Quantization and Topological Aspects of Renormalization group flows in 2-dim. Quantum Field Theory

Spenta R. Wadia
(ICTS-TIFR)

Infosys Chandrasekharan
Random Geometry Colloquim
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Dedicated to the memory of
Miguel Angel Virasoro
(1940 - 2021)

Talk mainly based on:

S.R. Das, G. Mandal & SRW

"Stochastic differential equations on 2-dim theory space and Morse theory"

MPL A Vol.4 No.8 (1989)

Independent similar work:

C. Vafa:

"c Theorem and the Topology of 2-d QFts"

Phys. Lett B 212 (1988) 28-32

Quantum field theory in 2-dim.

Consider the euclidean space R^2, with coordinates (x_1, x_2) or equivalently $z = x_1 + i x_2$ and $\bar{z} = x_1 - i x_2$.

There exists a symmetric tensor $T_{\mu\nu}(z,\bar{z}) = T_{\nu\mu}(z,\bar{z})$, $\mu,\nu = 1,2$

which is conserved: $\partial_{x^\mu} T_{\mu\nu} = 0$

This is called the energy-momentum tensor and it is a local function of the 'elementary fields' $\phi_i(z,\bar{z})$ of the model, and coupling constants 'g^i'.

Both depend on length scale: $a \geq a_0$ which can be represented in terms of a 'fictitious time' $t = \ln\left(\frac{a}{a_0}\right)$, $0 \leq t < \infty$.

The Lagrangian

One can describe the 2-dim field theory by a Lagrangian which depends on $\phi_i(z,\bar{z})$ and g^i:

$$S_t = \int d^2x \, \mathcal{L}\left(\phi_{i_t}(x), g^i(t)\right)$$

$$\frac{\partial \mathcal{L}}{\partial g^i} = \phi_i \quad , \quad i = 1, 2, \ldots N \ldots\ldots$$

e.g. $\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \sum_{k=1}^{N} g^k \phi_k$

where $\phi_k = \phi^k$

The quantum theory at scale 't' is given by the Feynman path integral

$$Z_t = \int \mathcal{D}\phi_t(x) \, e^{-S_t}$$

Renormalization à la K.G. Wilson

As mentioned above the quantum theory is defined at a given length scale 't'.

If we change the scale $t \to t+\epsilon$ then we expect the fields

$$\{\phi_i\}_t \to \{\tilde{\Phi}_i\}_{t+\epsilon}$$

$$g^i(t) \longrightarrow g^i(t+\epsilon) \qquad i = 1, 2, \ldots$$

So that $Z_t = (\text{Const})\, Z_{t+\epsilon}$

Call the space $\{g^i\} = \mathcal{M}$

The 1-parameter flow in $\mathcal{M}$ is generated by a vector field $\beta^i(g)$

$$\frac{d}{dt} g^i(t) = \beta^i(g), \quad i = 1, 2, \ldots N \ldots$$

These are 1st order RG eqns. that represent a dynamical system in $\mathcal{M}$. (Gell-Mann, Low, Wilson)

Fixed points

$\beta^i(g) = 0 , \qquad g^i = g^{*i} , \quad i = 1, \dots, p.$

In general there can be fixed points, lines, surfaces etc. In this talk we will restrict ourselves to flows with fixed points.

Nbd of fixed points

The fixed points of RG flows describe scale invariant theories. In 2-dim. they are organized by the Virasoro algebra Vir(c):

$$\boxed{[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\,\delta_{n+m,0}}$$

and its reps. c is the 'central charge', associated with the fixed point.

The fields $\phi_i(z,\bar{z})$, can be chosen to carry a representation of Vir(c).

The fields $\phi^i(z,\bar{z})$, can be chosen to define a representation of Vir(c), specified by a pair of scaling dimensions $(h,\bar{h})$:

$$\boxed{\begin{array}{l} \phi_i(\lambda z, \lambda \bar{z}) = \lambda^{-(h_i+\bar{h}_i)}\phi_i(z,\bar{z}) \,, \quad \lambda > 0 \\[2ex] \phi_i(e^{i\theta}z, e^{-i\theta}\bar{z}) = e^{i\theta(h_i-\bar{h}_i)}\phi_i(z,\bar{z}) \\[1ex] h_i+\bar{h}_i \equiv \Delta_i \text{ is the `dimension' of } \phi_i \\[1ex] h_i-\bar{h}_i \equiv s_i \text{ is the spin} \end{array}}$$

$\phi_{i;h,\bar{h}}$ are called 'primary' fields.

