Universität Münster

Mathematics Münster
Cluster of Excellence

Random Walks on Hyperbolic Groups

Luzie Kupffer

based on joint work with Mahan Mj and Chiranjib Mukherjee

ICTS - 02.03.2016

living.knowledge

Setting: Graph G, random walk $(X_n)_n$ on G with transition probabilities $p(x, y) > 0$ only if $x \sim y$,

luzie.kupffer@uni-muenster.de

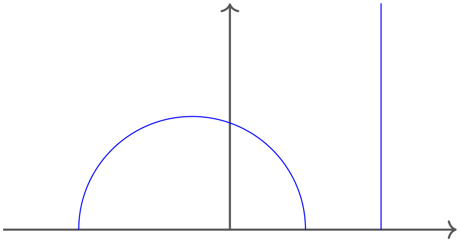
Setting: Graph G, random walk $(X_n)_n$ on G with transition probabilities $p(x,y) > 0$ only if $x \sim y$,

Theorem

If $G = \mathbb{Z}^d$ the random walk is

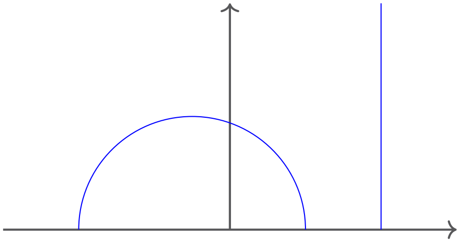
- recurrent if $d \leq 2$
- transient if $d \geq 3$

Hyperbolic Spaces

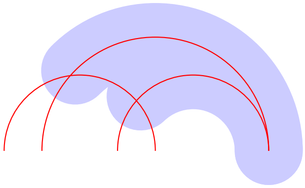


(a) $\mathbb{H}^2$

Hyperbolic Spaces



(a) $\mathbb{H}^2$



(b) thin triangles

Hyperbolic Space

In general:

- Metric Space (X, d)

Hyperbolic Space

In general:

- Metric Space (X, d)
- Gromov product: $(x, y)_z = \frac{1}{2}(d(x, z) + d(y, z) - d(x, y))$

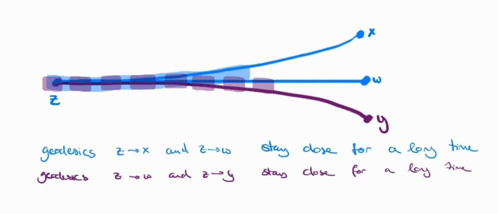
Hyperbolic Space

In general:

- Metric Space (X, d)
- Gromov product: $(x, y)_z = \frac{1}{2}(d(x, z) + d(y, z) - d(x, y))$
- (X, d) is δ-hyperbolic, if $(x, y)_z \geq \min\{(x, w)_z, (y, w)_z\} - \delta$

In general:

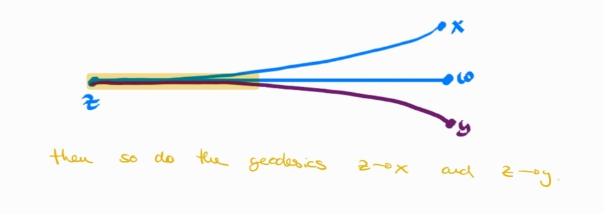
- Metric Space (X, d)
- Gromov product: $(x, y)_z = \frac{1}{2}(d(x, z) + d(y, z) - d(x, y))$
- (X, d) is δ-hyperbolic, if $(x, y)_z \geq \min\{(x, w)_z, (y, w)_z\} - \delta$



Hyperbolic Space

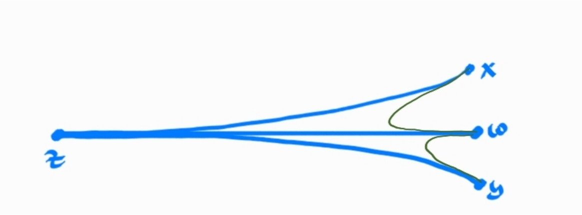
In general:

- Metric Space (X, d)
- Gromov product: $(x, y)_z = \frac{1}{2}(d(x, z) + d(y, z) - d(x, y))$
- (X, d) is δ-hyperbolic, if $(x, y)_z \geq \min\{(x, w)_z, (y, w)_z\} - \delta$



In general:

- Metric Space (X, d)
- Gromov product: $(x, y)_z = \frac{1}{2}(d(x, z) + d(y, z) - d(x, y))$
- (X, d) is δ-hyperbolic, if $(x, y)_z \geq \min\{(x, w)_z, (y, w)_z\} - \delta$



In general:

- Metric Space (X, d)
- Gromov product: $(x, y)_z = \frac{1}{2}(d(x, z) + d(y, z) - d(x, y))$
- (X, d) is δ-hyperbolic, if $(x, y)_z \geq \min\{(x, w)_z, (y, w)_z\} - \delta$
- boundary at infinity $\partial X = \{[\gamma] : \gamma \text{ geodesic ray } \}$

EXAMPLES:

- $\mathbb{Z}^d$ is not hyperbolic (for $d \geq 2$)

Hyperbolic Spaces

EXAMPLES:

- $\mathbb{Z}^d$ is not hyperbolic (for $d \geq 2$)
- trees are hyperbolic

Hyperbolic Spaces

EXAMPLES:

