Maths Circle: Random Walks (Part IV)

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The Ballot Problem

Consider an election where there are exactly two candidates (A and B, say) and $a+b$ voters. Assume that each voter votes for either A or B (but not both) with equal probability independently of other voters and no voter is allowed to abstain from voting. Suppose it is known that Candidate A has received a votes and Candidate B has received b votes in this election, where $a > b$ (i.e., A has won). This session will focus on the answer to the following question.

Question B *What is the probability that A was winning the election throughout the counting process?*

We shall try to answer Question B with the help of random walks. Let S_m denote the number of votes in favour of A minus the number of votes in favour of B after m ballots have been counted. Clearly $S_0 = 0$. Observe that S_m is the position of a random walker at time m (why?). Define $n = a+b$ and $k = a-b$. Note that both n and k are positive integers. It is given to us that the random walker is on the integer k at time n (i.e., $S_n = k$). We have to find the *conditional* probability of the event that the random walker stayed on strictly positive integers during times $1, 2, \ldots, n$. In other words, we have to find the conditional probability

$$P\big(S_1 > 0, S_2 > 0, \ldots, S_n > 0 \,\big|\, S_n = k\big).$$

Since $k = a - b > 0$, it follows that

$$P\big(S_1 > 0, S_2 > 0, \ldots, S_n > 0 \,\big|\, S_n = k\big) = \frac{P\big(S_1 > 0, S_2 > 0, \ldots, S_{n-1} > 0, S_n = k\big)}{P(S_n = k)}.$$

By the assumptions on the election, the coin (that the random walker is tossing) is fair and the coin tosses are independent of each other. Also $k = a-b$ has the same parity as $n = a+b$ and $|k| = k = a-b < a+b = n$. Therefore by Problem 13,

$$P(S_n = k) = P\big(E_{n,k}\big) = \frac{N_{n,k}}{2^n} = \frac{n!}{2^n \left(\frac{n+k}{2}\right)! \left(\frac{n-k}{2}\right)!}.$$

On the other hand, using the classical definition of probability, we get

$$P\big(S_1 > 0, S_2 > 0, \ldots, S_{n-1} > 0, S_n = k\big) = \frac{\text{the number of random walk paths from } (0,0) \text{ to } (n,k) \text{ that lie strictly above the horizontal axis}}{2^n}$$

and hence the conditional probability

$$\begin{aligned} &P\big(S_1 > 0, S_2 > 0, \ldots, S_n > 0 \,\big|\, S_n = k\big) \\ &= \frac{P\big(S_1 > 0, S_2 > 0, \ldots, S_{n-1} > 0, S_n = k\big)}{P(S_n = k)} \\ &= \frac{\text{the number of random walk paths from } (0,0) \text{ to } (n,k) \text{ that lie strictly above the horizontal axis}}{\frac{n!}{\left(\frac{n+k}{2}\right)! \left(\frac{n-k}{2}\right)!}}. \end{aligned}$$

Problem 16 *Answer Question B with the help of Problem 15 by showing that*

$$P\big(S_1 > 0, S_2 > 0, \ldots, S_n > 0 \,\big|\, S_n = k\big) = \frac{k}{n} = \frac{a-b}{a+b}\,.$$

We have thus proved the following result known as Ballot Theorem.

Theorem 1 (Ballot Theorem: Whitworth (1878), Bertrand (1887)) *Consider an election where there are exactly two candidates (A and B, say) and $a+b$ voters. Assume that each voter votes for either A or B (but not both) with equal probability independently of other voters and no voter is allowed to abstain from voting. Suppose it is known that Candidate A has received a votes and Candidate B has received b votes in this election, where $a > b$ (i.e., A has won). Then the conditional probability that A was winning the election throughout the counting process is $\frac{a-b}{a+b}$.*

Note that the positive integer $a-b$ is the winning margin of A. Hence the fraction $\frac{a-b}{a+b}$ is the winning proportion of A. Therefore the Ballot Theorem says that the conditional probability that A was winning the election throughout the counting process is equal to the winning proportion of A.