

Main results – Feedback stabilization

In Theorem 1.1 we state that if (A, B) is exp. stabilizable in Z, then $(A_\varepsilon, B_\varepsilon)_{0<\varepsilon<1}$ is uniformly exp. stabilizable in Z_ε.

The pair $(A_\varepsilon, B_\varepsilon)$ is exp. stabilizable in Z_ε uniformly w. r. to $\varepsilon \in (0,1)$ if there exist $M \geq 1$ and $\omega_F > 0$ such that for all $\varepsilon \in (0,1)$, there exists $F_\varepsilon \in \mathcal{L}(Z_\varepsilon, U)$ such that

$$\|e^{(A_\varepsilon + B_\varepsilon F_\varepsilon)t}\|_{\mathcal{L}(Z_\varepsilon)} \leq Me^{-\omega_F t}, \forall t \geq 0, \forall \varepsilon \in (0,1).$$

In Theorem 1.2 we state that if $(A_\varepsilon, B_\varepsilon)_{0<\varepsilon<1}$ is uniformly exp. stabilizable in Z_ε then (A, B) is exp. stabilizable in Z.

Theorem 1.1. Let $F \in \mathcal{L}(Z, U)$ and $\omega_F > 0$ be such that $A + \omega_F I + BF$ is exponentially stable on Z. $(F_\varepsilon)_{0<\varepsilon<1} \subset \mathcal{L}(Z_\varepsilon, U)$ satisfies

$$\|FP - F_\varepsilon\|_{\mathcal{L}(Z_\varepsilon, U)} \leq \sigma(\varepsilon), \quad \forall \varepsilon \in (0,1), \quad \sigma(\varepsilon) \to 0 \text{ as } \varepsilon \to 0.$$

Set $A_F = A + BF$ and $A_{\varepsilon,F_\varepsilon} = A_\varepsilon + B_\varepsilon F_\varepsilon$.
Then, for all $\widetilde{\delta} \in (0,\delta)$, there exist $\varrho > 0$ and $\varepsilon_0 \in (0,1)$ such that $\{-\omega_{F,\varepsilon}\} + \mathbb{S}_{\pi/2+\widetilde{\delta}} \subset \rho(A_{\varepsilon,F_\varepsilon})$, with $\omega_{F,\varepsilon} = \omega_F - \varrho(\varepsilon^r + \sigma(\varepsilon))$, and

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \leq \frac{C}{|\lambda + \omega_{F,\varepsilon}|}, \quad \forall \lambda \in \{-\omega_{F,\varepsilon}\} + \mathbb{S}_{\pi/2+\widetilde{\delta}},$$

$$\|e^{A_{\varepsilon,F_\varepsilon} t}\|_{\mathcal{L}(Z_\varepsilon)} \leq C e^{-\omega_{F,\varepsilon} t}, \ \forall t \geq 0, \ \forall \varepsilon \in (0, \varepsilon_0).$$

Moreover, for all $\varepsilon \in (0,\varepsilon_0)$, we have

$$\|e^{A_F t} P - e^{A_{\varepsilon,F_\varepsilon} t} P_\varepsilon\|_{\mathcal{L}(H)} \leq C\, e^{-\omega_{F,\varepsilon} t} \left(\frac{\varepsilon^r}{t^{r/s}} + \sigma(\varepsilon) \right), \qquad \forall t \geq 0.$$

These results are true for $F_\varepsilon = FP$, with $\sigma \equiv 0$.

Remark. If

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \leq \frac{C}{|\lambda+\omega_{F,\varepsilon}|}, \quad \forall \lambda \in \{-\omega_{F,\varepsilon}\} + \mathbb{S}_{\pi/2+\widetilde{\delta}},$$

and if $A_{\varepsilon,F_\varepsilon}$ is the infinitesimal generator of an analytic semigroup on Z_ε, then

$$\|e^{A_{\varepsilon,F_\varepsilon} t}\|_{\mathcal{L}(Z_\varepsilon)} \leq Ce^{-\omega_{F,\varepsilon} t}, \; \forall t \geq 0, \; \forall \varepsilon \in (0, \varepsilon_0).$$

Beginning of the proof. We assume that

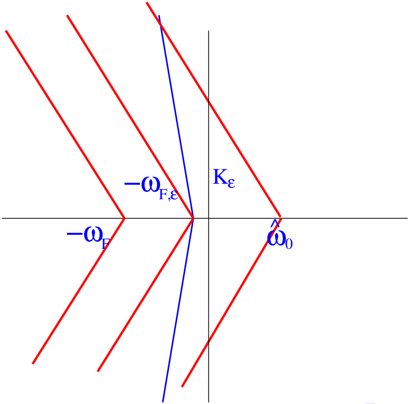
$$\|(\lambda I - A_F)^{-1}\|_{\mathcal{L}(Z)} \leq \frac{C_F}{|\lambda+\omega_F|}, \quad \forall \lambda \in \{-\omega_F\} + \mathbb{S}_{\pi/2+\delta}.$$

- We first prove

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \leq \frac{C}{|\lambda+\omega_F|}, \quad \forall\lambda \in \{\widehat{\omega}_0\} + \mathbb{S}_{\pi/2+\delta}, \quad \forall\varepsilon \in (0,\varepsilon_0),$$

where $\widehat{\omega}_0 > \max(\lambda - \lambda_0, -\omega_F)$, $\varepsilon_0 \in (0,1)$.

This provides the desired estimate for $|\lambda|$ large and $0 < \widetilde{\delta} < \delta$.