They and their descendents $L_{-n_1}L_{-n_2}\cdots L_{-n_k}\phi_{i;h,\bar{h}}(0)|0\rangle$ form the conformal block (Verma module) denoted by $[\phi_{i,h,\bar{h}}]$.

They form a closed operator algebra :

$$\phi^i \times \phi^j = \sum_k C_{ijk} [\phi^k]$$

C_{ijk} are symmetric in i, j, k.

In conclusion, the conformal field theory at the fixed point of the RG is characterised by:

1. A central charge $\underline{c}$ ($c > 0$ by unitarity)

2. A set of primary operators with scaling dim. $\underline{(h, \bar{h})}$ $h + \bar{h} \geq 0$ (unitarity)

3. Structure constants $\underline{C_{ijk}}$ of the operator algebra of primary operators.

e.g. Ising model at critical point:

$$c = \frac{1}{2}, \quad \phi_i = \left\{ \mathbb{1}_{(0,0)},\ \epsilon_{(\frac{1}{2}, \frac{1}{2})},\ \sigma_{(\frac{1}{16}, \frac{1}{16})} \right\}$$

$$\epsilon \times \epsilon = [\mathbb{1}], \quad \epsilon \times \sigma = [\sigma]$$

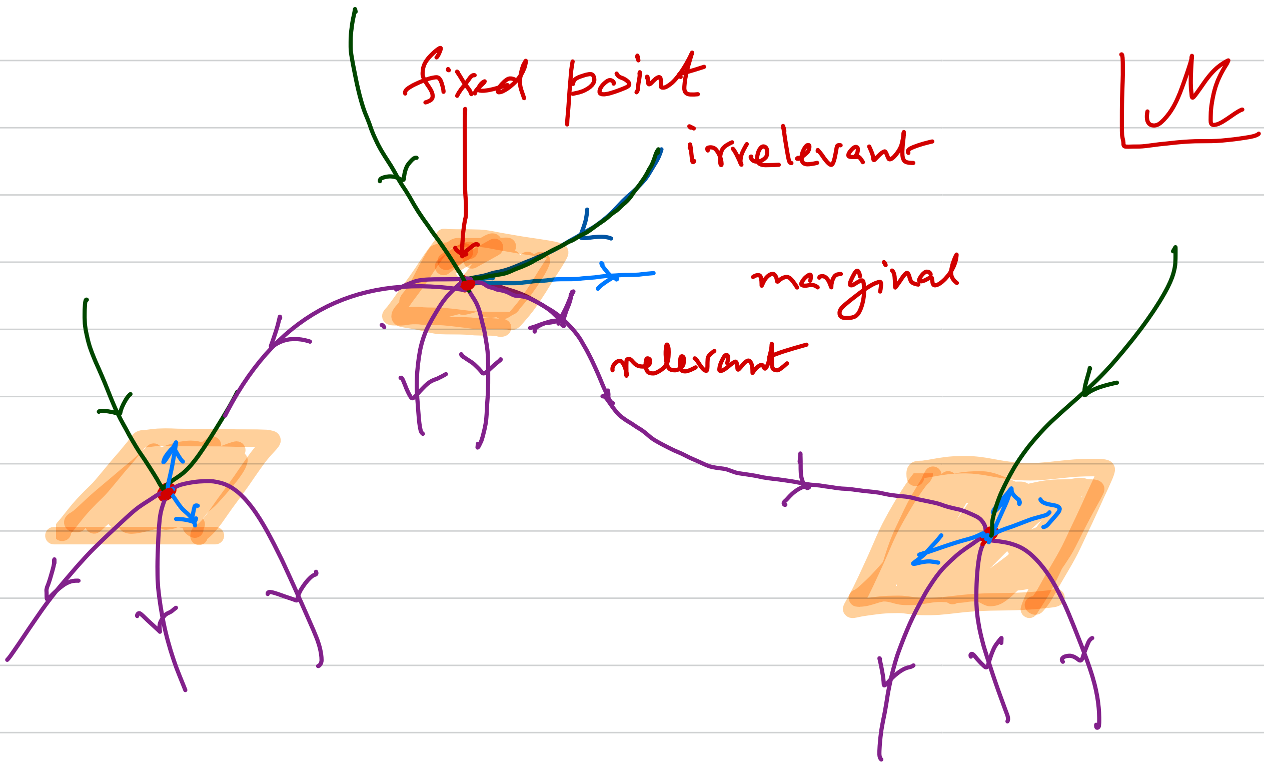
$$\sigma \times \sigma = [\mathbb{1}] + [\epsilon]$$

