Twisted and coupled constant scalar curvature Kähler (cscK) metrics on ruled surfaces

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Outline

1. Canonical metrics in Kähler Geometry

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2. (Generalized) Futaki invariant

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3. Ruled surfaces and main results

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2. (Generalized) Futaki invariant
3. Ruled surfaces and main results
4. A bound for Chen-Cheng invariant on ruled surfaces

Kähler-Einstein (KE) and cscK metrics

Suppose (M, ω) is a (compact) **Kähler manifold** of complex dimension n.

- We denote the **Ricci curvature** by $\mathrm{Ric}(\omega)$ and the **scalar curvature** by $R(\omega)$ for the Kähler metric (or form) ω.
- The **first Chern class** $c_1(M)$ of the manifold M is defined as

$$c_1(M) := \frac{1}{2\pi}[\mathrm{Ric}(\omega)] \in H^2(M, \mathbb{R}).$$

- The **space of Kähler potentials** with respect to ω is given by

$$\mathcal{H}_\omega := \left\{ \varphi \in C^\infty(M, \mathbb{R}) : \ \omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0 \right\}.$$

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- We say that the metric ω is **Kähler-Einstein** (**KE**) if $\mathrm{Ric}(\omega) = \lambda\omega$ for some $\lambda \in \mathbb{R}$.
- We say that the metric ω is a constant scalar curvature Kähler (**cscK**) metric if $R(\omega)$ is constant on M.

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- Note that any KE metric ω is cscK. Converse is also true if the first Chern class $c_1(M)$ is a constant multiple of the Kähler class $[\omega]$.

Twisted constant scalar curvature Kähler (cscK) equation

- Let (M, ω_0) be a (compact) Kähler n-manifold with a closed real $(1,1)$-form χ.

[1] Stoppa, J.: *Twisted constant scalar curvature Kähler metrics and Kähler slope stability*, J. Differential Geom. **83** (2009), no. 3, 663–691.

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- Let (M, ω_0) be a (compact) Kähler n-manifold with a closed real $(1,1)$-form χ.
- Following Stoppa,[1] a metric $\omega \in [\omega_0]$ is called a χ-**twisted cscK metric** if it satisfies

$$R(\omega) - \Lambda_\omega \chi = \underline{R} - \underline{\chi}, \tag{2.1}$$

where $R(\omega) := \Lambda_\omega \mathrm{Ric}(\omega)$ is the scalar curvature of ω and $\Lambda_\omega \chi := \frac{n\chi \wedge \omega^{n-1}}{\omega^n}$ is the trace function, and

$$\underline{R} := n\frac{2\pi c_1(M) \cdot [\omega_0]^{n-1}}{[\omega_0]^n}, \quad \underline{\chi} := n\frac{[\chi] \cdot [\omega_0]^{n-1}}{[\omega_0]^n}.$$

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- In particular, when $\chi = 0$ the equation (2.1) is nothing but the **cscK equation**

$$R(\omega) = \underline{R}. \tag{2.2}$$

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- If $[\chi] = 2\pi c_1(M)$, then $\underline{\chi} = \underline{R}$. By Calabi-Yau, $\chi = \mathrm{Ric}(\omega')$ for a unique Kähler metric $\omega' \in [\omega_0]$. Then ω' is a χ-twisted cscK metric (although it may not be cscK).
- If $[\chi] = [\omega_0]$, then $\underline{\chi} = n$ and $\chi = \omega_\phi := \omega_0 + \sqrt{-1}\partial\overline{\partial}\phi$ for some $\phi \in \mathcal{H}_{\omega_0}$. In this case, the twisted cscK equation becomes

$$R(\omega) - \Lambda_\omega \omega_\phi = \underline{R} - n.$$

So, any cscK metric in $[\omega_0]$ is a self-twisted cscK metric.

