# Topological Objects Near Tc: Instanton-dyons in Theory and on Lattice

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# Overview

- Topological objects:
  - Why are topological objects interesting
  - How does topological objects look like
  - Why do we need Instanton-dyons
  - How does Instanton-dyons look like
- Topological objects on the Lattice:
  - How do we see topological object on the lattice
  - The overlap Dirac operator
  - Expected temperature dependence of Topological objects
- Comparison between lattice and theory:
  - How does the shape of zero-modes compare
  - Results from fits to lattice data
- Ensembles of dyons
  - Effective interactions
  - Temperature dependence of Polyakov loop and Chiral Condensate

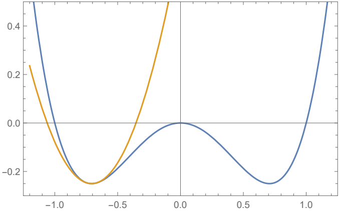
# Topological objects

# Path Integrals

- We want to describe a system governed by a path integral

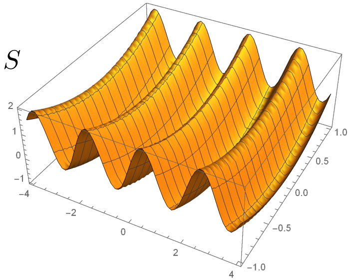
$$Z = \int D\psi D\bar{\psi} DA_\mu \exp(-S(\psi, \bar{\psi}, A_\mu)) \tag{1}$$

- Tough integral, with many different minimas contributing
- Perturbative results expands around minimum $A_\mu = 0$



# Why are topological objects interesting

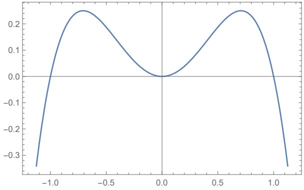
- Topological objects correspond to real time transition between different minima




- True state of system, like energy, changed by transitions, like in the double well potential
  - Important to include effect of transitions in order to understand the system
  - Perturbative calculation that expands around 1 minima will not see full picture

# Why are topological objects interesting 2

- In complex time, topological objects are local minimas
- In the below example the instanton is the path that start at one maximum and just barely reaches the other maximum



# Why are topological objects interesting 3

- In QCD in complex time, action is always positive
- Action bounded by ( [Dmitri Diakonov Arxiv:hep-ph/0906.2456]):

$$0 \leq \int dx^4 Tr\left(F_{\mu\nu} - \tilde{F_{\mu\nu}}\right)^2 \quad (2)$$

$$\tilde{F^a_{\mu\nu}} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^a_{\alpha\beta} \quad (3)$$

- Minimum is found when action S

$$S = \frac{8\pi^2}{g^2}|Q_t| \quad (4)$$

$$Q_t = \frac{1}{16\pi^2}\int dx^4 Tr[F_{\mu\nu}\tilde{F_{\mu\nu}}] \quad (5)$$

- $Q_t$ is the 4-dimensional topological charge number
- QCD in complex time therefore contains several minimums with an action proportional to $\frac{8\pi^2}{g^2}$
- Contribution of each local minimum has to be included

# Semi-classical description

- Expand around local minimum as $A_\mu = A_{\mu,top} + a_\mu$
- Integrate out $a_\mu$

$$Z \approx \int DA_\mu \exp[-S(A_{\mu,top}) - a_\mu(x)M(x,x')a_\mu(x')] \quad (6)$$

$$= \frac{1}{\sqrt{Det(M)}} e^{-\frac{8\pi^2}{g^2}|Q_t|} \quad (7)$$

- Certain directions in phase space does not change the action $\rightarrow$ eigenvalue of M is 0
- Zero-mode contribution proportional to Volume they move in

$$Z = C(g)e^{-\frac{8\pi^2}{g^2}|Q_t|}V \quad (8)$$

# Semi-classical description 2

- Well separated $Q_t = 2$ the same as 2 $Q_t = 1$ solutions
- Sum over all possible minimums

$$Z = \sum_n \frac{(C(g)e^{-\frac{8\pi^2}{g^2}}V)^n}{n!} \quad (9)$$

- Largest contribution not necessary around $n = 0$ due to large factor of volume
- Depend on coupling constant $g$ which is small at large temperature and large at small temperature
- $n!$ is to remove double counting