Relevant, Irrelevant, Marginal fields

1) $h+\bar{h} < 2$ (relevant, unstable ...)

2) $h+\bar{h} > 2$ (irrelevant, stable ...)

3) $h=\bar{h}=0$ (marginal, neutral)



'Cartoon of the 'Space of theories'

The 'tangent space' is spanned by the 'primary operators' of the CFT.

The trajectories connecting the fixed points are solutions of $\dot{g}^i = \beta^i(g)$.

CFT near fixed point :

$$\beta^i(g^*) = 0 , \quad \text{define } \bar{g}^i = (g^i - g^{i*})$$

Hamiltonian is parametrized as :

$$H = H^* + \sum_i \bar{g}^i \phi_i$$

Zamolodchikov introduced a 'metric'

$$G_{ij} = \left[|z|^{\Delta_i + \Delta_j} \phi_i(z,\bar{z}) \, \phi_j(0,0) \right] \Big|_{|z| = x_0}$$

where $x_0 \ni \frac{x_0}{a} \gg 1$.

Near a fixed point one can choose a basis of primary fields $\Rightarrow$

$$G_{ij} = \delta_{ij} + O(\bar{g}^2)$$

$$\beta^i = \sum_i (h_i + \bar{h}_i - 2) \bar{g}^i + \sum_{j,k} C_{ijk} \bar{g}_j \bar{g}_k + O(\bar{g}^3)$$

$$= - G^{ij} \frac{\partial \mathcal{C}}{\partial g^j} ,$$ $\mathcal{C}$ is the c-function (see later).

$$\mathcal{C} = c - \frac{1}{2} \sum_i (h_i + \bar{h}_i - 1)(\bar{g}^i)^2 - \frac{1}{3} \sum_{ijk} C_{ijk} \bar{g}^i \bar{g}^j \bar{g}^k$$

The C-function (A. B. Zamolodchikov)

Using the fact components of the conserved tensor $T_{\mu\nu}(z,\bar z)$, one can construct

$$\boxed{C = 2\left(F - \frac{1}{2}G - \frac{3}{16}H\right)}$$

$$\langle T_{zz}(z,\bar z)\, T_{zz}(0,0)\rangle = \frac{F(z\bar z/a_0)}{z^4}$$

$$\langle T_{z\bar z}(z,\bar z)\, T_{zz}(0,0)\rangle = \frac{G(z\bar z/a_0)}{z^3\bar z}$$

$$\langle T_{z\bar z}(z,\bar z)\, T_{z\bar z}(0,0)\rangle = \frac{H(z\bar z/a_0)}{z^2\bar z^2}$$

$$\boxed{\frac{d}{dt}C = -\frac{3}{4}H, \qquad t = \ln\left(\frac{z\bar z}{a_0}\right)}$$

Properties of the $\mathcal{C}$ function :

1) In unitary theories, $H \geq 0$

$\Rightarrow \frac{d}{dt}\mathcal{C} = -\frac{3}{4} H \leq 0$ and

$\frac{d}{dt}\mathcal{C} = \beta^i(g)\frac{\partial}{\partial g^i}\mathcal{C} \leq 0$, decresing function of 't'.

2) $\beta^i(g^*) = 0 \Rightarrow H = 0$, because $T_{z\bar{z}} = 0$;

$\Rightarrow \frac{d}{dt}\mathcal{C} = 0$, $\mathcal{C}(g^*) = c$, the central charge.