Ideas of Proof - III

- To prove

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \leq \frac{C}{|\lambda+\omega_F|}, \quad \forall\lambda \in \{\widehat{\omega}_0\} + \mathbb{S}_{\pi/2+\delta},$$

we use $A_{\varepsilon,F_\varepsilon} = A_\varepsilon + B_\varepsilon F_\varepsilon$, and prove that

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \leq c_0\|(\lambda I - A_\varepsilon)^{-1}\|_{\mathcal{L}(Z_\varepsilon)}$$

$$\leq \frac{C}{|\lambda-\omega_0|}, \quad \forall\lambda \in \{\widehat{\omega}_0\} + \mathbb{S}_{\pi/2+\delta}.$$

- Indeed, we have

$$(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1} = (I - T_\varepsilon(\lambda))^{-1}(\lambda I - A_\varepsilon)^{-1}$$

where

$$T_\varepsilon(\lambda) = (\lambda I - A_\varepsilon)^{-1}B_\varepsilon F_\varepsilon P_\varepsilon.$$

We conclude with an estimate of $T_\varepsilon(\lambda)$.

To estimate $\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}\|_{\mathcal{L}(Z_\varepsilon)}$ for $\lambda \in K_\epsilon$, we first prove that, for some $\widehat{\lambda} \in \{\widehat{\omega}_0\} + \mathbb{S}_{\pi/2+\delta}$ and all $\mu \in K_\varepsilon$, we have

$$\|(\widehat{\lambda} I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon - (\widehat{\lambda} I - A_F)^{-1}P\|_{\mathcal{L}(H)} \leq C(\varepsilon^r + \sigma(\varepsilon))$$

$$\leq \frac{1}{2(1+|\mu|\|P\|)^2(1+\|((\widehat{\lambda}+\mu)I - A_F)^{-1}P\|)}. \quad (\lambda = \widehat{\lambda} + \mu)$$

With a New Gap Theorem - Theorem 1.3. - we deduce that

$$\|(\lambda I - A_{\varepsilon,F_\varepsilon})^{-1}P_\varepsilon\|_{\mathcal{L}(H)} \leq \frac{C}{|\lambda + \omega_F|}, \quad \forall \lambda \in \{\widehat{\lambda}\} + K_\varepsilon.$$

Theorem 1.3. If $\mathbb{A}_1$ and $\mathbb{A}_2$ both admit a bounded inverse in Z_1 and Z_2 respectively, if $\lambda \in \mathbb{C}$ belongs to the resolvent set of $\mathbb{A}_2$, and if

$$\|\mathbb{A}_1^{-1} P_1 - \mathbb{A}_2^{-1} P_2\|_{\mathcal{L}(H)}$$

$$< \frac{1}{2(1+|\lambda| \max(\|P_1\|,\|P_2\|)^2(1+\|(\mathbb{A}_2-\lambda I)^{-1} P_2\|_{\mathcal{L}(H,Z_2)})},$$

then $\mathbb{A}_1 - \lambda I$ admits a bounded inverse in Z_1, and

$$\|(\lambda I - \mathbb{A}_1)^{-1}\|_{\mathcal{L}(Z_1)} \leq 1 + 2\|(\lambda I - \mathbb{A}_2)^{-1} P_2\|_{\mathcal{L}(H,Z_2)}.$$

In Kato, when $Z_1 = Z_2 = H$, the gap of $\mathbb{A}_2$ from $\mathbb{A}_1$ is

$$\delta(\mathbb{A}_2, \mathbb{A}_1) = \sup_{\|z_2\|_{\mathcal{D}(\mathbb{A}_2)}=1} \inf_{z_1 \in \mathcal{D}(\mathbb{A}_1)} \{\|z_2 - z_1\|_H + \|\mathbb{A}_2 z_2 - \mathbb{A}_1 z_1\|_H\}.$$

The symmetric gap is

$$\widehat{\delta}(\mathbb{A}_2, \mathbb{A}_1) = \max[\delta(\mathbb{A}_1, \mathbb{A}_2), \delta(\mathbb{A}_2, \mathbb{A}_1)].$$

Gap for noncomforming approximation

$$\delta((\mathbb{A}_2, P_2), (\mathbb{A}_1, P_1))$$

$$= \sup_{\|z_2\|_H + \|\mathbb{A}_2 z_2 + (I - P_2)\zeta\|_H = 1} \inf_{z_1 \in \mathcal{D}(\mathbb{A}_1)} \{\|z_2 - z_1\|_H + \|P_1(\mathbb{A}_2\, z_2 + (I - P_2)\zeta) - \mathbb{A}_1\, z_1\|_H\}.$$

If $\mathbb{A}_1$ and $\mathbb{A}_2$ both admit a bounded inverse in Z_1 and Z_2 respectively, we have

$$\delta((\mathbb{A}_2, P_2), (\mathbb{A}_1, P_1)) \leq \|\mathbb{A}_1^{-1} P_1 - \mathbb{A}_2^{-1} P_2\|_{\mathcal{L}(H)}.$$

Theorem 1.2. Let $(F_\varepsilon)_{0<\varepsilon<1} \subset \mathcal{L}(Z_\varepsilon, U)$ and $\omega_F > 0$ satisfy

$$\|F_\varepsilon P_\varepsilon\|_{\mathcal{L}(Z,U)} \le C, \quad \forall \varepsilon \in (0,1),$$

$$((e^{t(A_\varepsilon + \omega_F I + B_\varepsilon F_\varepsilon)})_{t\ge 0})_{0<\varepsilon<1} \text{ is unif. exp. stable on } Z_\varepsilon.$$

Let $(F^{(\varepsilon)})_{0<\varepsilon<1} \subset \mathcal{L}(Z, U)$ satisfying

$$\|F_\varepsilon P_\varepsilon - F^{(\varepsilon)}\|_{\mathcal{L}(Z,U)} \le \sigma(\varepsilon), \quad \forall \varepsilon \in (0,1).$$