# Semi-classical description 3

- Topological objects affect the interactions of the fermions

$$D_\mu = \partial_\mu + igA_\mu \quad (10)$$

$$Z \propto \int DA_\mu Det(D+m)\exp(-S(A_\mu)) \quad (11)$$

- Contribution of Dirac operator given by $\Pi_i(m+i\lambda_i)$
- The amount of zero modes ($\lambda_i = 0$) are equal to $|Q_t|$
- Objects like Chiral condensate depends on inverse of Dirac operator

# Anti-Instantons and Chiral Condendate

- Anti-topological solutions are anti-selfdual $(F^a_{\mu\nu} = -\tilde{F^a_{\mu\nu}})$
- Cannot be self-dual and anti-selfdual at the same time
- Breaks fermionic zero-modes

- Models of weakly interacting Instantons and anti-instantons has shown to create a region of small eigenvalues due to the breaking of the zero mode, creating a chiral condensate

- Chiral condensate given by eigenvalue density at zero
- Banks-Casher relation

$$\Sigma = \pi \lim_{\lambda \to 0} \lim_{m \to 0} \lim_{V \to \infty} \rho(\lambda) \tag{12}$$

# How does topological objects look like

- The simplest solution one can construct is the $Q_t = 1$ configuration at zero temperature. The instanton
- One way to find the instanton is to find a selfdual $(F^a_{\mu\nu} = \tilde{F^a_{\mu\nu}})$ object that is a local minima
- [Dmitri Diakonov Arxiv:hep-ph/0906.2456]

$$A^a_\mu = \frac{2\bar{\eta}^a_{\mu\nu} x_\nu \rho^2}{x^2(x^2+\rho^2)} \tag{13}$$

$$F^a_{\mu\nu}F^a_{\mu\nu} \propto \frac{\rho^4}{(x^2+\rho^2)^4} \tag{14}$$

- This is a local object with a maximum at the origin, with a size defined by $\rho$
- The Dirac operator in the presence of this field has 1 zero mode, that sit also at the origin
- $\eta$ 't Hooft symbol

# Why do we need Instanton-dyons

- Instantons need to be generalized to finite temperature
  - This is called the Caloron
  - Caloron is an infinity sum of Instantons separated by $1/T$
- Instantons have Polyakov loop expectation value 1

$$P = Tr(Path(\exp(i\int_0^{1/T} A_4 dt))) \quad (15)$$

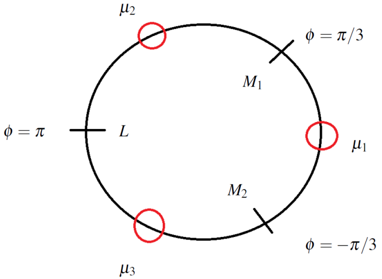
- In pure gauge, the Polyakov loop is related to confinement
- With fermions, the connections is not as clear, but the topological object still need to follow the behavior of the Polyakov loop (chicken or egg question)

# How does Instanton-dyons look like

- The generalized Caloron is constructed through ADHM construction, by requiring the Polyakov loop to be able to take any value [Thomas C. Kraan and Pierre van Baal arXiv:hep-th/$9805168v1$]
- This is introduced through $N_c$ angles $\mu_i$ that correspond to the eigenvalues of the Polyakov loop