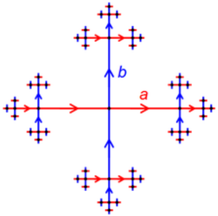
- $\mathbb{Z}^d$ is not hyperbolic (for $d \geq 2$)
- trees are hyperbolic
- Cayley graphs of hyperbolic groups

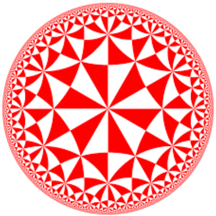
luzie.kupffer@uni-muenster.de

Hyperbolic Groups

- Group G, generating set S, i.e. for all $g \in G$ exist $n \in \mathbb{N}, s_1, ..., s_n \in S$ such that $g = s_1 ... s_n$
- Cayley graph $Cay(G, S)$ with vertices $V = G$ and edges $E = \{(g, gs) : g \in G, s \in S\}$
- G is a hyperbolic group if $Cay(G, S)$ with graph metric is hyperbolic

- Group G, generating set S, i.e. for all $g \in G$ exist $n \in \mathbb{N}, s_1, ..., s_n \in S$ such that $g = s_1 ... s_n$
- Cayley graph $Cay(G, S)$ with vertices $V = G$ and edges $E = \{(g, gs) : g \in G, s \in S\}$
- G is a hyperbolic group if $Cay(G, S)$ with graph metric is hyperbolic

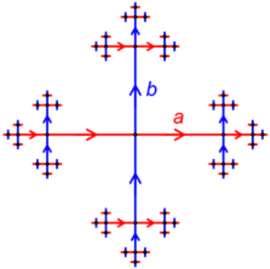




Have: (non-elementary) hyperbolic group G, with symmetric finite generating set S, symmetric measure μ supported on S

Have: (non-elementary) hyperbolic group G, with symmetric finite generating set S, symmetric measure μ supported on S

- Random walk $(X_n)_{n\in\mathbb{N}}$ with transition probabilities $p(x, y) = \mu(y^{-1}x)$

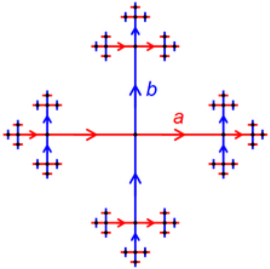


Have: (non-elementary) hyperbolic group G, with symmetric finite generating set S, symmetric measure μ supported on S

- Random walk $(X_n)_{n\in\mathbb{N}}$ with transition probabilities $p(x,y) = \mu(y^{-1}x)$
- **Green metric** d_G, defined by

$$d_G(x,y) := -\log \frac{G(x,y)}{G(e,e)}$$

where $G(x,y) = \sum_n \mathbb{P}_x[X_n = y]$ is the Green function



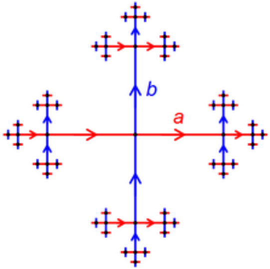
Have: (non-elementary) hyperbolic group G, with symmetric finite generating set S, symmetric measure μ supported on S

- Random walk $(X_n)_{n\in\mathbb{N}}$ with transition probabilities $p(x,y)=\mu(y^{-1}x)$
- **Green metric** d_G, defined by

$$d_G(x,y) := -\log\frac{G(x,y)}{G(e,e)}$$

where $G(x,y)=\sum_n \mathbb{P}_x[X_n=y]$ is the Green function

- (G,d_G) is hyperbolic [Blachère, Haïssinsky, Mathieu '11]



luzie.kupffer@uni-muenster.de

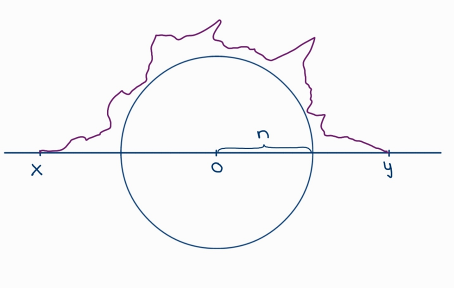
- $(X_n)_{n\in\mathbb{N}}$ converges to a point $X_\infty \in \partial G$ a.s. ([Kaimanovich '00])
- The hitting measures $(\nu_g)_{g\in G}$ on ∂G are a family of conformal measures with densities

$$\frac{\mathrm{d}\nu_g}{\mathrm{d}\nu_o}(\xi) = \mathrm{e}^{-\limsup_{\xi_n\to\xi} d_G(o,\xi_n) - d_G(g,\xi_n)}$$

- the paths of the random walk lie close to the geodesic they converge to, i.e

$$\mathbb{P}[d_G(X_n, [o, X_\infty)) \geq D] \asymp e^{-cD}$$

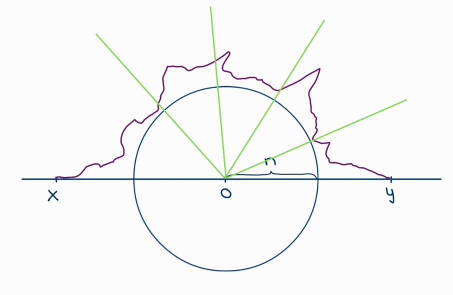
(upper bound: [Blachère, Haïssinsky, Mathieu '11], lower bound: [KMM '24])



Theorem (Gouëzel '14, KMM '24)