3) In the nbd of a fixed point

$$\mathcal{C} = c - \frac{1}{2}\sum_i (h_i + \bar{h}_i - 1)(\bar{g}^i)^2 - \frac{1}{3}\sum_{ijk} C_{ijk}\bar{g}^i\bar{g}^j\bar{g}^k$$

4) If we assume that $\exists\, G_{ij}$ in 'TS',

$\beta^i = -G^{ij}\frac{\partial}{\partial g^j}\mathcal{C}$,

(Principles to determine G_{ij} ?)

in all of 'TS', then

$$\frac{d}{dt}\mathcal{C} = -G^{ij}\frac{\partial \mathcal{C}}{\partial g^i}\frac{\partial \mathcal{C}}{\partial g^j}$$

Quntization of 'Theory space'

Proposal :

Let us discuss a simpler example first :

Euclidean space : i.e. $G_{ij} = \delta_{ij}$

$$\frac{d}{dt} g^i(t) = -\frac{\partial \mathbb{C}}{\partial g^i}$$

Q : Can we invent a quantum mechanics for which the flow eqns. are classical sols ?

A : Yes.

$$\frac{d}{dt} g^i(t) = -\frac{\partial \mathbb{C}}{\partial g^i} + \sqrt{\hbar}\, \eta^i(t)$$

$$\langle \eta^i(t)\, \eta^j(t') \rangle = \delta_{ij}\, \delta(t-t')$$

Langevin eqn. with $\hbar \sim D$ (diffusion const)

Henceforth we will call $\mathbb{C}(g) = h(g)$, in anticipation of adopting $h(g)$ as a Morse function on $\mathcal{M}$.

Supersymmetric QM in a Riemannian manifold

All correlation functions of the Langevin theory can be calculated by an exact map to supersymetric QM (Parisi-Sourlas)

The generalization to a stochastic process on a Riemannian manifold was done in (DMW). I will not discuss this route and connection to the Kolmogorov backward and forward equations involving the operators:

$$K_b = (\hbar \nabla_i - \beta_i)\nabla^i$$

$$K_f = \nabla^i(\hbar \nabla_i + \beta_i)$$

but go directly to the associated Schrödinger operator, which turns out to be the Hamiltonian of SUSY QM.

Supersymmetric (SUSY) QM for Stochastic processes

Introduce fermions ψ^i and ψ^{*i}, $i=1,2,\ldots$

$$(\psi^i)^2 = 0 = (\psi^{i*})^2, \quad \psi^i\psi^{*j} + \psi^{*j}\psi^i = G^{ij}$$

Define the linear operators:

$$Q = \sum_i \psi^i(\hbar\nabla_i + \beta_i), \quad Q^* = -\sum_i \psi^{*i}(\hbar\nabla_i - \beta_i)$$

then $\quad Q^2 = Q^{*2} = 0$ iff $\beta_i = -\partial_i h$

Morse function

$$H = \frac{1}{2}(QQ^* + Q^*Q)$$

$$= \hbar^2\Delta + G^{ij}\partial_i h\,\partial_j h + \hbar\nabla_i\partial_j h[\psi^{*i},\psi^j]$$

Witten arrives at this H as a deformation of the Hodge Laplacian:

$H_t = d_t d_t^* + d_t^* d_t$, $\quad d_t = e^{-ht} d\, e^{+ht}$

and $d_t^* = e^{+ht} d\, e^{-ht}$, and studies the spectrum in the 2 limits $t\to 0$ & $t\to\infty$.

Harmonic forms $\quad$ Morse index

The Lagrangian corresponding to this Hamiltonian is:

$$\mathcal{L} = \frac{1}{2} G_{ij} \left(\frac{d}{dt} q^i \frac{d}{dt} q^j - \frac{\partial h}{\partial q^i} \frac{\partial h}{\partial q^j} \right) + \hbar \nabla_i \nabla_j h \, \bar{\psi}^i \psi^j$$

$$+ i \bar{\psi}^i \frac{D}{Dt} \psi^j G_{ij} + \frac{1}{4} R_{ijkl} \bar{\psi}^i \psi^k \bar{\psi}^j \psi^l$$

where $\frac{D}{Dt} \psi^i = \frac{d}{dt} \psi^i + \Gamma^i_{jk} \frac{dq^j}{dt} \psi^k$

(Alvarez-Gaume, Witten, ...)

The quantum theory is defined (in 'euclidean time') by:

$$Z = \int \mathcal{D}\mu(q, \bar{\psi}, \psi) \, e^{-\frac{1}{\hbar} \int_{-\infty}^{+\infty} dt \, \mathcal{L}(q, \bar{\psi}, \psi)}$$

Instantons interpolate between critical points of h.