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J-equation (or Donaldson's equation)

- To obtain the twisted cscK equation (2.1), we twist the cscK equation (2.2) with the following important equation

$$\Lambda_\omega \chi = \underline{\chi}, \quad \iff \quad \underline{\chi} \cdot \omega^n = n\omega^{n-1} \wedge \chi, \tag{2.3}$$

which is called the ***J*-equation**. Here, $\chi > 0$ is a Kähler metric.

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- Donaldson[2] introduced this equation and gave the moment map interpretation. Later, X.X. Chen[3] re-discovered this equation while studying the lower bound the Mabuchi energy[4] and showed that solutions of the J-equation (2.3) are exactly critical points a functional $\mathcal{J}^{\chi}_{\omega_0}$ defined on the space of $\mathcal{H}_{\omega_0}$.

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- Observe that if the Kähler metric ω solves the J-equation (2.3) for the fixed Kähler metric $\chi > 0$, then

$$\omega \text{ is a } \chi\text{-twisted cscK metric} \quad \Longleftrightarrow \quad \omega \text{ is a cscK metric.}$$

But there may exists a Kähler metric solving the twisted cscK equation (2.1) which is neither a cscK metric nor solves the J-equation (2.3).

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- A N-tuple of Kähler metrics $(\omega_i)_{i=1}^N$ on M, where $N \in \mathbb{N}$, is called **coupled cscK metric** if the following conditions are satisfied

$$\mathrm{Ric}(\omega_1) = \mathrm{Ric}(\omega_2) = \cdots = \mathrm{Ric}(\omega_N), \tag{2.4}$$

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Here, $\omega_{\mathrm{sum}} := \sum_{i=1}^N \omega_i$, $\underline{R}$ is the average scalar curvature of the Kähler class $[\omega_1]$ and

$$\underline{\omega_{\mathrm{sum}}} = n\frac{[\omega_{\mathrm{sum}}]\cdot[\omega_1]^{n-1}}{[\omega_1]^n}.$$

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- For $2\pi c_1(M) = [\omega_{\mathrm{sum}}]$, it is reduced to the **coupled Kähler-Einstein (cKE)** equation:

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Holomorphy potentials for vector fields

- Let (M, ω) be a compact Kähler manifold of complex dimension n. For any smooth function $f \in C^\infty(M, \mathbb{C})$, we define a vector field on M of type $(1,0)$ as follows

$$\nabla^{1,0}_\omega f := g^{k\bar{l}} \frac{\partial f}{\partial \overline{z}^l} \frac{\partial}{\partial z^k},$$

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- For any N-tuple $(\alpha_1, \cdots, \alpha_N)$ of Kähler classes on M, define the following sub-space of holomorphic vector fields $H^0(M, T^{1,0}M)$:

$$\mathfrak{h}(\alpha_1, \cdots, \alpha_N) = \left\{ \xi \in H^0(M, T^{1,0}M) \middle| \ \xi = \nabla^{1,0}_{\omega_j} f_j, \ f_j \in C^\infty(M, \mathbb{C}) \ \forall \ j = 1, \cdots, N \right\}$$

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- The space $\mathfrak{h}(\alpha_1, \cdots, \alpha_N)$ is independent of the choice of the metrics $\omega_i \in \alpha_i$. The tuple $(f_1, \cdots, f_N)$ is called a N-tuple of holomorphy potentials for $\xi \in \mathfrak{h}(\alpha_1, \cdots, \alpha_N)$ with respect to $(\omega_1, \cdots, \omega_N)$.

Generalized or coupled Futaki invariant for coupled cscK problem

- Datar-Pingali [2] introduced the **coupled (or generalized) Futaki invariant** $\mathrm{Fut}_c(\alpha_1, \cdots, \alpha_N, -) : \mathfrak{h}(\alpha_1, \cdots, \alpha_N) \longrightarrow \mathbb{C}$ which is defined by