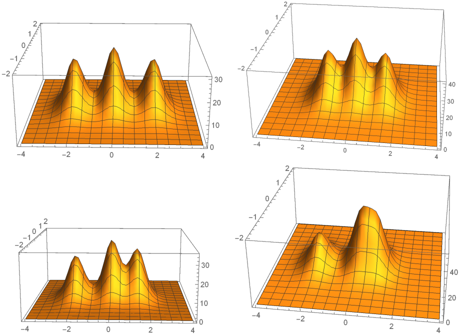
$$P = \exp(2\pi i * diag(\mu_1, \mu_2, ..., \mu_{Nc}))$$

$$\psi(\tau + 1/T) = \exp(i\phi)\psi(\tau)$$




- These angles are periodic, and therefore live on a circle
- Each region depends on individual coordinates that can be interpreted as a position of an object
- Caloron therefore can be seen as being composed of $N_c$ objects, which we call Instanton-dyons

- We therefore get a picture of $N_c$ object inside one Caloron Solution
- Figures: Density of the instanton-dyons in x-y plane




- The sum of Instanton-dyons Add up to 1 Caloron
  - $Q_t = 1$
  - $S = 8\pi^2/g^2$
  - 1 fermionic zero mode
  - When well separated, each dyon holds a fraction proportional to $\nu_i = \mu_{i+1} - \mu_i$

# Topological objects on the Lattice

# How do we see topological object on the lattice

- Direct:
  - Measure $F\tilde{F}$ on lattice
  - Pros:
    - See all Topological objects
  - Problem:
    - No simply way to use Link
    - UV noise needs to be removed with cooling
    - Cooling affects topological objects
- Indirect:
  - Find fermionic zero-modes
  - Compare shape from lattice with zero-modes from analytic formula
  - Pros:
    - No need for cooling
  - Problem:
    - Only see some topological objects

# The overlap Dirac operator

- Needs exact zero-modes $->$ overlap Dirac operator

$$D_{ov} = 1 - \gamma_5 sign(H_W) \;, H_W = \gamma_5(M - aD_W) \tag{16}$$

- Obey the Ginsparg-Wilson relation within numerical precision $(10^{-9})$

$$\gamma_5 D_{ov}^{-1} + D_{ov}^{-1}\gamma_5 \quad = \quad \gamma_5 \tag{17}$$

- Operator is not local
- Expensive due to having to calculate the sign
- Can make boundary condition anything we want

# Expected temperature dependence of Topological objects

- Low temperature
  - Polyakov loop small
    - Polyakov loop eigenvalues close to 0, 1/3, 2/3
  - Coupling g large
    - Many dyons expected
- High temperature
  - Polyakov loop close to 1
    - Polyakov loop eigenvalues close to 0, 0, 0
    - Standard Caloron should work
  - Coupling g small
    - Few dyons expected

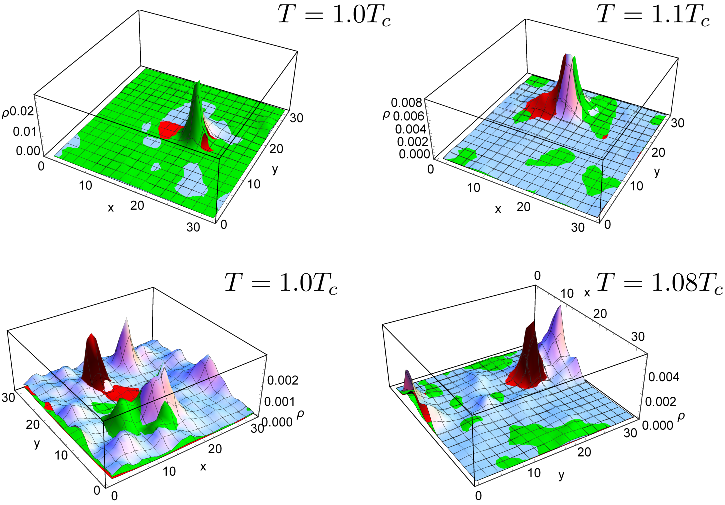
# The overlap Dirac operator 2

- We explore the range $T = 1 - 1.2T_c$
- Configurations was generated with Physical masses using domain wall fermions
- Size: $N_s = 32$ and $N_\tau = 8$
- We find the zero-modes using the overlap operator with zero fermion mass
- Zero-modes appear alone
- near zero-modes ($\lambda$ around $10^{-6}$) appears in pairs
- $N_c = 3$