Set $A_{F^{(\varepsilon)}} = A + BF^{(\varepsilon)}$ and $A_{\varepsilon,F_\varepsilon} = A_\varepsilon + B_\varepsilon F_\varepsilon$.
Then, for all $\widetilde{\delta} \in (0, \delta)$, there exist $\varrho > 0$ and $\varepsilon_0 \in (0,1)$ such that $\{-\omega_{F,\varepsilon}\} + \mathbb{S}_{\pi/2+\widetilde{\delta}} \subset \rho(A_{F^{(\varepsilon)}})$, with $\omega_{F,\varepsilon} = \omega_F - \varrho(\varepsilon^r + \sigma(\varepsilon))$, and

$$\|(\lambda I - A_{F^{(\varepsilon)}})^{-1}\|_{\mathcal{L}(Z)} \le \frac{C}{|\lambda + \omega_{F,\varepsilon}|}, \ \forall \lambda \in \{-\omega_{F,\varepsilon}\} + \mathbb{S}_{\pi/2+\widetilde{\delta}},$$

$$\|e^{A_{F^{(\varepsilon)}} t}\|_{\mathcal{L}(Z)} \le C e^{-\omega_{F,\varepsilon} t}, \ \forall t \ge 0, \qquad \forall \varepsilon \in (0, \varepsilon_0),$$

$$\|e^{A_{\varepsilon,F_\varepsilon} t} P - e^{A_{F^{(\varepsilon)}} t} P_\varepsilon\|_{\mathcal{L}(H)} \le C\, e^{-\omega_{F,\varepsilon} t} \left(\frac{\varepsilon^r}{t^{r/s}} + \sigma(\varepsilon) \right), \quad \forall t \ge 0.$$

These results are true for $F^{(\varepsilon)} = F_\varepsilon P_\varepsilon$, with $\sigma \equiv 0$.

- LQR problem for $(A, B, \mathcal{C}|_Z)$: $\min \int_0^\infty (\|\mathcal{C}z(t)\|_Y^2 + \|u(t)\|^2)dt$.

(A, B) is stab. in Z and $(A, \mathcal{C}|_Z)$ is detectable in Z.

The solution is $u(t) = -B^*\Pi\, z(t)$, where Π solves the A.R.E.

$$\Pi \in \mathcal{L}(Z), \quad \Pi = \Pi^* \geq 0, \quad B^*\Pi \in \mathcal{L}(Z, U),$$
$$\Pi A + A^*\Pi - \Pi BB^*\Pi + P^*\mathcal{C}^*\mathcal{C}P = 0.$$

- LQR for $(A_\varepsilon, B_\varepsilon, \mathcal{C}|_{Z_\varepsilon})$: $\min \int_0^\infty (\|\mathcal{C}z_\varepsilon(t)\|_Y^2 + \|u_\varepsilon(t)\|^2)dt$.

If $(A_\varepsilon, B_\varepsilon)$ is stab. in Z_ε and $(A_\varepsilon, \mathcal{C}|_{Z_\varepsilon})$ is detectable in Z_ε,

the solution is $u_\varepsilon(t) = -B_\varepsilon^*\Pi_\varepsilon\, z_\varepsilon(t)$, where Π_ε solves the A.R.E.

$$\Pi_\varepsilon \in \mathcal{L}(Z_\varepsilon), \quad \Pi_\varepsilon = \Pi_\varepsilon^* \geq 0, \quad B_\varepsilon^*\Pi_\varepsilon \in \mathcal{L}(Z_\varepsilon, U),$$
$$\Pi_\varepsilon A_\varepsilon + A_\varepsilon^*\Pi_\varepsilon - \Pi_\varepsilon B_\varepsilon B_\varepsilon^*\Pi_\varepsilon + P_\varepsilon^*\mathcal{C}^*\mathcal{C}P_\varepsilon = 0.$$

Stabilizability of $(A_\varepsilon, B_\varepsilon)$

• We assume that $A + \omega_\Pi I - BB^*\Pi$ is exponentially stable in Z, for some $\omega_\Pi > 0$. $(F = -B^*\Pi,\ \omega_\Pi = \omega_F)$

• With Theorem 1.1, we prove that $A_{\varepsilon,\Pi} = A_\varepsilon - B_\varepsilon B^*\Pi P$ is exponentially stable in Z_ε, uniformly with respect to $\varepsilon \in (0, \varepsilon_0)$, for some $\varepsilon_0 \in (0, 1)$:

$$\|e^{tA_{\varepsilon,\Pi}}\|_{\mathcal{L}(Z_\varepsilon)} \leq C\, e^{-\omega_{\Pi,\varepsilon}\, t}, \quad \forall t \geq 0, \quad \forall \varepsilon \in (0, \varepsilon_0),$$

where

$$\omega_{\Pi,\varepsilon} = \omega_\Pi - \varrho\, \varepsilon^r.$$

Detectability of $(A_\varepsilon, \mathcal{C}|_{Z_\varepsilon})$

- We assume that $\mathcal{C} \in \mathcal{L}(H, Y)$, Y is a Hilbert space, and that

$$(A, \mathcal{C}|_Z) \quad \text{is detectable in } Z,$$

i.e.

$$(A^*, (\mathcal{C}|_Z{}^*) \quad \text{is stabilizable in } Z.$$

- With Theorem 1.1, we prove that

$$(A_\varepsilon, \mathcal{C}|_{Z_\varepsilon} = \mathcal{C}_\varepsilon = \mathcal{C}) \quad \text{is detectable in } Z_\varepsilon,$$

uniformly with respect to $\varepsilon \in (0, \varepsilon_0)$, for some $\varepsilon_0 \in (0, 1)$. More precisely, from Theorem 1.1 it follows that there exists $F \in \mathcal{L}(Y, Z)$ such that

$$\left(e^{t(A_\varepsilon + P_\varepsilon F \mathcal{C}_\varepsilon)}\right)_{t \geq 0}$$

is exponentially stable on Z_ε uniformly in $\varepsilon \in (0, \varepsilon_0)$.

