Asymptotics of invariant measures of mean-field Markov models on countably infinite state spaces

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- Particle transitions: at time t, a $z \to z'$ transition occurs at rate $\lambda_{z,z'}(\mu^N(t))$. Mean-field interaction.
- $\{(X_n^N(t), 1 \leq n \leq N), t \geq 0\}$ is a Markov process on $\mathcal{Z}^N$. $\{\mu^N(t), t \geq 0\}$ is a Markov process on $M_1(\mathcal{Z})$.

A countable-state mean-field model

- μ^N is a Markov process on $M_1(\mathcal{Z})$ with infinitesimal generator

$$L^N f(\xi) = \sum_{(z,z')\in\mathcal{E}} N\xi(z)\lambda_{z,z'}(\xi)\left[f\left(\xi + \frac{\delta_{z'}}{N} - \frac{\delta_z}{N}\right) - f(\xi)\right].$$

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- Goal: study a large deviation principle (LDP) for the family $\{\wp^N, N \geq 1\}$.

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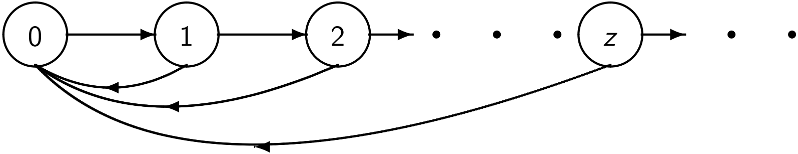

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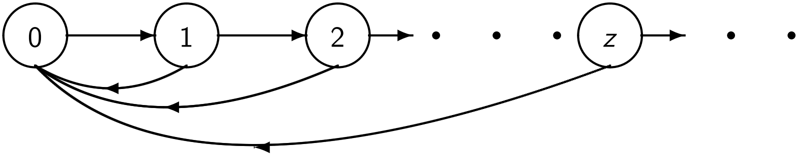

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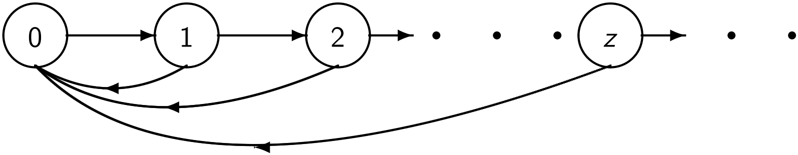

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  Assume that $\lambda_{z,z'}$ are Lipschitz continuous.
  Then $\{(\mu^N(t), 0 \leq t \leq T)\}$ converges in probability to the solution to the McKean-Vlasov equation:

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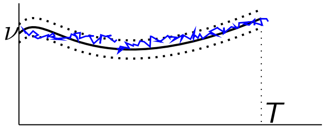
- μ^N: a random perturbation of the above ODE.



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    $$P(\text{dist}(X^N, \Phi(s)) \geq \delta) \leq \exp\{-N(s - \gamma)\}$$

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The process-level LDP for $\mu^N(\cdot)$

Theorem (Léonard (1995), Borkar and Sundaresan (2012))

Let $\nu_N \to \nu$ weakly. Then $\mu^N_{\nu_N}$ satisfies the LDP on $D([0,T], M_1(\mathcal{Z}))$ with rate function $S_{[0,T]}(\cdot|\nu)$ defined as follows. If $\mu_0 = \nu$ and $[0,T] \ni t \mapsto \mu_t \in M_1(\mathcal{Z})$ is absolutely continuous,

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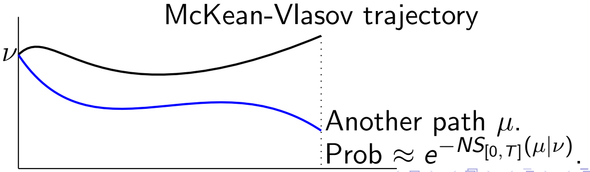

Alternative representation for $S_{[0,T]}(\mu|\nu)$

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&S_{[0,T]}(\mu|\nu) \\
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- Consider the Freidlin-Wentzell quasipotential