$$\begin{aligned} \mathrm{Fut}_c(\alpha_1, \cdots, \alpha_N, \xi) := \frac{1}{\alpha_1^n} \int_M f_1 \left(R(\omega_1) - \Lambda_{\omega_1} \omega_{\mathrm{sum}} - \underline{R}_{\alpha_1} + \underline{\omega_{\mathrm{sum}}} \right) \omega_1^n \\ + \sum_{j=2}^{N} \int_M f_j \left(\frac{\omega_j^n}{\alpha_j^n} - \frac{\omega_1^n}{\alpha_1^n} \right), \end{aligned} \tag{3.1}$$

where $\alpha_j^n := [\omega_j]^n := \int_M \omega_j^n$, $\omega_{\mathrm{sum}} := \sum_{i=1}^N \omega_i$, and the averages $\underline{R}_{\alpha_1}$, $\underline{\omega_{\mathrm{sum}}}$ are computed with respect to the Kähler class α_1, i.e.

$$\underline{R}_{\alpha_1} = n \frac{2\pi c_1(M) \cdot \alpha_1^{n-1}}{\alpha_1^n}, \quad \underline{\omega_{\mathrm{sum}}} = n \frac{\left(\sum_{i=1}^N \alpha_i\right) \cdot \alpha_1^{n-1}}{\alpha_1^n}.$$

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- It is known that Fut_c is independent of the choice of $\omega_i \in \alpha_i$ (for each i) and the choice of holomorphy potential $(f_1, \cdots, f_N)$ for $\xi \in \mathfrak{h}(\alpha_1, \cdots, \alpha_N)$ with respect to $(\omega_1, \cdots, \omega_N)$.

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- In particular, $\mathrm{Fut}_c \equiv 0$ if $(\alpha_i)_{i=1}^N$ admits a (N-tuple) coupled cscK metric.

Ruled surfaces (or Pseudo-Hirzebruch surfaces)

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- Let (L, h) be a degree -1 holomorphic line bundle over Σ equipped with a Hermitian metric h whose Chern curvature form

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- Let C denotes the Poincaré dual of a fibre of X, and let $D_\infty := \mathbb{P}(L \oplus \{0\})$, $D_0 := \mathbb{P}(\{0\} \oplus \mathcal{O})$ be the infinity divisor and zero divisor respectively on X.

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- We have $H^2(X, \mathbb{R}) = \mathbb{R}\mathrm{C} \oplus \mathbb{R}D_\infty$. Thanks to the works of Fujiki (1992), LeBrun and Singer (1993) and Tonnesen-Friedman (1998), a real $(1,1)$ cohomology class $\beta \in H^{1,1}(X, \mathbb{R}) := H^2(X, \mathbb{R}) \cap H^{1,1}(X, \mathbb{C})$ is Kähler if and only if

$$\beta^2 > 0, \quad \beta \cdot \mathrm{C} > 0, \quad \beta \cdot D_\infty > 0, \quad \beta \cdot D_0 > 0.$$

In particular, the Kähler cone (i.e. the set of all Kähler classes) in X is given by

$$\mathcal{K}(X) = \{a\mathrm{C} + bD_\infty \in H^{1,1}(X, \mathbb{R}) : \ a > 0, \ b > 0\}.$$

Existence of twisted cscK metrics on ruled surfaces

We have the following existence of solutions to the twisted cscK equations on ruled surfaces.

Theorem 4.1 (M. [7, Theorem 1.1])

Suppose $\Omega_a = 2\pi(\mathrm{C} + aD_\infty)$ and $\Omega_b = 2\pi(\mathrm{C} + bD_\infty)$ are two normalized Kähler classes on the ruled surface $X = \mathbb{P}(L \oplus \mathcal{O})$ defined above with $b \geq \frac{a(5a+4)}{2(a+1)}$ and $a > 0$. Then there exists a Kähler metric $\omega \in \Omega_a$ whose "momentum profile" ϕ is given by

$$\phi(\tau) = \frac{a\tau - \tau^2}{a}, \quad \tau \in [0, a]$$

such that ω is a χ-twisted cscK metric for some Kähler metric χ in the class Ω_b.

Existence of twisted cscK metrics on ruled surfaces

We have the following existence of solutions to the twisted cscK equations on ruled surfaces.