In this setting

$$\mathbb{P}_x[(X_n)_n \text{ hits } y, \text{ stays outside } B_n(o)] \lesssim e^{-d_G(y,y_n)} e^{-c_1 \exp(c_2 n)} \mathbb{P}_x[(X_n)_n \text{ hits } y]$$



Theorem (Gouëzel '14, KMM '24)

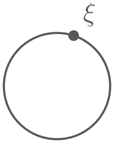
In this setting

$$\mathbb{P}_x[(X_n)_n \text{ hits } y, \text{ stays outside } B_n(o)] \lesssim e^{-d_G(y,y_n)} e^{-c_1 \exp(c_2 n)} \mathbb{P}_x[(X_n)_n \text{ hits } y]$$

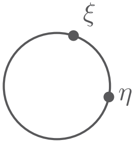
Patterson-Sullivan measures and Geodesic flow

Setup: X a hyperbolic space with boundary ∂X and a group of isometries G acting on X

Setup: X a hyperbolic space with boundary ∂X and a group of isometries G acting on X
Goal: Study the extension of the G-action to ∂X. In particular, is there a measure Θ for which the action is ergodic?

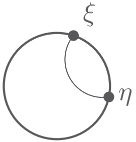


Setup: X a hyperbolic space with boundary ∂X and a group of isometries G acting on X
Goal: Study the extension of the G-action to ∂X. In particular, is there a measure Θ for which the action is ergodic?



Patterson-Sullivan measures and Geodesic flow

Setup: X a hyperbolic space with boundary ∂X and a group of isometries G acting on X
Goal: Study the extension of the G-action to ∂X. In particular, is there a measure Θ for which the action is ergodic?



- conformal measures $(\nu_g)_{g\in G}$ on ∂X with $\frac{\mathrm{d}\nu_g}{\mathrm{d}\nu_h}(\xi) = \mathrm{e}^{c \limsup_{\xi_n \to \xi} d(h,\xi_n) - d(g,\xi_n)}$
- lead to a (quasi)-invariant measure on $\partial^2 G$

$$\mathrm{d}\Theta(\xi,\eta) \asymp \mathrm{e}^{2c(\xi,\eta)_0}\mathrm{d}(\nu_0 \otimes \nu_0)$$

- For this measure the action is ergodic, i.e. for $A \subset \partial^2 G$ invariant it is

$$\Theta(A) = 0 \quad \text{or} \quad \Theta(A^c) = 0.$$

Question: Can we use bi-infinite random walk paths to study the coarse geometric setting?

The space of bi-infinite random walk paths

- $(X_n)_n$ random walk started at g with distribution $\mathbb{P}_g$

The space of bi-infinite random walk paths

- $(X_n)_n$ random walk started at g with distribution $\mathbb{P}_g$
- get a bi-infinite path by shifting X_n "backwards" to preserve geometry of the path

The space of bi-infinite random walk paths

- $(X_n)_n$ random walk started at g with distribution $\mathbb{P}_g$
- get a bi-infinite path by shifting X_n "backwards" to preserve geometry of the path
- regard limit of $X_m^{-1}(X_{n-m})_n$ with distribution $\mathbb{P}_g \otimes \mathbb{P}_g$

The space of bi-infinite random walk paths

- $(X_n)_n$ random walk started at g with distribution $\mathbb{P}_g$
- get a bi-infinite path by shifting X_n "backwards" to preserve geometry of the path
- regard limit of $X_m^{-1}(X_{n-m})_n$ with distribution $\mathbb{P}_g \otimes \mathbb{P}_g$
- $(G^{\mathbb{Z}}, \sum_g \mathbb{P}_g \otimes \mathbb{P}_g)$

The space of bi-infinite random walk paths

- $(X_n)_n$ random walk started at g with distribution $\mathbb{P}_g$
- get a bi-infinite path by shifting X_n "backwards" to preserve geometry of the path
- regard limit of $X_m^{-1}(X_{n-m})_n$ with distribution $\mathbb{P}_g \otimes \mathbb{P}_g$
- $(G^{\mathbb{Z}}, \sum_g \mathbb{P}_g \otimes \mathbb{P}_g)$
- restrict to one copy per path:

$$\mathcal{D} := \bigcup_{g \in G} \left\{ (x_z)_z : x_0 = g, (x_z)_{z=-\infty}^{-1} \notin B_{d_G(0,g)}(0)^c, (x_z)_{z=0}^{\infty} \notin \mathring{B}_{d_G(0,g)}(0) \right\}$$

with measure $\mathbb{Q}(A) = \sum_g \bar{\mathbb{P}}_g((X_z)_z \in A, \mathcal{D})$

The space of bi-infinite random walk paths

- $(X_n)_n$ random walk started at g with distribution $\mathbb{P}_g$
- get a bi-infinite path by shifting X_n "backwards" to preserve geometry of the path
- regard limit of $X_m^{-1}(X_{n-m})_n$ with distribution $\mathbb{P}_g \otimes \mathbb{P}_g$
- $(G^{\mathbb{Z}}, \sum_g \mathbb{P}_g \otimes \mathbb{P}_g)$
- restrict to one copy per path:

$$\mathcal{D} := \bigcup_{g \in G} \left\{ (x_z)_z : x_0 = g, (x_z)_{z=-\infty}^{-1} \notin B_{d_G(0,g)}(0)^c, (x_z)_{z=0}^{\infty} \notin \mathring{B}_{d_G(0,g)}(0) \right\}$$

with measure $\mathbb{Q}(A) = \sum_g \bar{\mathbb{P}}_g((X_z)_z \in A, \mathcal{D})$

- Θ is push forward of $\mathbb{Q}$ under the boundary map

Theorem (K.,Mj,Mukherjee, 24+)

There exists a measure Θ on $\partial^2 G$ with the following properties:

- For any $(\xi, \eta) \in \partial^2 G$, if we denote by $(\cdot,\cdot)^G_\cdot$ the Gromov product with respect to the Green metric,

$$\mathrm{d}\Theta(\xi,\eta) \asymp \mathrm{e}^{2(\xi,\eta)^G_0}\mathrm{d}(\nu_0 \otimes \nu_0)(\xi,\eta).$$

- Θ is (quasi)-invariant and ergodic under the action of G on $\partial^2 G$.