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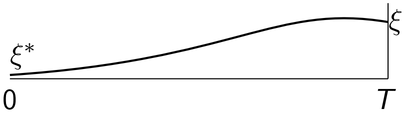

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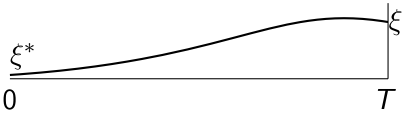

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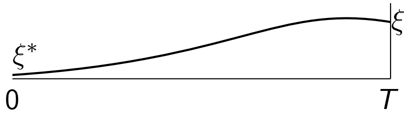

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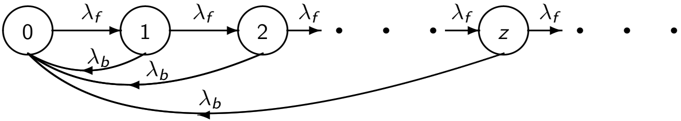
- Small noise diffusions (Freidlin and Wentzell (1984)), finite-state mean-field models (Borkar and Sundaresan (2012)), reaction-diffusion equations (Sowers (1992), Cerrai and Röckner (2004)), stochastic wave equation (Martirosyan (2017)).

LDP for $\wp^N$: V need not be the rate function

- Consider the non-interacting MAC system



The stationary law of each particle is

$$\xi^*(z) = \frac{\lambda_b}{\lambda_f + \lambda_b} \left(\frac{\lambda_f}{\lambda_f + \lambda_b} \right)^z, \, z \in \mathcal{Z}.$$

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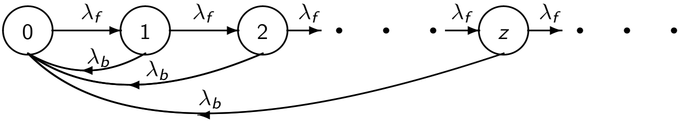

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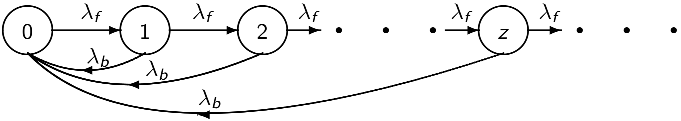

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- Let $\iota(z) = z$, $\vartheta(z) = z \log z$.
- $H(\xi\|\xi^*) < \infty \Leftrightarrow \langle \xi, \iota \rangle < \infty$ while $V(\xi) < \infty \Leftrightarrow \langle \xi, \vartheta \rangle < \infty$.
- If $\xi \in M_1(\mathcal{Z})$ is such that $\langle \xi, \iota \rangle < \infty$ and $\langle \xi, \vartheta \rangle = \infty$, then $V(\xi) = \infty$ but $H(\xi\|\xi^*) < \infty$.

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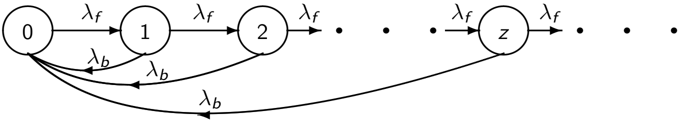

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- In particular, $V \neq H(\cdot\|\xi^*)$.

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- There are barriers to moving from ξ^* to ξ which are crossed in the stationary regime, but not in any finite time.

Assumptions and main result

- Assumptions:
  - There exist positive constants $\overline{\lambda}$ and $\underline{\lambda}$ such that

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Theorem

Under the above assumptions, the family $\{\wp^N, N \geq 1\}$ satisfies the LDP on $M_1(\mathcal{Z})$ with rate function V.

Remarks

- Main difficulty: For any $\xi \in M_1(\mathcal{Z})$ with $V(\xi) < \infty$, we can produce a sequence $\xi_n \to \xi$ in $M_1(\mathcal{Z})$ as $n \to \infty$, but $\langle \xi_n, \vartheta \rangle = \infty$, so that $V(\xi_n) = \infty$ for all n.

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- Earlier, we could not restrict attention to $\mathscr{M} := \{\xi \in M_1(\mathcal{Z}) : \langle \xi, \vartheta \rangle < \infty\}$.
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