Theorem 4.1 (M. [7, Theorem 1.1])

Suppose $\Omega_a = 2\pi(\mathrm{C} + aD_\infty)$ *and* $\Omega_b = 2\pi(\mathrm{C} + bD_\infty)$ *are two normalized Kähler classes on the ruled surface* $X = \mathbb{P}(L \oplus \mathcal{O})$ *defined above with* $b \geq \frac{a(5a+4)}{2(a+1)}$ *and* $a > 0$*. Then there exists a Kähler metric* $\omega \in \Omega_a$ *whose "momentum profile"* ϕ *is given by*

$$\phi(\tau) = \frac{a\tau - \tau^2}{a}, \quad \tau \in [0, a]$$

such that ω *is a* χ*-twisted cscK metric for some Kähler metric* χ *in the class* Ω_b*.*

We have the following solvability criteria for the J-equation on ruled surfaces.

Theorem 4.2 (M. [7, Theorem 1.4])

Suppose $\Omega_a = 2\pi(\mathrm{C} + aD_\infty)$ *and* $\Omega_b = 2\pi(\mathrm{C} + bD_\infty)$ *are two normalized Kähler classes on the ruled surface* $X = \mathbb{P}(L \oplus \mathcal{O})$*, where* $a, b > 0$*. If* $b \geq \frac{a^2}{2(a+1)}$*, then there exist Kähler metrics* $\omega \in \Omega_a$ *and* $\chi \in \Omega_b$ *solving the* J*-equation* (2.3)*.*

Non-existence of coupled cscK metrics on ruled surfaces

Our next result is the following.

Theorem 4.3 (M. [7, Theorem 1.3])

The pair of (normalized) Kähler classes $(\Omega_a = 2\pi(\mathrm{C} + aD_\infty),\ \Omega_b = 2\pi(\mathrm{C} + bD_\infty))$, *where both* $a, b > 0$, *on the ruled surface* $X = \mathbb{P}(L \oplus \mathcal{O})$ *does not admit any coupled cscK metric.*

Non-existence of coupled cscK metrics on ruled surfaces

Our next result is the following.

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The pair of (normalized) Kähler classes $(\Omega_a = 2\pi(\mathrm{C} + aD_\infty),\ \Omega_b = 2\pi(\mathrm{C} + bD_\infty))$, where both $a, b > 0$, on the ruled surface $X = \mathbb{P}(L \oplus \mathcal{O})$ does not admit any coupled cscK metric.

As a corollary of Theorem 4.3 and Theorem 4.2 and using some scaling argument, we have the following result.

Corollary 4.1 (M. [7, Corollary 1.5])

Consider any two Kähler classes $\alpha := a_1\mathrm{C} + a_2D_\infty$ and $\beta := b_1\mathrm{C} + b_2D_\infty$, where $a_i, b_i > 0$ for $i = 1, 2$, on the ruled surface $X = \mathbb{P}(L \oplus \mathcal{O})$. If $b_1 = 2\pi$, i.e. the class β is in the normalized form $\beta = 2\pi(\mathrm{C} + \frac{b_2}{2\pi}D_\infty)$, then the pair (α, β) does not admit any coupled cscK metric on X. Moreover, when $b_1 \neq 2\pi$ the pair (α, β) does not admit a coupled cscK metric on X if the following holds

$$\frac{b_2}{b_1} > \frac{a_2^2}{2a_1(a_1 + a_2)}. \tag{4.1}$$

Calabi ansatz/symmetry

- On $X \setminus (D_\infty \cup D_0)$ we consider the logarithm of the fiber-wise norm
$$s = \ln |(z,w)|_h^2 = \ln h(z) + \ln |w|^2$$
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where z being a local coordinate on Σ and w being a fibre coordinate on L.
- Following Calabi (1982) and Hwang-Singer (2002), consider the following closed real $(1, 1)$-form
$$\omega = p^* \omega_\Sigma + \sqrt{-1}\partial\bar{\partial}u(s), \qquad (4.2)$$
where $u \in C^\infty(\mathbb{R}, \mathbb{R})$ and $p : X \to \Sigma$ is the projection map.