Theorem (K.,Mj,Mukherjee, 24+)

There exists a measure Θ on $\partial^2 G$ with the following properties:

- For any $(\xi, \eta) \in \partial^2 G$, if we denote by $(\cdot, \cdot)_\cdot^G$ the Gromov product with respect to the Green metric,

$$\mathrm{d}\Theta(\xi, \eta) \asymp \mathrm{e}^{2(\xi,\eta)_0^G} \mathrm{d}(\nu_0 \otimes \nu_0)(\xi, \eta).$$

- Θ is (quasi)-invariant and ergodic under the action of G on $\partial^2 G$.

Theorem (K.,Mj,Mukherjee, 24+)

There exists a measure Θ on $\partial^2 G$ with the following properties:

- For any $(\xi,\eta) \in \partial^2 G$, if we denote by $(\cdot,\cdot)_\cdot^G$ the Gromov product with respect to the Green metric,

$$\mathrm{d}\Theta(\xi,\eta) \asymp \mathrm{e}^{2(\xi,\eta)_0^G} \mathrm{d}(\nu_0 \otimes \nu_0)(\xi,\eta).$$

- Θ is (quasi)-invariant and ergodic under the action of G on $\partial^2 G$.

Remarks

- *We have three explicit constructions for Θ which are equivalent in the measure theoretic way.*

Universität Münster

MM

Theorem (K.,Mj,Mukherjee, 24+)

There exists a measure Θ on $\partial^2 G$ with the following properties:

- For any $(\xi,\eta) \in \partial^2 G$, if we denote by $(\cdot,\cdot)^G_\cdot$ the Gromov product with respect to the Green metric,

$$\mathrm{d}\Theta(\xi,\eta) \asymp \mathrm{e}^{2(\xi,\eta)^G_0}\mathrm{d}(\nu_0 \otimes \nu_0)(\xi,\eta).$$

- Θ is (quasi)-invariant and ergodic under the action of G on $\partial^2 G$.

Remarks

- *We have three explicit constructions for Θ which are equivalent in the measure theoretic way.*
- *Ergodicity is shown in the construction which captures a flow along random walk paths. This Θ is also fully invariant.*

Thank you!

The Space of all Random Walk Paths

Quotients are identified with a fundamental domain:

The Space of all Random Walk Paths

Quotients are identified with a fundamental domain:

- $\{\text{all RW paths}\}/G \simeq \{\text{all RW paths with } X_0 = 0\}$ with measure $\bar{\mathbb{P}}_0$

The Space of all Random Walk Paths

Quotients are identified with a fundamental domain:

- $\{\text{all RW paths}\}/G \simeq \{\text{all RW paths with } X_0 = 0\}$ with measure $\bar{\mathbb{P}}_0$
- almost sure fundamental domain for $\mathbb{Z}$ action

The Space of all Random Walk Paths

Quotients are identified with a fundamental domain:

- $\{\text{all RW paths}\}/G \simeq \{\text{all RW paths with } X_0 = 0\}$ with measure $\bar{\mathbb{P}}_0$
- almost sure fundamental domain for $\mathbb{Z}$ action
    - restrict to paths with two ends on $\partial^2 G$

The Space of all Random Walk Paths

Quotients are identified with a fundamental domain:

- $\{$all RW paths$\}/G \simeq \{$all RW paths with $X_0 = 0\}$ with measure $\bar{\mathbb{P}}_0$
- almost sure fundamental domain for $\mathbb{Z}$ action
  - restrict to paths with two ends on $\partial^2 G$
  - for each path $(x_z)_{z\in\mathbb{Z}}$ identify point g where it first gets closest to 0

The Space of all Random Walk Paths

Quotients are identified with a fundamental domain:

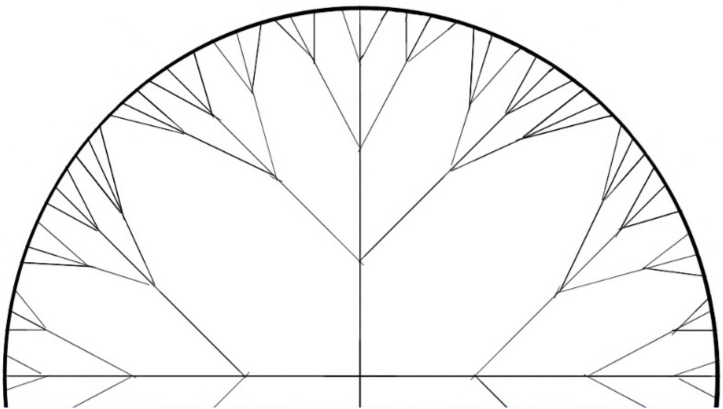
- $\{$all RW paths$\}/G \simeq \{$all RW paths with $X_0 = 0\}$ with measure $\bar{\mathbb{P}}_0$
- almost sure fundamental domain for $\mathbb{Z}$ action
  - restrict to paths with two ends on $\partial^2 G$
  - for each path $(x_z)_{z\in\mathbb{Z}}$ identify point g where it first gets closest to 0
  - choose representative for $[(x_z)_z]$ with $x_0 = g$ is first visit to g.