Preliminaries to prove convergence rate for feedback gains

$$\widehat{z}(t) = e^{t(A-BB^*\Pi)}Pz_0, \quad \widehat{u}(t) = -B^*\Pi\widehat{z}(t), \quad (\mathcal{P})$$

$$\widehat{z}_\varepsilon(t) = e^{t(A_\varepsilon-B_\varepsilon B_\varepsilon^*\Pi_\varepsilon)}P_\varepsilon z_0, \quad \widehat{u}_\varepsilon(t) = -B_\varepsilon^*\Pi_\varepsilon\widehat{z}_\varepsilon(t), \quad (\mathcal{P}_\varepsilon)$$

$$\widetilde{z}(t) = e^{t(A-BB_\varepsilon^*\Pi_\varepsilon P_\varepsilon)}Pz_0, \quad \widetilde{u}(t) = -B_\varepsilon^*\Pi_\varepsilon P_\varepsilon\widetilde{z}(t),$$

$$\widetilde{z}_\varepsilon(t) = e^{t(A_\varepsilon-B_\varepsilon B^*\Pi P)}P_\varepsilon z_0, \quad \widetilde{u}_\varepsilon(t) = -B^*\Pi P\widetilde{z}_\varepsilon.$$

With Theorem 1.1, we have

$$\|\widetilde{z}_\varepsilon(t)\|_{Z_\varepsilon} \leq C\, e^{-\omega_{\Pi,\varepsilon}t}\|z_0\|_H, \quad \omega_{\Pi,\varepsilon} = \omega_{\Pi,\varepsilon} - \varrho\varepsilon^r,$$

$$\|\widehat{z}(t) - \widetilde{z}_\varepsilon(t)\|_H \leq C\, \frac{\varepsilon^r}{t^{r/s}} e^{-\omega_{\Pi,\varepsilon}t}\|z_0\|_H.$$

With Theorem 1.2, we will obtain a similar result for $\|\widetilde{z}(t)\|_Z$ and for $\|\widehat{z}_\varepsilon(t) - \widetilde{z}(t)\|_H$, but we first need to prove the uniform exponential stability of $e^{(A_\varepsilon-B_\varepsilon B_\varepsilon^*\Pi_\varepsilon)t}$.

$F_\varepsilon = -B_\varepsilon^* \Pi_\varepsilon$. There exist $\omega^* > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$\sup_{\varepsilon \in (0,\varepsilon_0)} \|e^{(A_\varepsilon - B_\varepsilon B_\varepsilon^* \Pi_\varepsilon)t}\|_{\mathcal{L}(Z_\varepsilon)} \leq C e^{-\omega^* t}, \quad \forall t \geq 0.$$

Idea of proof. Step 1.

$$\widehat{z}_\varepsilon = e^{(A_\varepsilon - B_\varepsilon B_\varepsilon^* \Pi_\varepsilon)t} P_\varepsilon z_0, \quad \widehat{u}_\varepsilon = -B_\varepsilon^* \Pi_\varepsilon \widehat{z}_\varepsilon,$$

$$\mathcal{I}_\varepsilon(\widehat{z}_\varepsilon, \widehat{u}_\varepsilon) = \frac{1}{2}(\Pi_\varepsilon P_\varepsilon z_0, P_\varepsilon z_0)_H$$

$$(\Pi_\varepsilon P_\varepsilon z_0, P_\varepsilon z_0)_H \leq 2\,\mathcal{I}_\varepsilon(\widetilde{z}_\varepsilon, \widetilde{u}_\varepsilon) \leq c\|z_0\|_H^2.$$

We deduce that

$$\sup_{0<\varepsilon<\varepsilon_0} \|P_\varepsilon^* \Pi_\varepsilon P_\varepsilon\|_{\mathcal{L}(H)} < \infty.$$

Uniform stability - II

Step 2. $(A_\varepsilon, \mathcal{C}|_{Z_\varepsilon}) = (A_\varepsilon, \mathcal{C}_\varepsilon)$ is detectable, uniformly in $\varepsilon \in (0, \varepsilon_0)$.

Thus, there exists $F \in \mathcal{L}(Y, Z)$ such that

$$\left(e^{t(A_\varepsilon + P_\varepsilon F \mathcal{C}_\varepsilon)}\right)_{t \geq 0}$$

is an analytic semig. on Z_ε, exponen. stable unif. in $\varepsilon \in (0, \varepsilon_0)$.

With

$$A_{\varepsilon,\Pi_\varepsilon} = A_{\varepsilon,F} - P_\varepsilon F \mathcal{C}_\varepsilon - B_\varepsilon B_\varepsilon^* \Pi_\varepsilon,$$

$$e^{tA_{\varepsilon,\Pi_\varepsilon}} P_\varepsilon z_0 = e^{tA_{\varepsilon,K}} P_\varepsilon z_0 - \int_0^t e^{(t-\tau)A_{\varepsilon,K}} P_\varepsilon P_\varepsilon F \widehat{z}_\varepsilon(\tau)\, d\tau$$

$$- \int_0^t e^{(t-\tau)A_{\varepsilon,F}} P_\varepsilon B_\varepsilon \widehat{u}_\varepsilon(\tau)\, d\tau,$$

we prove

$$\|e^{(\cdot)A_{\varepsilon,\Pi_\varepsilon}} P_\varepsilon z_0\|_{L^2(0,\infty;H)} \leq C\|z_0\|_H.$$

Uniform stability - III

Step 3. • We prove

$$\sup_{\varepsilon\in(0,\varepsilon_0)} \|e^{(\cdot)A_{\varepsilon,\Pi_\varepsilon}}\|_{\mathcal{L}(H,L^2(0,\infty;H))} < \infty.$$

This is not sufficient to have the exponential uniform stability.

