Maths Circle: Random Walks (Part III)

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Parthanil Roy, IIT Bombay

In the last session, we learnt about the classical definition of probability. In this session, we shall first learn about conditional probability and independence. Later, we shall retun to random walks.

Conditional Probability and Independence

Recall that in the previous session we defined the sample space as the set of all possible outcomes of a random experiment. Events are nothing but subsets of a sample space and are typically denoted by A, B, C, etc. For two events A and B, we define $A \cap B$ to be the event that both A and B occur, and $A \cup B$ to be the event that at least one of A and B occur. On the other hand, A^c denotes the event that A does not occur.

Definition 1 *If A and B are two events with $P(B) > 0$ then the conditional probability of A given B is denoted by $P(A|B)$ and defined by*

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Note that $P(A|B)$ gives the (conditional) probability of occurrence of A when it is known that B has already occurred. Therefore, if $P(B) = 0$, then $P(A|B)$ is *not defined*.

Multiplication Rule. *Directly from the definition of conditional probability it follows that*

$$P(A \cap B) = P(B)P(A|B)$$

whenever $P(B) > 0$.

Definition 2 *Two events A and B are said to be independent if*

$$P(A \cap B) = P(A)P(B).$$

If $P(B) \neq 0$ then A and B, are independent if and only if

$$P(A|B) = P(A).$$

If $P(B) = 0$ then A and B are always independent (why?).

Problem 11 *Suppose it is given that two events A and B are independent. Show that the following hold:*

(a) *A^c and B are independent.*

(b) *A and B^c are independent.*

(c) *A^c and B^c are independent.*

Definition 3 *Events A_1, A_2 and A_3 are called independent if*

$$\begin{aligned} P(A_1 \cap A_2) &= P(A_1)P(A_2), \\ P(A_1 \cap A_3) &= P(A_1)P(A_3), \\ P(A_2 \cap A_3) &= P(A_2)P(A_3) \ \textit{and} \\ P(A_1 \cap A_2 \cap A_3) &= P(A_1)P(A_2)P(A_3). \end{aligned}$$

Problem 12 *Suppose three events A, B and C are independent. Show the following:*

(a) *A and $B \cup C$ are independent.*

(b) *$A \cup B^c$ and C are independent.*

Definition 4 *Suppose $n \geq 2$. Events $A_1, A_2, \ldots, A_n$ are called independent if for each $k \in \{2, 3, \ldots, n\}$ and for each subset $\{i_1, i_2, \ldots, i_k\} \subseteq \{1, 2, \ldots, n\}$,*

$$P\big(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}\big) = P\big(A_{i_1}\big)P\big(A_{i_2}\big) \ldots P\big(A_{i_k}\big).$$

Back to Random Walks

Problem 13 *Fix $n \in \mathbb{N}$ and $k \in \mathbb{Z}$. Suppose that a particle starts at 0 and takes n random walk steps from there. Assume that the coin is fair and tosses are independent of each other. Show that all the 2^n many outcomes of the underlying sample space (considered in Problem 9) are equally likely. Using this and Problem 10, show that*

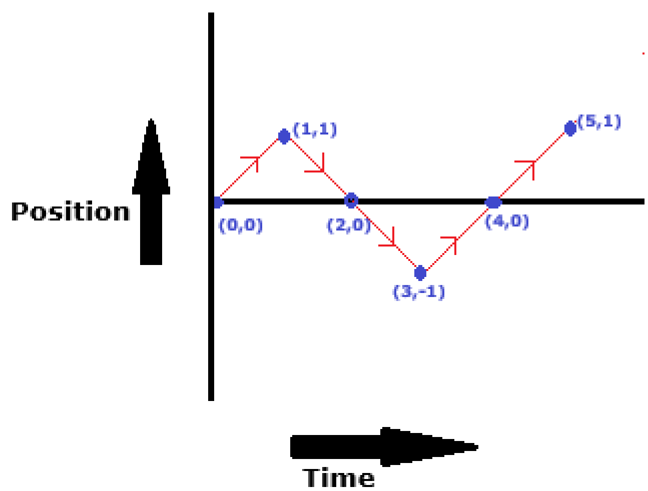
$$P\big(E_{n,k}\big) = \frac{N_{n,k}}{2^n} = \begin{cases} \frac{n!}{2^n\left(\frac{n+k}{2}\right)!\left(\frac{n-k}{2}\right)!} & \text{if } |k| \leq n \text{ and } k \equiv n \ (mod\ 2), \\ \\ 0 & \text{otherwise,} \end{cases}$$

where $E_{n,k}$ denotes the event that the random walker is on the integer k at time n

Recall the graphical representation of a random walk path as described in the session on 11th January 2025. This is carried out by joining the points

$$\big(0, 0\big),\ \big(1, S_1\big),\ \big(2, S_2\big),\ \ldots\ \big(n, S_n\big)$$

on the Cartesian plane. Obviously, the horizontal axis represents time and the vertical axis represents the position of the particle at that time. For instance, if $n = 5$ and the sequence of outcomes is $HTTHH$, then the corresponding random walk path can be represented graphically as follows:



Now we venture into a more intricate counting problem involving random walk paths. From Problem 7 of the last session, we already know that the total number of random walk paths from $(0,0)$ to (n,k) is

$$N_{n,k} = \begin{cases} \frac{n!}{\left(\frac{n+k}{2}\right)!\left(\frac{n-k}{2}\right)!} & \text{if } |k| \leq n \text{ and } k \equiv n \ (mod\ 2), \\ & \\ 0 & \text{otherwise.} \end{cases}$$

How many of these paths lie stricly above the horizontal axis (if $k \in \mathbb{N}$)? Note that any such random walk path has to pass through $(1,1)$ (why?). Therefore, it is enough to count the number of random walk paths from $(1,1)$ to (n,k) that do not touch or cross the horizontal axis. This isn't an easy counting problem at all. It is actually possible to count the number of random walk paths from $(1,1)$ to (n,k) that touch or cross the horizontal axis thanks to the following exercise.

Problem 14 (Reflection Principle) *Fix $k_0, k_1, n_0, n_1 \in \mathbb{N}$ such that $n_1 > n_0$. Show that the number of random walk paths from $\big(n_0, k_0\big)$ to $\big(n_1, k_1\big)$ that touch or cross the horizontal axis is equal to the total number of random walk paths from $\big(n_0, -k_0\big)$ to $\big(n_1, k_1\big)$.*

In order to solve Problem 14, draw a picture and try to use a one-to-one correspondence argument with the help of an appropriate reflection. Thanks to this problem, it is now possible to answer our original question.

Problem 15 *Fix $n, k \in \mathbb{N}$.*

(a) *Using Problem 14 and Problem 8, count the number of random walk paths from $(1,1)$ to (n,k) that touch or cross the horizontal axis.*

(b) *Using (a), count the number of random walk paths from $(0,0)$ to (n,k) that lie strictly above the horizontal axis.*