The Space of all Random Walk Paths

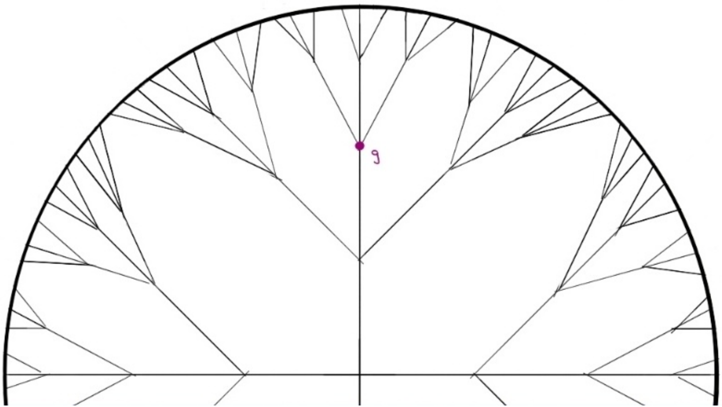
Quotients are identified with a fundamental domain:

- $\{\text{all RW paths}\}/G \simeq \{\text{all RW paths with } X_0 = 0\}$ with measure $\bar{\mathbb{P}}_0$
- almost sure fundamental domain for $\mathbb{Z}$ action
  - restrict to paths with two ends on $\partial^2 G$
  - for each path $(x_z)_{z\in\mathbb{Z}}$ identify point g where it first gets closest to 0
  - choose representative for $[(x_z)_z]$ with $x_0 = g$ is first visit to g.
  - What is $\mathbb{Q}$ restricted to this fundamental domain?
    What is its push forward Θ under the boundary map?