• To have the exponentially uniformly in $\varepsilon \in (0, \varepsilon_0)$, we have to prove some additional bound

$$\|e^{tA_{\varepsilon,\Pi_\varepsilon}}\|_{\mathcal{L}(H)} \le C\, e^{\lambda_0 t}, \quad \forall t \ge 0.$$

• There exist $\omega^* > 0$ and $\varepsilon_0 \in (0, 1)$ such that

$$\sup_{\varepsilon\in(0,\varepsilon_0)} \|e^{(A_\varepsilon - B_\varepsilon B_\varepsilon^* \Pi_\varepsilon)t}\|_{\mathcal{L}(Z_\varepsilon)} \le Ce^{-\omega^* t}, \quad \forall t \ge 0.$$

With Theorem 1.2, we have

$$\|\widetilde{z}(t)\|_Z \le C\, e^{-t\omega^*/2}\|z_0\|_H, \quad \omega_{\Pi,\varepsilon} = \omega_{\Pi,\varepsilon} - \varrho\varepsilon^r,$$

$$\|\widehat{z}_\varepsilon(t) - \widetilde{z}(t)\|_H \le C\, \frac{\varepsilon^r}{t^{r/s}} e^{-t\omega^*/2}\|z_0\|_H.$$

Convergence rate for the feedback gains

We have

$$\|P^*\Pi P - P_\varepsilon^*\Pi_\varepsilon P_\varepsilon\|_{\mathcal{L}(H)} \le C\varepsilon^r\,, \quad \forall\varepsilon\in(0,\varepsilon_0), \quad \text{if } r<s,$$

$$\|P^*\Pi P - P_\varepsilon^*\Pi_\varepsilon P_\varepsilon\|_{\mathcal{L}(H)} \le C\varepsilon^s\,|\ln(\varepsilon)|, \quad \forall\varepsilon\in(0,\varepsilon_0), \quad \text{if } r=s,$$

and

$$\|B^*\Pi P - B_\varepsilon^*\Pi_\varepsilon P_\varepsilon\|_{\mathcal{L}(H,U)} \le C\varepsilon^r|\ln\varepsilon|, \quad \forall\varepsilon\in(0,\varepsilon_0).$$

Convergence rate for the feedback gains - Idea of proof

$$\frac{1}{2}|((P^*\Pi P - P_\varepsilon^*\Pi_\varepsilon P_\varepsilon)z_0, z_0)_H| = |\mathcal{I}(\widehat{z}, \widehat{u}) - \mathcal{I}_\varepsilon(\widehat{z}_\varepsilon, \widehat{u}_\varepsilon)|$$

$$\leq |\mathcal{I}(\widetilde{z}, \widetilde{u}) - \mathcal{I}_\varepsilon(\widehat{z}_\varepsilon, \widehat{u}_\varepsilon)| + |\mathcal{I}(\widehat{z}, \widehat{u}) - \mathcal{I}_\varepsilon(\widetilde{z}_\varepsilon, \widetilde{u}_\varepsilon)|$$

$$\leq C\|\widehat{z}_\varepsilon - \widetilde{z}\|_{L^p(H)} \left(\|\widehat{z}_\varepsilon\|_{L^{p'}(H)} + \|\widetilde{z}\|_{L^{p'}(H)}\right)$$

$$+C\|\widetilde{z}_\varepsilon - \widehat{z}\|_{L^p(H)} \left(\|\widetilde{z}_\varepsilon\|_{L^{p'}(H)} + \|\widehat{z}\|_{L^{p'}(H)}\right).$$

From which, with the estimates on $\widehat{z}_\varepsilon - \widetilde{z}$ and $\widetilde{z}_\varepsilon - \widehat{z}$, we deduce

$$((P^*\Pi P - P_\varepsilon^*\Pi_\varepsilon P_\varepsilon)z_0, z_0)_H \leq C\|z_0\|_H^2\, \varepsilon^r \quad \text{if } r < s.$$

Convergence rates for the closed-loop systems

- $\widehat{z}(t) = e^{t(A+BF)}y_0, \quad F = -B^*\Pi.$

  $\|e^{t(A+BF)}\|_{\mathcal{L}(Z)} \leq Ce^{-\omega_\Pi t}, \quad \forall t \geq 0.$
- $\widetilde{z}(t) = e^{t(A+BF^{(\varepsilon)})}y_0, \quad F^{(\varepsilon)} = -B_\varepsilon^*\Pi_\varepsilon P_\varepsilon.$
- $\widehat{z}_\varepsilon(t) = e^{t(A_\varepsilon+B_\varepsilon F_\varepsilon)}y_0, \quad F_\varepsilon = -B_\varepsilon^*\Pi_\varepsilon.$

For all $\varepsilon \in (0, \varepsilon_0)$, we have

$$\|\widehat{z}(t) - \widehat{z}_\varepsilon(t)\|_H \leq C\,\frac{e^{(-\omega_\Pi+\varrho\varepsilon^r|\ln\varepsilon|)t}}{t^{r/s}}\,\varepsilon^r|\ln\varepsilon|\|z_0\|_H,$$

$$\|\widehat{z} - \widehat{z}_\varepsilon\|_{L^p(0,\infty;H)} \leq C_p\,\varepsilon^{r/p}|\ln\varepsilon|^{r/p}\|z_0\|_H, \quad \forall p \in (1,\infty),$$

$$\|\widehat{z}_\varepsilon - \widetilde{z}\|_{L^p(0,\infty;H)} \leq C_p\,\varepsilon^{r/p}\|z_0\|_H, \quad \forall p \in (1,\infty).$$

Part IV - Applications

- Numerical approximation of the Oseen system with a boundary control (lectures 2 and 3)
- Approximation of the Oseen system by the pseudo-compressibility method with internal control (see below)
- Approximation of the Oseen system by the pseudo-compressibility method with a boundary control (lecture 3)
- Numerical approximation of the Boussinesq system (lecture 3)
- Stabilization of FSI systems (lecture 4) and their numerical approximation (under investigation lecture 4)