The dominant trajectories that contribute to the path integral are the extreme of the bosonic part of $\mathcal{L}$. (Set the fermions to zero).

Instantons

Bosonic action is

$$S = \frac{1}{2}\int_{-\infty}^{+\infty} dt\, G_{ij}\left(\frac{dq^i}{dt}\frac{dq^j}{dt} - \frac{\partial h}{\partial g^i}\frac{\partial h}{\partial g^j}\right)$$

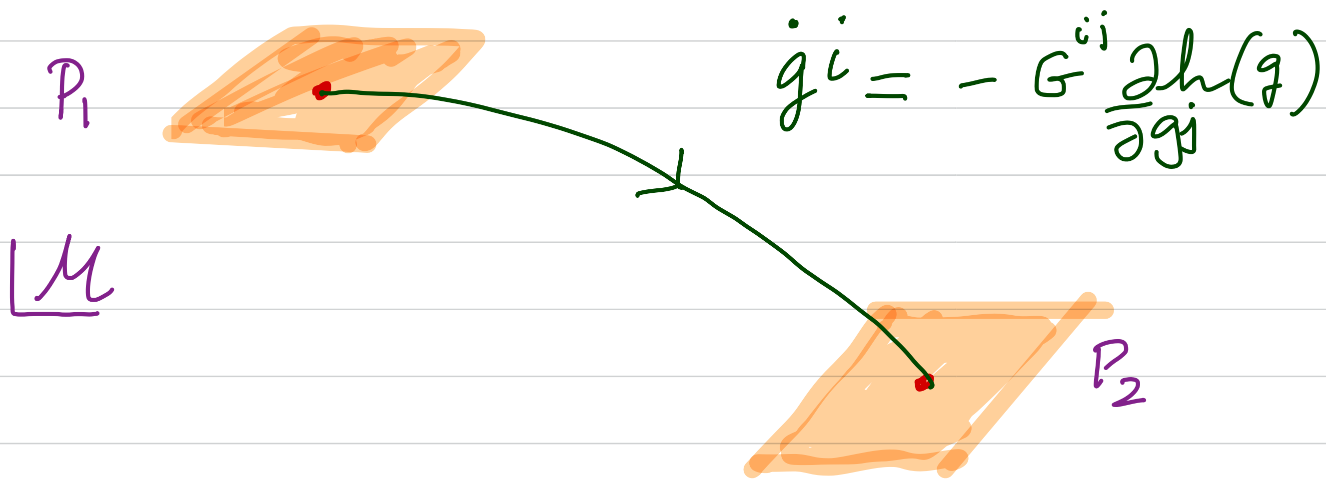
$$= \frac{1}{2}\int_{-\infty}^{+\infty} dt \left|\frac{dg^i}{dt} \pm G^{ij}\frac{\partial h}{\partial g^j}\right|^2 \mp [h(+\infty) - h(-\infty)]$$

$$\Rightarrow \quad S = \mp [h(\infty) - h(-\infty)]$$

$$\text{iff } \left(\frac{dg^i}{dt} \pm G^{ij}\frac{\partial h}{\partial g^j}\right) = 0$$

The RG eqn describes instantons that interpolate between the fixed points $\beta^i = G^{ij}\frac{\partial h}{\partial g^j} = 0$.

Note that the 2-instanton eqns. are related by $h \to -h$ ($-h$ is the inverted Morse function).



$P_{1,2}$ are fixed points.

Action cost of this transition is

$$\Delta S(P_1 \to P_2) = h(P_2) - h(P_1)$$
$$= \mathcal{C}(P_2) - \mathcal{C}(P_1) > 0$$

where $\mathcal{C}(P_2) = c_2$ and $\mathcal{C}(P_1) = c_1$ the central charges of the CFTs at P_1 and P_2.

In QM this contributes a non-perturbative 'tunneling' amplitude $\sim e^{-\frac{1}{\hbar}(c_2 - c_1)}$.

Morse index of a 'critical' point

As we have seen the critical points of the 'theory space' are given by the vanishing of the vector field

$$\beta_i = -\partial_i C = -\partial_i h .$$

(For this discussion we will assume for simplicity that the critical points are isolated, and there are no critical manifolds of dim. > 0.)