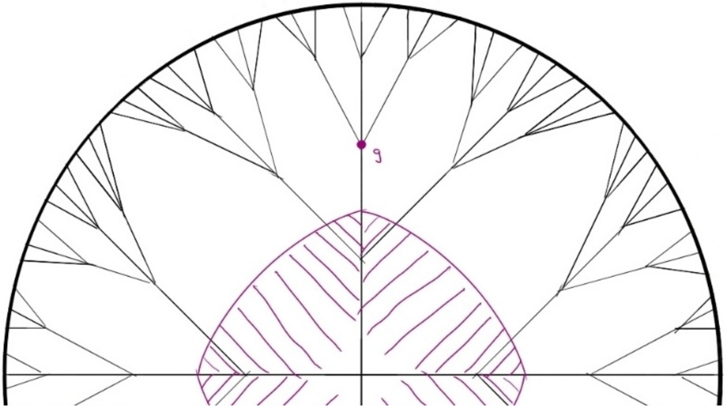
Θ if $G = \mathbb{F}_2$



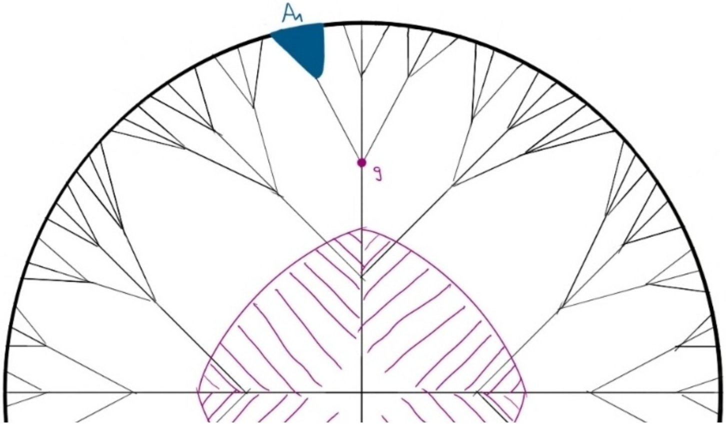
Look at $\mathbb{F}_2$



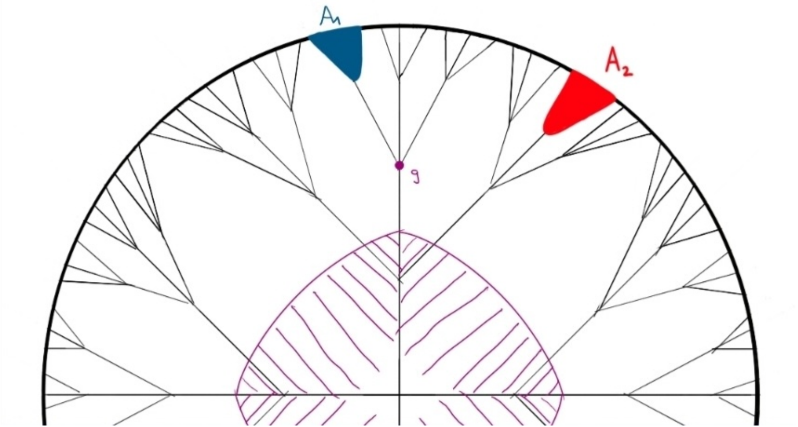
paths with origin g



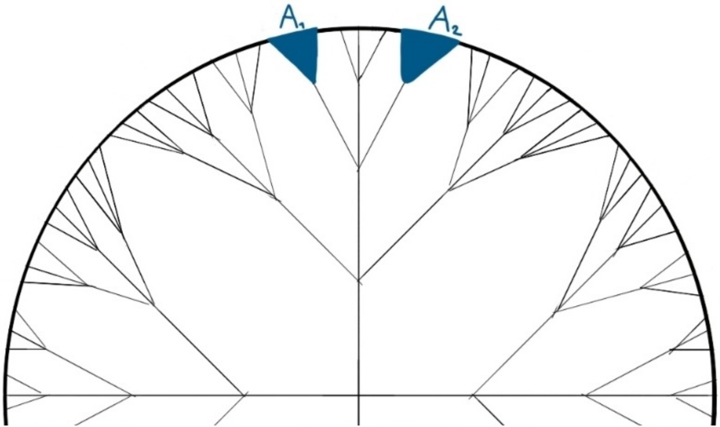
never go inside this ball



can converge to a point inside A_1

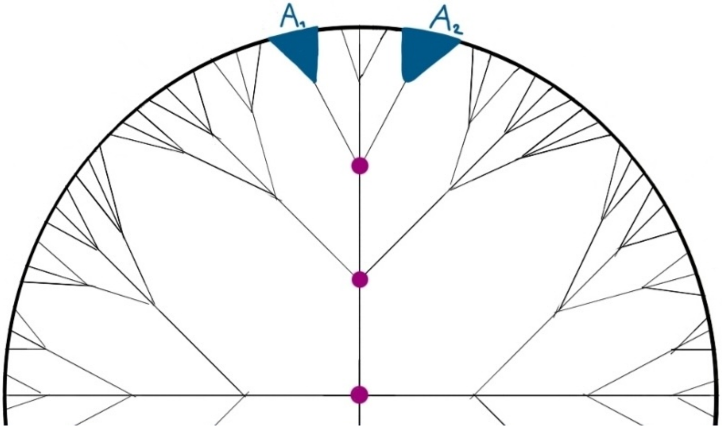


but not to a point inside A_2

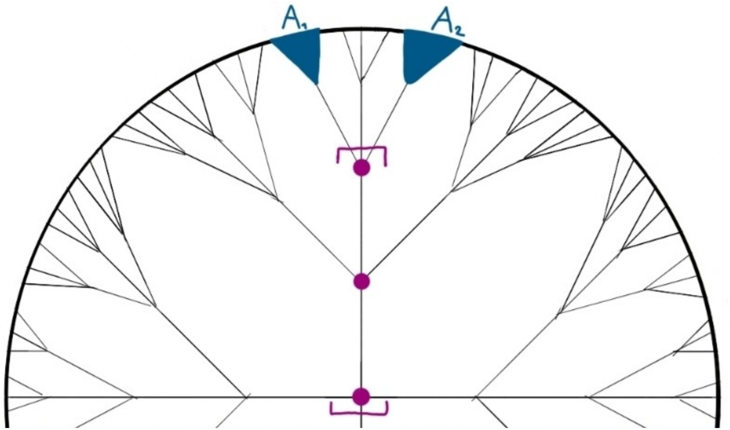


Want $(x_{-n})_n$ to converge in A_1 and $(x_n)_n$ to converge in A_2

Θ if $G = \mathbb{F}_2$



then we can have these starting points



Origins g lying on $[0, x]$ where geodesics to A_1 and A_2 run parallel

Θ if $G = \mathbb{F}_2$

We want that for

$$\Theta(A_1 \times A_2) = \sum_{g \in G} \bar{\mathbb{P}}((X_{-n})_n \text{ converges in } A_1, (X_n)_n \text{ converges in } A_2, \text{' stay outside balls '})$$

$$\mathrm{d}\Theta(\xi, \eta) \asymp \mathrm{e}^{2(\xi,\eta)_0^G} \mathrm{d}(\nu_0 \otimes \nu_0)(\xi, \eta).$$

We use that

Θ if $G = \mathbb{F}_2$

We want that for

$$\Theta(A_1 \times A_2) = \sum_{g \in G} \bar{\mathbb{P}}((X_{-n})_n \text{ converges in } A_1, (X_n)_n \text{ converges in } A_2, \text{' stay outside balls '})$$

$$\mathrm{d}\Theta(\xi,\eta) \asymp \mathrm{e}^{2(\xi,\eta)_0^G} \mathrm{d}(\nu_0 \otimes \nu_0)(\xi,\eta).$$

We use that

- we only have to sum over $g \in [0, x]$

Θ if $G = \mathbb{F}_2$

We want that for

$$\Theta(A_1 \times A_2) = \sum_{g \in G} \bar{\mathbb{P}}((X_{-n})_n \text{ converges in } A_1, (X_n)_n \text{ converges in } A_2, \text{' stay outside balls '})$$

$$\mathrm{d}\Theta(\xi, \eta) \asymp \mathrm{e}^{2(\xi,\eta)_0^G} \mathrm{d}(\nu_0 \otimes \nu_0)(\xi, \eta).$$

We use that

- we only have to sum over $g \in [0, x]$
- 'staying outside ball' changes probability by constant factor independent of g