- Ω is either a bounded domain in $\mathbb{R}^3$ either of class C^2, or a bounded polyhedral convex domain.
- $(w_s, \rho_s) \in (H^1(\Omega))^3 \times L^2_0(\Omega)$ is a stationary solution of the N.S.E:

$$(w_s \cdot \nabla) w_s - \nu \, \Delta w_s + \nabla \rho_s = f_s, \quad \operatorname{div} w_s = 0 \quad \text{in} \quad \Omega,$$
$$w_s = g_s \quad \text{on} \quad \Gamma = \partial\Omega.$$

- The controlled Navier-Stokes system

$$\frac{\partial w}{\partial t} + (w \cdot \nabla) w - \nu \, \Delta w + \nabla \rho = f_s + \chi_{\mathcal{O}} \, u, \quad \text{in} \quad Q = \Omega \times (0, \infty),$$
$$\operatorname{div} w = 0 \quad \text{in} \quad Q, \quad w = g_s \quad \text{on} \quad \Sigma = \Gamma \times (0, \infty),$$
$$w(0) = w_0 = w_s + y_0 \quad \text{in} \quad \Omega,$$

y_0 is a perturbation in the I.C.

The nonlinear system satisfied by $(y,p) = (w,\rho) - (w_s,\rho_s)$ is

$$\frac{\partial y}{\partial t} + (w_s \cdot \nabla)y + (y \cdot \nabla)w_s + \kappa(y \cdot \nabla)y - \nu\,\Delta y + \nabla p = \chi_{\mathcal{O}}\, u \quad \text{in} \quad Q,$$

$$\operatorname{div} y = 0 \quad \text{in} \quad Q,$$

$$y = 0 \quad \text{on} \quad \Sigma,$$

$$y(0) = y_0 \quad \text{in} \quad \Omega,$$

with $\kappa = 1$. The associated linearized system is obtained by setting $\kappa = 0$.

The controlled Oseen system

$$\frac{\partial y}{\partial t} + (w_s \cdot \nabla)y + (y \cdot \nabla)w_s - \nu\,\Delta y + \nabla p = \chi_{\mathcal{O}}\, u \quad \text{in} \quad Q,$$

$$\operatorname{div} y = 0 \quad \text{in} \quad Q, \quad y = 0 \quad \text{on} \quad \Sigma, \quad y(0) = y_0 \quad \text{in} \quad \Omega.$$

The Leray projector $P \in \mathcal{L}(H, Z)$, $H = (L^2(\Omega))^3$, $Z = V_n^0(\Omega)$.

$V_n^0(\Omega) = \{y \in L^2(\Omega; \mathbb{R}^3) \mid \operatorname{div} y = 0,\; y \cdot n = 0 \text{ on } \Gamma\}$

The Oseen operator $(A, \mathcal{D}(A))$, and the control op. B

$Ay = P(\nu\,\Delta y - (w_s \cdot \nabla)y - (y \cdot \nabla)w_s)$,
$\mathcal{D}(A) = V_n^0(\Omega) \cap (H_0^1(\Omega) \cap H^2(\Omega))^3, \quad B = P\chi_{\mathcal{O}}.$

The control Oseen system

$$y' = Ay + Bu, \quad y(0) = y_0.$$

Approximation by the pseudo-compressible model

- Pseudo-compressible approximation

$$\frac{\partial y_\varepsilon}{\partial t} - \nu \Delta y_\varepsilon + (y_\varepsilon \cdot \nabla) w_s^\varepsilon + (w_s^\varepsilon \cdot \nabla) y_\varepsilon + \nabla p_\varepsilon = \chi_{\mathcal{O}}\, u \quad \text{in } Q,$$

$$\operatorname{div} y_\varepsilon + \varepsilon p_\varepsilon = 0 \text{ in } Q, \qquad y_\varepsilon = 0 \text{ on } \Sigma, \qquad y_\varepsilon(0) = y_0 \text{ in } \Omega.$$

w_s^ε is an approximation of w_s.

- The equation for y_ε can be solved first

$$\frac{\partial y_\varepsilon}{\partial t} - \nu \Delta y_\varepsilon + (y_\varepsilon \cdot \nabla) w_s^\varepsilon + (w_s^\varepsilon \cdot \nabla) y_\varepsilon - \frac{1}{\varepsilon} \nabla \operatorname{div} y_\varepsilon = \chi_{\mathcal{O}}\, u \quad \text{in } Q.$$

- Find u in feedback form $u = Fy$ ($u_\varepsilon = F_\varepsilon y_\varepsilon$) able to stabilize the incomp. model (resp. pseudo-compressible model).
- Study convergence results $F_\varepsilon \to F$, $y_\varepsilon \to y$.
- Prove that the feedback $F_\varepsilon P$ also stabilizes the original system.

$$-\nu\Delta v + \nabla p = f \text{ in } Q,$$

$$\mathrm{div}\, v = 0 \text{ in } \Omega, \qquad v = 0 \text{ on } \Gamma.$$

- The pseudo-compressible Stokes system

$$-\nu\Delta v_\varepsilon + \nabla p_\varepsilon = f \text{ in } Q,$$

$$\mathrm{div}\, v_\varepsilon + \varepsilon p_\varepsilon = 0 \text{ in } \Omega, \qquad v_\varepsilon = 0 \text{ on } \Gamma.$$

- Temam (1977), Bercovier (1978) - Approximation error for the stationary Stokes equation

$$\|v - v_\varepsilon\|_{(H^1(\Omega))^3} + \|p - p_\varepsilon\|_{L^2(\Omega)} \le C\,\varepsilon\,\|f\|_{(H^{-1}(\Omega))^3}.$$