- Will explore 3 boundary conditions:
  - $\phi/(2\pi) = 1/2$, $1/6$, $5/6$
  - Each value of $\phi$ centered in region of suspected dyon
  - $L$, $M_1$, $M_2$

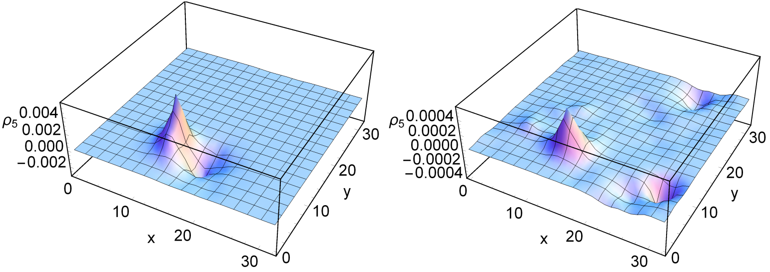
# Zero-modes of the overlap Dirac operator

- Zero-mode density $\rho$ examples (sum over t)
- Boundary condition is, red $\phi = 1/2$, blue $\phi = 1/6$, green $\phi = -1/6$



# Near-zero-modes of the overlap Dirac operator

- We can also look at near-zero-modes
- Chiral density $\rho_5$
- Smallest eigenvalues expected to be dominated by topological objects




- Left: $\phi = 1/2$, right $\phi = 1/6$
- $T = 1.1T_c$
- Shows difference in distribution of different dyons

- Rasmus Larsen, Sayantan Sharma, Edward Shuryak, arxiv:hep-lat/$1912.09141$

# Comparison between lattice and theory

# Analytic fitted to lattice zero-modes

- Fit analytic formula to the lattice calculated zero-modes by minimizing $\chi^2$
- Error bars assumed to be same for all data (Since there are no error bars from the modes)
- Fitted with $9^3 * 8$ points (entire t direction)
- Fit are only done on cases where individual peaks are observed
- Configurations have $|Q_t| \leq 3$
  - Even though $Q_t$ is 1, the zero-modes can (and most is) comprised of several peaks, though often far away from each other (we fit around largest peak)
  - Even though $Q_t$ is 1, we expect many more dyons in the system, but difference in number of dyons and anti-dyons to be 1 (for each type of dyon)

# Analytic density of dyon zero-modes

- Need to calculate zero-modes of Dirac operator from Instanton-dyons and from Lattice
- We will look at density of zero-modes, since it is gauge invariant
- Instanton-dyon zero-mode found from solving:

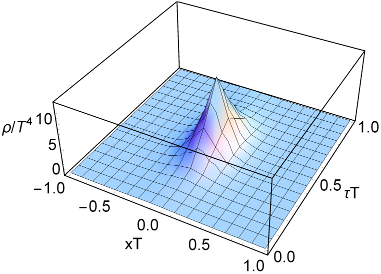
$$\rho(x) = -\frac{1}{4\pi^2}\partial_\mu^2 f_x(\phi,\phi) \ , \tag{18}$$

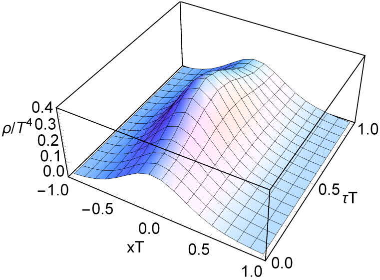
- [Pierre van Baal Arxiv:hep-lat/9907001]
- $f_x$ lives on the circle, and constants are dependent on the dyon positions

$$\left(D_\phi^2 + r^2(x,\phi) + \sum_{m=1}^{N_c} \delta_m(\phi)\right) f_x(\phi,\phi') = \delta(\phi-\phi') \ . \tag{19}$$