We want that for

$$\Theta(A_1 \times A_2) = \sum_{g \in G} \bar{\mathbb{P}}((X_{-n})_n \text{ converges in } A_1, (X_n)_n \text{ converges in } A_2, \text{' stay outside balls '})$$

$$\mathrm{d}\Theta(\xi, \eta) \asymp \mathrm{e}^{2(\xi,\eta)_0^G} \mathrm{d}(\nu_0 \otimes \nu_0)(\xi, \eta).$$

We use that

- we only have to sum over $g \in [0, x]$
- 'staying outside ball' changes probability by constant factor independent of g
- $$\frac{\mathrm{d}\nu_g}{\mathrm{d}\nu_h}(\xi) = \mathrm{e}^{\limsup_{\xi_n \to \xi} d_G(\xi_n, 0) - d_G(\xi_n, g)}$$

The argument for the tree depends on three things:

General Hyperbolic Groups

The argument for the tree depends on three things:

1. We only have to regard starting points on $[0, x]$,

General Hyperbolic Groups

The argument for the tree depends on three things:

1. We only have to regard starting points on $[0, x]$,
2. on $[0, x]$ the restriction changes the probability by a constant factor

General Hyperbolic Groups

The argument for the tree depends on three things:

1. We only have to regard starting points on $[0, x]$,
2. on $[0, x]$ the restriction changes the probability by a constant factor
3. the Green's metric is additive along geodesics in the tree

In general hyperbolic groups we deal with these by:

General Hyperbolic Groups

The argument for the tree depends on three things:

1. We only have to regard starting points on $[0, x]$,
2. on $[0, x]$ the restriction changes the probability by a constant factor
3. the Green's metric is additive along geodesics in the tree

In general hyperbolic groups we deal with these by:

- (2) and (3) get avoided by looking at upper and lower bounds

General Hyperbolic Groups

The argument for the tree depends on three things:

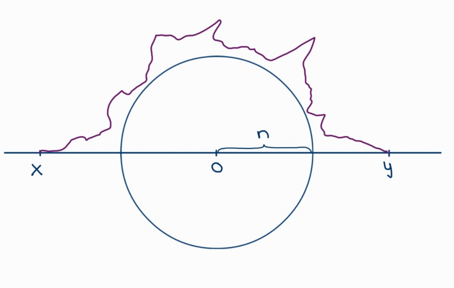
1. We only have to regard starting points on $[0, x]$,
2. on $[0, x]$ the restriction changes the probability by a constant factor
3. the Green's metric is additive along geodesics in the tree

In general hyperbolic groups we deal with these by:

- (2) and (3) get avoided by looking at upper and lower bounds
- study how forcing the random walk path to walk around a ball changes the probability

Main Ingredients

- Results on deviation from the geodesic
- Influence of being forced to walk around a ball on hitting measure

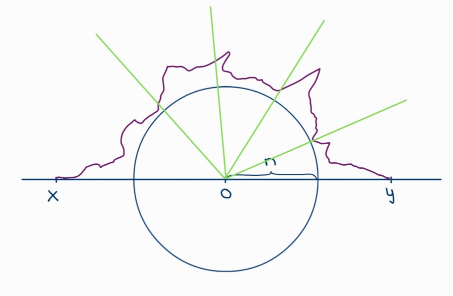


Theorem: [Gouëzel '14, KMM '24]
In this setting

$$\mathbb{P}_x[(X_n)_n \text{ hits } y, \text{ stays outside } B_n(o)]$$
$$\lesssim e^{-d_G(y,y_n)} e^{-c_1 \exp(c_2 n)} \mathbb{P}_x[(X_n)_n \text{ hits } y]$$

Main Ingredients

- Results on deviation from the geodesic
- Influence of being forced to walk around a ball on hitting measure



Theorem: [Gouëzel '14, KMM '24]
In this setting

$$\mathbb{P}_x[(X_n)_n \text{ hits } y, \text{ stays outside } B_n(o)]$$
$$\lesssim e^{-d_G(y,y_n)} e^{-c_1 \exp(c_2 n)} \mathbb{P}_x[(X_n)_n \text{ hits } y]$$

Lemma (K.,Mj,Mukherjee, 24+)

Let f be a continuous function on $G_0^{\mathbb{Z}}$ which is integrable w.r.t. $\bar{\mathbb{P}}_0$. Then a.s.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n} f(\tau_l(x_z)_z) = \int_{G_0^{\mathbb{Z}}} f \, d\bar{\mathbb{P}}_0.$$

Outlook

- Applications to classical questions

Outlook

- Applications to classical questions
- Extensions to more general spaces

Outlook

- Applications to classical questions
- Extensions to more general spaces
- Use Θ and its connections to the random walk paths to define a point process of random lines

S. Blachère and P. Haïssinsky and P. Mathieu

Harmonic measures versus quasiconformal measures for hyperbolic groups,Annales Scientifiques de l'École Normale Supérieure, **44**(2011)

U. Bader and A. Furman,

Some ergodic properties of metrics on hyperbolic groups,https://arxiv.org/abs/1707.02020 (2018)

S. Gouëzel,

Local limit theorem for symmetric random walks in Gromov-hyperbolic groups,
Journal of the American Mathematical Society, **27**, (2014)

L. Kupffer, M. Mj and C. Mukherjee,

Bi-infinite random walks and geodesic flow on hyperbolic groups,– (2024+)

S.J. Patterson,

The limit set of a Fuchsian group, *Acta mathematica* **136**, (1976), 241-273

D. Sullivan,

The density at infinity of a discrete group of hyperbolic motions, *Publications mathematiques de l'IHES* **50**, (1979) 171-202

Universität Münster MM

Lemma (K.,Mj,Mukherjee, 24+)

Let f be a continuous function on $G_0^{\mathbb{Z}}$ which is integrable w.r.t. $\bar{\mathbb{P}}_0$. Then a.s.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n} f(\tau_l(x_z)_z) = \int_{G_0^{\mathbb{Z}}} f \, d\bar{\mathbb{P}}_0.$$

Proof.

