Prologue: (Rational points on conics and beyond)

The purpose of this module is to explore rational points on geometric curves through algebraic means. Existence of such rational points, as we will see, gives rise to solutions of Diophantine equations. This is a beautiful topic where geometry and algebra work hand in hand; geometry provides intuition while algebraic methods, such as Linear Algebra, provide powerful tools.

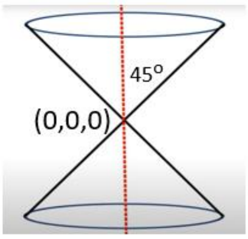
Our exploration will start in a most elementary way, but over several sessions we will scale and explore much higher levels of sophistication both conceptually and methodologically.

So hopefully hang tight and easy, and enjoy.

There are five PROBLEMS in the worksheet. The problem statements are intentionally elaborate to help you develop a feel and to give you hints. Also you can first skim over all the problems and start where you please.

Problem 4 is the most challenging in my opinion. Problem 5 is fun and creates space for the next session. Try that too.

1. To begin, let us revisit the familiar setting of conics in plane geometry. Conics are geometric objects that are also wonderfully described by algebraic equations which help to explore many of their properties. We will keep weaving algebra and geometry.



a. When two surfaces intersect, you expect the intersection to be some sort of a curve. In the case of intersection of a cone (apex at the origin as shown above) and a plane that you can move, tilt etc. LIST AND CLASSIFY ALL THE CURVES that you may get. Your answer should be a classification of these curves in geometric terms.

b. Convince yourself (DO the detailed work) that algebraically this intersection will give rise to a quadratic equation tin two variables

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F \ = 0$$

where A, B, C…. are real numbers. (Later we will or may allow these to belong to a Field.)

For later convenience we will rewrite the above equation as

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F \ = 0$$

Later we may consider the case where A, B, C are all rational, in which case the we will call the resulting conics, if any, rational conics.

For convenience later, we will arrange the coefficients in a 3x3 matrix and call it the Q matrix as follows (Note the symmetry of the matrix)

$$Q = \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix}$$

2. In this problem we will explore how conics transform under coordinate transformations. Recall that under translation and rotation in a plane, all distances are preserved, and a close curve like a triangle is transported to a congruent triangle. We want to similarly understand how conic sections transform under these transformations. Later we may add scaling to these. Also keep in mind that there are two types of quadratic expressions in x and y; those that are factorizable in linear expressions and those which are not. We will call the later non-degenerate conics.

a. For example, consider the following curves:

   i. $x^2 - 2xy + y^2 = 0,$

   ii. $x^2 - 3xy + 2y^2 - x + 3y - 2 = 0,$

   iii. $x^2 - 2xy + y^2 - 3x = 0$

   iv. $x^2 - 2xy + y^2 - 3x - 3y = 0$

   v. $x^2 - 2xy + y^2 - 3x - 3y + 3 = 0$

   (You can use Geogebra or equivalent to see how the curves are changing)

   Are all these curves "same"? Whether they are or are not, how can we know without plotting just by examining the coefficients? We will examine this in greater detail in the next question.

a. In part a.) suggest a few simple changes to the fifth equation that will change the above curve to a closed curve. What kind of curve will it be?

b. From the particular examples iii, iv, and v above, what is your conjecture as to which of the six coefficients A, B,... determine the nature of the curve?

c. In part a.) suppose we scale one of the axes, will that produce new types of curves? Will they be congruent to the old curves? If yes, why.. If not, why not.

As a preparation for the next problem, go back to the equation

$$Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$$

Can you identify any invariant quantities involving A, B, C that is important for the shape of the curve? By invariant we mean that this combination is not changed if we translate or rotate.

3. So, clearly, for the general quadratic equations the coefficients carry information about the curve and there are quantities made out of these coefficients that are invariant under these transformations.

Given that these are linear transformation, Linear Algebra and Matrices are natural framework for these. We will explore these now.

First, a deep notational and conceptual improvement. The given equation is not homogeneous; it has quadratic as well as linear and constant terms, We can use homogeneous coordinates (x,y,z). Replacing x and y by x/z, and y/z the quadratic equation in x, y becomes $Ax^2 + 2Bxy + Cy^2 + 2Dxz + 2Eyz + Fz^2 = 0$ and it is exactly the original equation when z= 1.

a.) Show that the original equation can be rewritten using matrix notation as

$$\begin{pmatrix} x & y & 1 \end{pmatrix} \begin{pmatrix} A & B & D \\ B & C & E \\ D & E & F \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = 0$$

or

$$\mathbf{x}^T Q \mathbf{x} = 0 \quad \text{where} \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Obviously a few questions will arise: what is the determinant of this matrix, what are the invariants under rotation and translation. Before doing that lets us fix the homogeneous case once and for all when the quadratic equation is factorizable into linear equations.

b. Calculate Q for a few simple cases such as

i.) (x-y+1) (x+y) = 0

ii.) (x+y-1)(x+1) = 0

iii.) (x−2y+1)(x−y−2)=0

Do you see a common feature (Hint calculate the determinant)

c. We will denote det Q as J_3. Later we will see the this is an invariant of this problem

FOR NOW try to prove the quadratic is factorizable Implies J_3= 0.Then prove the converse. (Try and we will discuss if you find this challenging)

4. **NOTE: THIS IS THE MOST challenging part of this worksheet. We will discuss these in detail.**

   Now we return to the transformations of the plane under rotation and translation but using rational coordinates as before. Let us call represent such a transformation by a 3x3 matrix T. Consider the following specific T as an example

$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 5 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   a. How does a two dim vector in a plane is transform by the above T.
   b. Give a general form of T. For this purpose you can use 2x2 rotation matrix R given by either by

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ or by $$R(t) = \frac{1}{1+t^2}\begin{pmatrix} 1-t^2 & -2t \\ 2t & 1-t^2 \end{pmatrix}$$

   c. Show that det T is 1 and that under T, Q transforms as

$$Q' = T^T Q T$$ where T^T is the transport of the T matrix.

   d. Show that the following quantities are invariant under this transformation J_1, J_2 , J_3 given by $J_1 = AC-B^2$, $J_2 = A + C$ and $J_3 = \det Q$.
   e. Therefore complete the classification of non-degenerate conics and show that all such conics can be written either as

$$A'x'^2 + C'y'^2 + F' = 0$$

   or as $$C'y'^2 + 2D'x = 0$$

   f. Discuss the group structure of T. Compare with the group of rotation of three dimensional vectors, Is T a subgroup of the later?
   g. EXTEND the above discussion to include scaling and discuss the group structure of T.

5. Finally we come to Rational points. Since we will be discussing this in great detail in the second lecture of this series, here are a few teasers.
    a. Show that Pythagorean triplets correspond to rational point on the unit circle. Show that there are infinite number of rational points on the unit circle and gave a way to construct this starting with one such point. THINK GEOMETRICALLY.
    b. Show that there are ellipses and hyperbolas which are rational conics but do not have a single rational point