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- If u is a strictly convex function, then ω is a Kähler form on $X \setminus (D_\infty \cup D_0)$. Now, in order to extend it to X and to belong in $\Omega_a := 2\pi(\mathrm{C} + \mathrm{aD}_\infty)$, we must have
$$\lim_{s\to-\infty} u'(s) = 0, \quad \lim_{s\to\infty} u'(s) = a. \tag{4.3}$$

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- Let $\tau := u'(s) \in [0, a]$ and $\phi(\tau) := u''(s)$. Then ϕ satisfies the following conditions
$$\phi(0) = \phi(a) = 0, \quad \phi'(0) = 1, \quad \phi'(a) = -1. \tag{4.4}$$
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Also, ϕ is positive on $(0, a)$ since u is a strictly convex function.
- In general, any smooth function $\phi : [0, a] \to \mathbb{R}$ with $\phi > 0$ on $(0, a)$ and satisfies the above boundary conditions (4.4) is called a momentum profile.

1. Canonical metrics in Kähler Geometry

2. (Generalized) Futaki invariant

3. Ruled surfaces and main results

4. A bound for Chen-Cheng invariant on ruled surfaces

Chen's continuity path related to cscK problem, and Chen-Cheng Invariant

- Chen[7] introduced the following continuity path to find a cscK metric in a given Kähler class $[\omega_0]$ on n-dimensional compact Kähler manifold (M, ω_0):

$$tR(\omega_{\varphi_t}) - (1-t)\Lambda_{\omega_{\varphi_t}}\chi = t\underline{R} - (1-t)\underline{\chi}, \tag{5.1}$$

where $t \in [0,1]$, χ is a Kähler form and $\omega_{\varphi_t} := \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi_t$ is a path in $[\omega_0]$.

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- He also defined the following quantity related to the above continuity path

$$R([\omega_0], \chi) := \sup\left\{t_0 \in [0,1] : \text{ (5.1) can be solved for any } 0 \le t \le t_0\right\},$$

and conjectured that it is an invariant of $[\chi]$,

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Theorem 5.1 (Chen-Cheng, Theorem 1.7)

Let χ be a Kähler form. If the Mabuchi K-energy is bounded from below on a compact Kähler manifold $(M, [\omega_0])$, then

$$R([\omega_0],[\chi]) = 1 \quad \Longleftrightarrow \quad \text{the J-equation } \Lambda_{\omega_\varphi}\chi = \underline{\chi} \text{ is solvable.}$$

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A bound for Chen-Cheng invariant on ruled surfaces

We have the following result about bound of the Chen-Cheng invariant on ruled surfaces.

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Proposition 5.1 (M. [7, Proposition 1.8])

Suppose $\Omega_a = 2\pi(\mathrm{C} + aD_\infty)$ *and* $\Omega_b = 2\pi(\mathrm{C} + bD_\infty)$ *are two normalized Kähler classes on the ruled surface* $X = \mathbb{P}(L \oplus \mathcal{O})$ *defined above with* $b \geq \frac{a^2}{2(1+a)}$ *and* $a > 0$. *Then we have*

$$\frac{2b(a+1) - a^2}{3a^2 + 4a + 2b(a+1)} \leq R(\Omega_a, \Omega_b) < 1.$$

In particular, we have

$$\frac{a+2}{5a+6} \leq R(\Omega_a, \Omega_a) < 1$$

for any normalized Kähler class $\Omega_a = 2\pi(\mathrm{C} + aD_\infty)$, *where* $a > 0$, *on the ruled surface* X.

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Remark 5.1

Note that

$$\frac{2b(a+1) - a^2}{3a^2 + 4a + 2b(a+1)} \geq \frac{1}{2} \quad \Longleftrightarrow \quad b \geq \frac{a(5a+4)}{2(a+1)}.$$

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Thank you for your attention!