- Temam (1977), Hebecker (1982), Shen (1995) (instationary Stokes equation)

$$\|v - v_\varepsilon\|_{L^2(H^1(\Omega))\cap L^\infty(L^2(\Omega))} + \|\mathrm{div}\, v_\varepsilon\|_{L^\infty(L^2(\Omega))}$$
$$\le C\,\varepsilon^{1/2}\left(\|f\|_{L^2((H^{-1}(\Omega))^3)} + \|y_0\|_{(L^2(\Omega))^3}\right).$$

Uniform coervivity condition

The stationary solution (w_s, ρ_s) belongs to $(H^1(\Omega))^3 \times L^2(\Omega)$.
For all v_s satisfying the H^1-bound

$$\|v_s\|_{(H^1(\Omega))^3} \leq \|w_s\|_{(H^1(\Omega))^3} + 1,$$

we set

$$a_{v_s}(z, \zeta) = \int_\Omega \left(\nu \nabla z : \nabla \zeta + (v_s \cdot \nabla) z \cdot \zeta + (z \cdot \nabla) v_s \cdot \zeta\right) \, dx,$$

for all $z \in (H^1(\Omega))^3$, $\zeta \in (H^1(\Omega))^3$.

We can choose $\omega_0 > 0$ such that

$$\omega_0 \|z\|^2_{(L^2(\Omega))^3} + a_{v_s}(z, z) \geq \frac{\nu}{2} \|z\|^2_{(H^1(\Omega))^3},$$

for all $z \in (H^1(\Omega))^3$ and all v_s satisfying the H^1-bound.

Analyticity - Resolvent estimate

The operator $(A, \mathcal{D}(A))$ is the infinitesimal generator of an analytic semigroup on $Z = V_n^0(\Omega)$. There exists a sector $\{\omega_0\} + \mathbb{S}_{\pi/2+\delta}$, with $\delta \in]0, \pi/2[$, such that

$$\{\omega_0\} + \mathbb{S}_{\pi/2+\delta} \subset \rho(A),$$

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(Z)} \leq \frac{C}{|\lambda - \omega_0|} \quad \text{for all } \lambda \in \{\omega_0\} + \mathbb{S}_{\pi/2+\delta}.$$

We assume that $\|w_s^\varepsilon - w_s\|_{(H^1(\Omega))^3} \leq C_s\,\varepsilon, \quad \forall \varepsilon \in (0,1).$
We set $\varepsilon_0 = 1/C_s$. The pseudo-compressible Oseen operator A_ε is

$$\mathcal{D}(A_\varepsilon) = (H^2(\Omega) \cap H_0^1(\Omega))^3,$$
$$A_\varepsilon y = \nu\Delta y - (y\cdot\nabla)w_s^\varepsilon - (w_s^\varepsilon\cdot\nabla)y + \tfrac{1}{\varepsilon}\nabla(\mathrm{div}\, y).$$

The pseudo-compressible system can be rewritten in the form

$$y_\varepsilon' = A_\varepsilon y_\varepsilon + B_\varepsilon u, \quad y_\varepsilon(0) = y_0, \quad \text{with } B_\varepsilon = \chi_{\mathcal{O}}.$$

For all $\varepsilon \in (0,\varepsilon_0)$, the operator $(A_\varepsilon, \mathcal{D}(A_\varepsilon))$ is the infinitesimal generator of an analytic semigroup on $(L^2(\Omega))^3$. We have

$$\{\omega_0\} + \mathbb{S}_{\pi/2+\delta} \subset \rho(A_\varepsilon),$$
$$\|(\lambda I - A_\varepsilon)^{-1}\|_{\mathcal{L}(Z_\varepsilon)} \leq \tfrac{C}{|\lambda-\omega_0|} \quad \text{for all } \lambda \in \{\omega_0\} + \mathbb{S}_{\pi/2+\delta},$$

for all $\varepsilon \in (0,\varepsilon_0)$.

- The following bounds hold, uniformly in $\varepsilon \in (0, \varepsilon_0)$:

$$\|y\|_{(H^2(\Omega))^3} + \frac{1}{\varepsilon}\|\mathrm{div}\, y\|_{H^1(\Omega)} \leq C\|(\lambda_0 I - A_\varepsilon)y\|_{(L^2(\Omega))^3}, \ \forall z \in \mathcal{D}(A_\varepsilon),$$
$$\|\phi\|_{(H^2(\Omega))^3} + \frac{1}{\varepsilon}\|\mathrm{div}\, \phi\|_{H^1(\Omega)} \leq C\|(\lambda_0 I - A_\varepsilon^*)\phi\|_{(L^2(\Omega))^3}, \ \forall \phi \in \mathcal{D}(A_\varepsilon^*).$$

proved by rewriting the divergence eq. as for the imcompressible case.

- The following approximation property holds:

$$\|(\lambda_0 I - A)^{-1}P - (\lambda_0 I - A_\varepsilon)^{-1}\|_{\mathcal{L}((L^2(\Omega))^3)} \leq C\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0).$$

- The control operators B and B_ε satisfy

$$\|(\lambda_0 I - A)^{-1}B - (\lambda_0 I - A_\varepsilon)^{-1}B_\varepsilon\|_{\mathcal{L}((L^2(\Omega))^3)} \leq C\varepsilon, \quad \forall \varepsilon \in (0, \varepsilon_0),$$

$$(\lambda_0 I - A)^{-1}B - (\lambda_0 I - A_\varepsilon)^{-1}B_\varepsilon = [(\lambda_0 I - A)^{-1} - (\lambda_0 I - A_\varepsilon)^{-1}]\chi_{\mathcal{O}}.$$

Convergence rate of A towards A_ε

- $y = (\lambda_0 I - A)^{-1} P f$ is solution of

$$\lambda_0 y - \nu\Delta y + (y\cdot\nabla)w_s + (w_s\cdot\nabla)y + \nabla p = f \quad \text{in } \Omega,$$
$$\operatorname{div} y = 0 \text{ in } \Omega, \qquad y = 0 \quad \text{on } \Gamma.$$