Critical points in 2-dim QFT are organised by the Virasoro algebra.

The Morse index of a critical point P is just the number of 'relevant' operators at P i.e. with $h + \bar{h} < 2$.

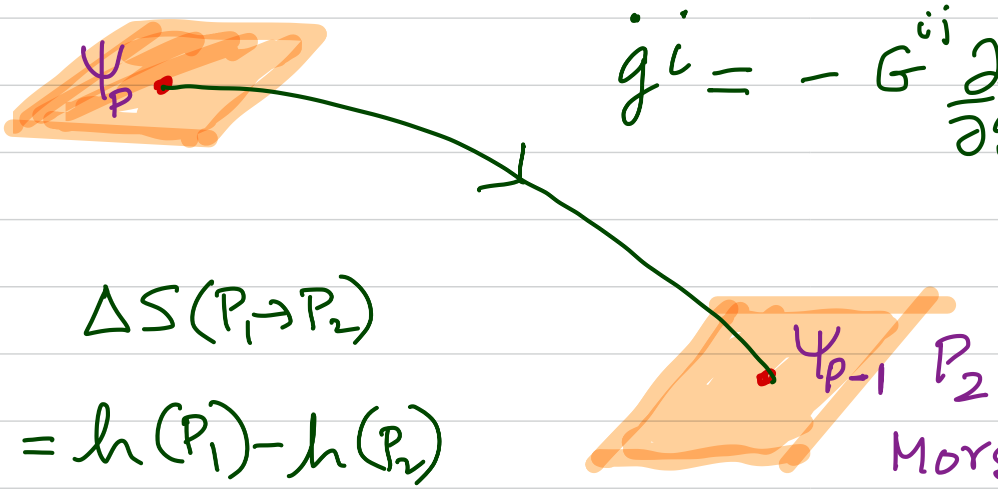
We are able to make the identification of the dim. of the space of relevant operators at the point P, because of the existence of the C-function.

Let us continue our discussion of the QM in the 'instanton' approximation: (Witten, DMW..)

Morse index : p

P_1

$\mathcal{M}$



$$\dot{q}^i = -G^{ij}\frac{\partial h(q)}{\partial q^j}$$

$$\Delta S(P_1 \to P_2)$$
$$= h(P_1) - h(P_2)$$
$$= (c_1 - c_2) > 0$$

One can explicitly construct the wave functions Ψ_p and Ψ_{p-1} at P_1 + P_2

The wave functions localized at P_1 & P_2:

$$\Psi_p(g_i^* + \bar{g}_i, \psi_i) = \Psi_0(\bar{g})\, \psi_1^* \psi_2^* \cdots \psi_p^* |0\rangle$$

where 'p' is the Morse index,

Ψ_p is a p-form.

$\Psi_0(\bar{g}_i)$ is the gaussian wave function localized at the critical point.

This beautiful understanding of Morse theory is due to E. Witten, who introduced the spectral theory of the Schrödinger Operator to discuss Morse theory.

Ref :- E. Witten, Supersymmetry & Morse theory (1982)

- R. Bott, Morse theory indomitable (1988)

Topology of the space of RG flows

Elementary results for closed, compact manifolds with isolated fixed points.

Given a Morse function $h(g)$,

$\partial_i h(g_i^*) = 0$, critical point P

$$\left| \frac{\partial^2}{\partial_i \partial_j} h(g_i^*) \right| \neq 0$$

The number of -ve eigenvalues '$m(P)$' of the Hessian at P is the Morse index of P.

Consider the following 2-polynomials:

$M(t) = \sum t^{m(P)}$ Morse polynomial

$P(t) = \sum B_p t^p$ Poincaré Polynomial

B_p = Betti number.

Morse theorems:

1. $B_p \leq m(p)$

The p^{th} Betti number sets lower bound on the number of critical points with Morse index $m(p)$.

2. $P(t) - M(t) = (1+t)Q(t)$

where $Q(t) = \sum_i a_i t^i$, $a_i \geq 0$

3. Lacunary principle:

If no two critical points of the Morse function have consecutive Morse indices, then $M(t) = P(t)$

e.g. for the space of paths between 2 points on S^n; $M(t) = P(t) = (1 - t^{n-1})^{-1}$

for Lie groups see R. Bott.