# Zero-modes of Instanton-dyons

- zero-mode: Short range behavior dependent on distance between dyons
- zero-mode: Long range behavior dependent on $\mu - \phi$
- Left: Dyons close
- Right: Dyons Far

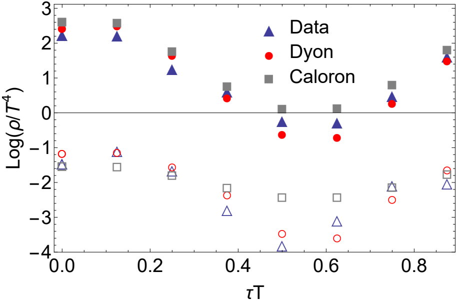



- Periodic in $\tau$ direction

# How does the Shape of zero-modes compare

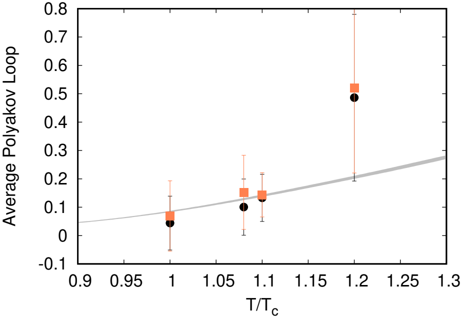
- We fit the analytic formula to the lattice **data** zero-modes by minimizing $\chi^2$
- Figure: 1D slice along $\tau$ direction of 4D density fit
- Filled points: (x,y,z) at the center of the zero-mode
- Open points: (x,y,z) far away from the center of the zero-mode




- Short distance (upper): Dyon and Caloron able to explain reasonably
- Long distance (lower): Dyon describes behavior around minimum better

# How is the predicted Polyakov Loop

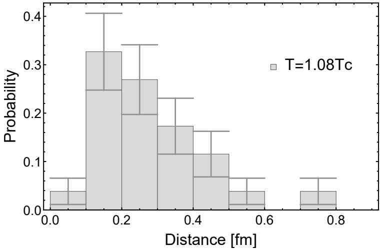
- From the $\chi^2$ fit we obtain position of dyons and eigenvalue of Polyakov loop for the specific configuration
- Can recreate Polyakov loop expectation value from fit
- Low statistics for $T = 1.2T_c$
- Different points are for different starting conditions in $\chi^2$ fit

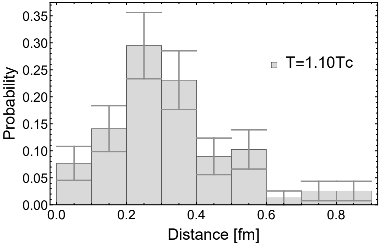



- Grey line is lattice results

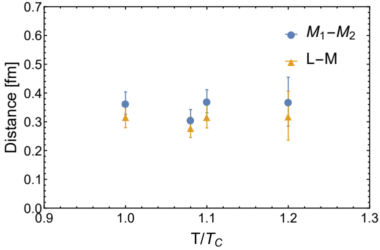
# Other results from fits

- Distance distribution to other dyons inside the caloron




- Average distance between dyons inside the caloron



# Ensembles of dyons

# Ensembles of dyons

- Describe the possible combinations of dyons and their contributions to the path integral

$$
\begin{aligned}
Z &= \sum_{n_L, n_{M1}, n_{M2}} \frac{(C(g,\nu_L) e^{-\nu_L \frac{8\pi^2}{g^2}} V_3)^{n_L}}{n_L!} \\
&* \frac{(C(g,\nu_{M1}) e^{-\nu_{M1} \frac{8\pi^2}{g^2}} V_3)^{n_{M1}}}{n_{M1}!} * \frac{(C(g,\nu_{M2}) e^{-\nu_{M2} \frac{8\pi^2}{g^2}} V_3)^{n_{M2}}}{n_{M2}!} \\
&* (Anti - dyon - contributions) * Exp[-\Delta S]
\end{aligned} \tag{20}
$$