- $y^\varepsilon = (\lambda_0 I - A_{w_s^\varepsilon})^{-1} P f$ is solution of

$$\lambda_0 y - \nu\Delta y + (y\cdot\nabla)w_s^\varepsilon + (w_s^\varepsilon\cdot\nabla)y + \nabla q = f \quad \text{in } \Omega,$$
$$\operatorname{div} y = 0 \text{ in } \Omega, \qquad y = 0 \quad \text{on } \Gamma.$$

- $y_\varepsilon = (\lambda_0 I - A_\varepsilon)^{-1} f$ is solution of

$$\lambda_0 y_\varepsilon - \nu\Delta y_\varepsilon + (y_\varepsilon\cdot\nabla)w_s^\varepsilon + (w_s^\varepsilon\cdot\nabla)y_\varepsilon + \nabla q_\varepsilon = f \quad \text{in } \Omega,$$
$$\operatorname{div} y_\varepsilon + \varepsilon q_\varepsilon = 0 \text{ in } \Omega, \qquad y_\varepsilon = 0 \quad \text{on } \Gamma.$$

$\|y - y^\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)}$ can be estimated with regularity results for the Oseen system and with the estimate on $\|w_s - w_s^\varepsilon\|_{H^1(\Omega;\mathbb{R}^3)}$.

Estimate of $\|y_\varepsilon - y^\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)}$

The differences $z_\varepsilon = y_\varepsilon - y^\varepsilon$ and $p_\varepsilon = q_\varepsilon - q$ obey

$$\lambda_0 z_\varepsilon - \nu\Delta z_\varepsilon + (z_\varepsilon\cdot\nabla)w_s^\varepsilon + (w_s^\varepsilon\cdot\nabla)z_\varepsilon + \nabla p_\varepsilon = 0 \ \text{ in } \Omega,$$
$$\operatorname{div} z_\varepsilon + \varepsilon p_\varepsilon = -\varepsilon q \text{ in } \Omega, \quad z_\varepsilon = 0 \ \text{ on } \Gamma.$$

With the adjoint system

$$\lambda_0\Phi_\varepsilon - \nu\Delta\Phi_\varepsilon + (\nabla w_s^\varepsilon)^T\Phi_\varepsilon - (w_s^\varepsilon\cdot\nabla)\Phi_\varepsilon + \nabla\psi_\varepsilon - \operatorname{div}(w_s^\varepsilon)\Phi_\varepsilon = y_\varepsilon - y^\varepsilon \ \text{ in } \Omega,$$
$$\operatorname{div}\Phi_\varepsilon + \varepsilon\psi_\varepsilon = 0 \text{ in } \Omega, \quad \Phi_\varepsilon = 0 \ \text{ on } \Gamma,$$

we obtain

$$\int_\Omega |y_\varepsilon - y^\varepsilon|^2 \mathrm{d}x = \varepsilon\int_\Omega q\psi_\varepsilon \mathrm{d}x$$
$$\le \varepsilon\|q\|_{L^2(\Omega)}\|\psi_\varepsilon\|_{L^2(\Omega)} \le C\,\varepsilon\|y_\varepsilon - y^\varepsilon\|_{L^2(\Omega)}.$$

Convergence rate for Riccati based feedback gains

- $F_\varepsilon = -\Pi_\varepsilon$. There exist $\omega^* > 0$ and $\varepsilon_0 \in (0,1)$ such that

$$\sup_{\varepsilon\in(0,\varepsilon_0)} \|e^{(A_\varepsilon - B_\varepsilon \Pi_\varepsilon)t}\|_{\mathcal{L}(Z_\varepsilon)} \leq Ce^{-\omega^* t}, \quad \forall t \geq 0.$$

- We have

$$\|\Pi P - \Pi_\varepsilon\|_{\mathcal{L}(H)} \leq C\varepsilon\,|\ln(\varepsilon)|, \quad \forall \varepsilon \in (0,\varepsilon_0),$$

and

$$\|B^*\Pi P - B_\varepsilon^*\Pi_\varepsilon\|_{\mathcal{L}(H,U)} \leq C\varepsilon|\ln\varepsilon|, \quad \forall \varepsilon \in (0,\varepsilon_0).$$

Convergence rates for the closed-loop systems

- $\widehat{z}(t) = e^{t(A+BF)}z_0, \quad F = -\Pi.$

  $\|e^{t(A+BF)}\|_{\mathcal{L}(Z)} \leq Ce^{-\omega_\Pi t}, \quad \forall t \geq 0.$
- $\widetilde{z}(t) = e^{t(A+BF_\varepsilon)}z_0, \quad F_\varepsilon = -\Pi_\varepsilon.$
- $\widehat{z}_\varepsilon(t) = e^{t(A_\varepsilon+B_\varepsilon F_\varepsilon)}z_0, \quad F_\varepsilon = -\Pi_\varepsilon.$

For all $\varepsilon \in (0, \varepsilon_0)$, we have

$$\|\widehat{z}(t) - \widehat{z}_\varepsilon(t)\|_H \leq C\,\frac{e^{(-\omega_\Pi+\varrho\varepsilon|\ln\varepsilon|)t}}{t}\,\varepsilon|\ln\varepsilon|\|y_0\|_H,$$

$$\|\widehat{z}_\varepsilon - \widehat{z}\|_{L^p(0,\infty;H)} \leq C_p\,\varepsilon^{1/p}|\ln\varepsilon|^{1/p}\|y_0\|_H, \quad \forall p \in (1,\infty),$$

$$\|\widehat{z}_\varepsilon - \widetilde{z}\|_{L^p(0,\infty;H)} \leq C_p\,\varepsilon^{1/p}\|y_0\|_H, \quad \forall p \in (1,\infty).$$

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Thank you for your attention