An example from critical phenomena in 2-dim.

Consider $c<1$ unitary CFTs in 2-dim. The critical points are well known & well studied

(Belavin, Polyakov, Zamolodchikov
Friedan, Qui, Shenker, V. Kac, Feigen-Fuks,
Dotsenko-Fateev, Zamolodchikov ...)

Summary of results:

1. $c<1$, unitary CFTs are specified by a +ve integer $m \geq 2$

2. $c = 1 - \dfrac{6}{m(m+1)}$ is the central charge

3. $h_{r,s}(m, m+1) = \dfrac{[r(m+1) - sm]^2 - 1}{4m(m+1)}$

$$1 \leq r \leq m-1, \quad 1 \leq s \leq m$$

is the spectrum of dimensions.

Example:

Ising model : $c = \frac{1}{2}$, $m = 3$

The independent h_{rs} are:

$$h_{11} = 0, \quad h_{21} = \frac{1}{2}, \quad h_{22} = \frac{1}{16}$$
$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$
$$\mathbb{1} \qquad\qquad \epsilon \qquad\qquad \sigma$$

(we have mentioned this before).

$m = 4 \rightarrow$ Tricitical ising model

$m = 5 \rightarrow$ 3-state Potts model ...

The key question is for a given 'm'

how does one determine the number of

relevant operators in the list $h_{rs}(m, m+1)$.

I do not know how to calculate the

number of solutions to $h_{rs}(m, m+1) < 1$.

Change track:

Zamolodchikov → Landau–Ginzburg description of $c<1$ unitary models.

They are described by a QFT of a single scalar field $\phi(z,\bar{z})$, with

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + V(\phi)$$

$$V(\phi) = \gamma_2 \phi^2 + \gamma_4 \phi^4 + \cdots + \gamma_m \phi^{2(m-1)}$$

Relevant operators:

$$:\phi^k: = \begin{cases} \Phi_{(k+1,k+1)}, & k=0,1,\ldots,(m-2) \\ \Phi_{(k+3-m,\,k+2-m)}, & k=m-1,m,\ldots,(2m-4) \end{cases}$$

The corresponding values of h_{rs} $(m,m+1)$ are:

$$h_{k+1,k+1} = \frac{k(k+2)}{4m(m+1)}, \quad k=0,\ldots,m-2$$

$$h_{k+3-3,\,k+2-m} = \frac{(k+2)(k+4)}{4m(m+1)}, \quad k=m-2,\ldots,2m-4$$

If we assume that this is a complete list of relevant operators, then there are exactly $2(m-2)$ relevant operators at the CFT with $c = 1 - 6/m(m+1)$.

$\Rightarrow$ Morse index is $2(m-2)$ and

$$M(t) = 1 + t^2 + t^4 + \cdots$$
$$= \frac{1}{1-t^2}$$

We can indeed apply the Lacunary principle and hence

$$P(t) = M(t) = \frac{1}{1-t^2}$$

(Identical result for S^3)

$$\Rightarrow \quad B_0 = B_2 = B_4 = B_6 \cdots = 1$$
$$B_3 = B_5 = B_7 \cdots = 0$$

(Same result was obtained by C. Vafa)

Conclusions, comments

1. We have tried to develop a unified model of the space of 2-dim. QFT using the Virasoro data at the critical points.

2. The 'quantization' naturally leads to Langevin–Kolmogorov–Fokker–Planck type formulation, and the natural emergence of SUSY QM for such systems. These can be formulated for a $\mathbb{R}$-manifold.

3. The key mathematical object is the Schrödinger (Hodge) operator in the Hilbert space of forms

4. The Zam. C-function = Morse function (Connects critical phenomena to topology)

4. The RG flows can be understood as 'instantons', in this theory

5. We can calculate the Poincaré polynomial for $c<1$ unitary theories.

$B_{p=\text{even}} = 1$, $B_{p=\text{odd}} = 0$.

6. The Morse polynomial, where there are marginal operators has been discussed by Bott + Witten (See S. Gukov, 2016).

7. Most of the results we have obtained did not require the explicit use of the metric $G_{ij}(g)$.

8. Q: Is there a way to infer this?
Perhaps by introducing additional structures on the space of flows?