Lemma (K.,Mj,Mukherjee, 24+)

Let f be a continuous function on $G_0^{\mathbb{Z}}$ which is integrable w.r.t. $\bar{\mathbb{P}}_0$. Then a.s.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n} f(\tau_l(x_z)_z) = \int_{G_0^{\mathbb{Z}}} f \, d\bar{\mathbb{P}}_0.$$

Proof.

□

Lemma (K.,Mj,Mukherjee, 24+)

Let f be a continuous function on $G_0^{\mathbb{Z}}$ which is integrable w.r.t. $\bar{\mathbb{P}}_0$. Then a.s.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n} f(\tau_l(x_z)_z) = \int_{G_0^{\mathbb{Z}}} f \, d\bar{\mathbb{P}}_0.$$

Proof.

- Limit exists, given by $\mathbb{E}[f|\mathcal{F}^{\tau_1}]$

□

Lemma (K.,Mj,Mukherjee, 24+)

Let f be a continuous function on $G_0^{\mathbb{Z}}$ which is integrable w.r.t. $\bar{\mathbb{P}}_0$. Then a.s.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n} f(\tau_l(x_z)_z) = \int_{G_0^{\mathbb{Z}}} f \, d\bar{\mathbb{P}}_0.$$

Proof.

- Limit exists, given by $\mathbb{E}[f|\mathcal{F}^{\tau_1}]$
- Similarly, when replacing τ_l with τ_{-l} get limit $\mathbb{E}[f|\mathcal{F}^{\tau_{-1}}]$

□

Ergodicity

Lemma (K.,Mj,Mukherjee, 24+)

Let f be a continuous function on $G_0^{\mathbb{Z}}$ which is integrable w.r.t. $\bar{\mathbb{P}}_0$. Then a.s.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n} f(\tau_l(x_z)_z) = \int_{G_0^{\mathbb{Z}}} f \, d\bar{\mathbb{P}}_0.$$

Proof.

- Limit exists, given by $\mathbb{E}[f|\mathcal{F}^{\tau_1}]$
- Similarly, when replacing τ_l with τ_{-l} get limit $\mathbb{E}[f|\mathcal{F}^{\tau_{-1}}]$
- Invariant σ-algebras agree, so limits agree.

□

Lemma (K.,Mj,Mukherjee, 24+)

Let f be a continuous function on $G_0^{\mathbb{Z}}$ which is integrable w.r.t. $\bar{\mathbb{P}}_0$. Then a.s.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n} f(\tau_l(x_z)_z) = \int_{G_0^{\mathbb{Z}}} f \, d\bar{\mathbb{P}}_0.$$

Proof.

- Limit exists, given by $\mathbb{E}[f|\mathcal{F}^{\tau_1}]$
- Similarly, when replacing τ_l with τ_{-l} get limit $\mathbb{E}[f|\mathcal{F}^{\tau_{-1}}]$
- Invariant σ-algebras agree, so limits agree.
- f only depends on finitely many indices, so limit doesn't depend on negative indices

□

Lemma (K.,Mj,Mukherjee, 24+)

Let f be a continuous function on $G_0^{\mathbb{Z}}$ which is integrable w.r.t. $\bar{\mathbb{P}}_0$. Then a.s.

$$\lim_{n\to\infty} \frac{1}{n}\sum_{l=0}^{n} f(\tau_l(x_z)_z) = \int_{G_0^{\mathbb{Z}}} f \, d\bar{\mathbb{P}}_0.$$

Proof.

- Limit exists, given by $\mathbb{E}[f|\mathcal{F}^{\tau_1}]$
- Similarly, when replacing τ_l with τ_{-l} get limit $\mathbb{E}[f|\mathcal{F}^{\tau_{-1}}]$
- Invariant σ-algebras agree, so limits agree.
- f only depends on finitely many indices, so limit doesn't depend on negative indices
- similarly, by looking at backwards shift, the limit doesn't depend on the positive indices

□

Lemma (K.,Mj,Mukherjee, 24+)

Let f be a continuous function on $G_0^{\mathbb{Z}}$ which is integrable w.r.t. $\bar{\mathbb{P}}_0$. Then a.s.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n} f(\tau_l(x_z)_z) = \int_{G_0^{\mathbb{Z}}} f \, d\bar{\mathbb{P}}_0.$$

Proof.

- Limit exists, given by $\mathbb{E}[f|\mathcal{F}^{\tau_1}]$
- Similarly, when replacing τ_l with τ_{-l} get limit $\mathbb{E}[f|\mathcal{F}^{\tau_{-1}}]$
- Invariant σ-algebras agree, so limits agree.
- f only depends on finitely many indices, so limit doesn't depend on negative indices
- similarly, by looking at backwards shift, the limit doesn't depend on the positive indices
- hence constant

□