- $\Delta S$ are corrections from interactions of all dyons and anti-dyons
- Typically assumes that amount of dyons and anti-dyon are the same
- Amount of $L$, $M_1$ and $M_2$ dyons not necessarily the same

# Interactions

- A dyon and anti-dyon is not a self-dual solution, creates difference compared to action of dyon

$$\Delta S_{class}^{d\bar{d}} = -\frac{S_0 C_{d\bar{d}}}{2\pi}(\frac{1}{rT} - 2.75\pi\sqrt{\nu_i\nu_j}e^{-1.408\pi\sqrt{\nu_i\nu_j}rT}) \quad (21)$$

- A repulsive core is used for all types of dyons

$$\Delta S_{class}^{core} = \frac{\nu V_0}{1 + e^{2\pi\nu T(r-r_0)}} \quad (22)$$

- Introducing holonomy cost action
- Assumed $\nu = \nu_{M1} = \nu_{M2}$

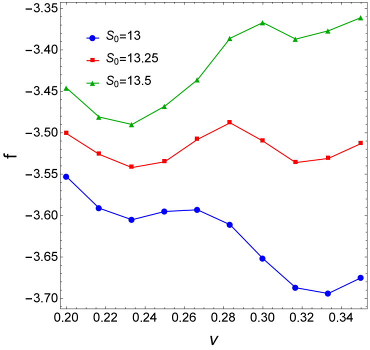
$$\Delta S_\nu = \frac{4\pi^2}{3}(2[\nu(1-\nu)]^2 + [2\nu(1-2\nu)]^2) - N_f\frac{4\pi^2}{3}(2\nu^4 - \nu^2) \quad (23)$$

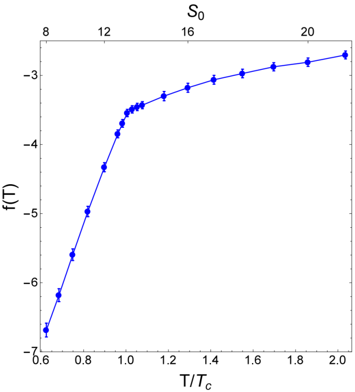
# Simulation

- Simulation of $\Delta S$ done using Monte-Carlo simulation with 120 dyons
- Input:
  - Holonomy $\nu = \nu_{M1} = \nu_{M2}$
  - Density of dyons $n_L$ and $n_M$
  - Action of one instanton $S_0 = \frac{8\pi^2}{g^2}$
- Temperature found from running of coupling constant g
- Dominating configuration has smallest free energy density $f = -ln(Z)/V$

# Pure Gauge Ensemble

- 2 minimums exist, one at P=0, $\nu = 0.33$ and one away from this
- At specific Temperature/Action of instantons, dominating minimum changes
- $S_0 = \frac{8\pi^2}{g^2}$, $f = -ln(Z)/V$, $\nu = \nu_{M1} = \nu_{M2}$

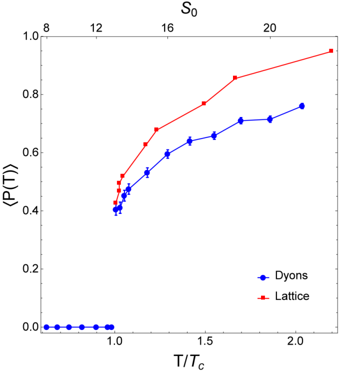


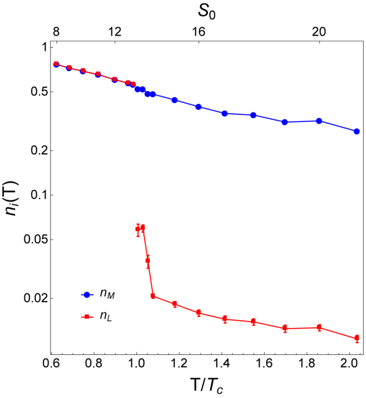




- Dallas DeMartini and Edward Shuryak: arxiv:2102.11321

# Pure Gauge: Polyakov loop and density

- 1.st order transition from jump in $\nu$
- Density of L dyons much smaller at deconfined phase, as it becomes heavier
- $S_0 = \frac{8\pi^2}{g^2}$

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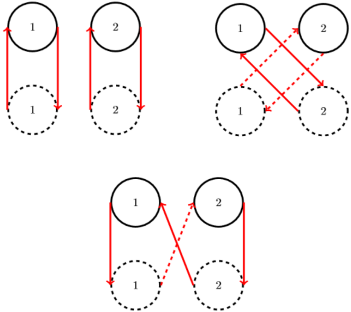
# Quarks in the ensemble

- Quarks included through the zero-mode contribution to the Dirac operator
- Introduces a jump from dyon to anti-dyon
- Amplitude to jump is described by matrix $T_{ij}$

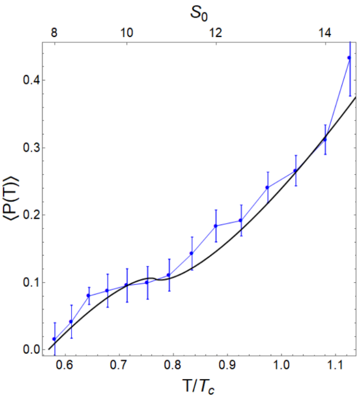
$$T_{ij} = \nu_L c \exp(-\sqrt{11.2 + (\pi \nu_L r T)^2})$$

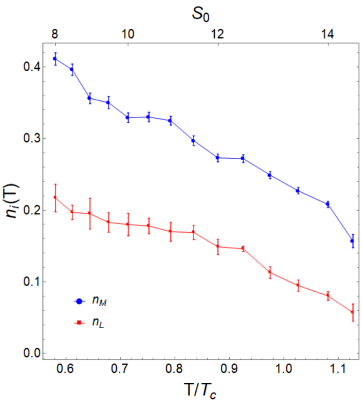
$$\Delta S = -N_f ln(det(T))$$



# Polyakov loop and density

- Transition with quarks are smooth
- Density of L dyons always smaller, due to quarks-zero-modes on L dyons

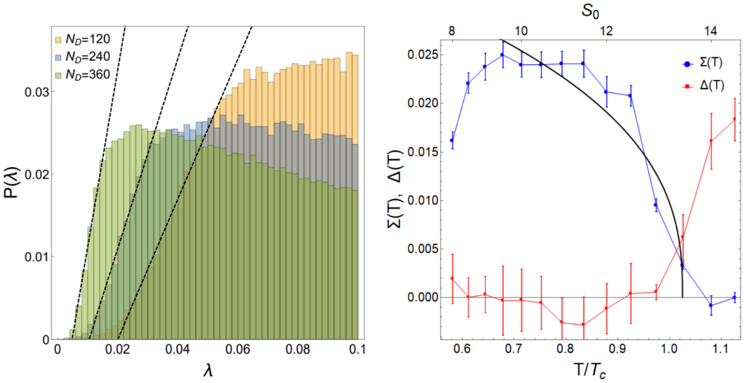



- Dallas DeMartini and Edward Shuryak: arxiv:2108.06353

# Chiral condensate and mass Gap

- Chiral condensate goes to zero and Mass gap appears




- Dallas DeMartini and Edward Shuryak: arxiv:2108.06353

# Summary

- Topological objects represent tunneling between real time vacuum states
- Need to include all (important) minima for precise description
- Instanton-dyons comes from the need to generalize to finite temperature and Polyakov loop different from 1
- Instanton split into $N_c$ fractions, each fraction is a dyon
- Zero-modes shape depend on dyon position and Polyakov loop
- Lattice zero-modes are in good agreement with dyon description
  - Shape well described, no obvious differences, though fluctuation of size $20\%$ observed
  - Polyakov loop reproduced, more statistics needed, especially at higher temperature
- A semi-classical ensemble of dyons in SU(3) done
  - Dominating minimum with smallest free energy changes with temperature
  - Correctly predicts the behavior of the chiral condensate and Polyakov loop