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Ph.D. Synopsis : Weyl covariance, Hydrodynamics and Gravity

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ABSTRACT: In this synopsis, we summarise some of our work on the hydrodynamics that emerges out of an underlying strongly coupled conformal field theory at a finite temperature and chemical potentials. The hydrodynamic transport in such theories can be studied using the methods of gauge-gravity duality which reformulate the questions about transport into questions in a classical gravity theory interacting with matter. Further, we show that the the underlying Weyl covariance of the conformal field theory finds a natural expression both in hydrodynamics and in gravity. We also describe a new non-dissipative transport phenomenon which was discovered using the above methods.

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1. Motivation

Our understanding of the physical world relies greatly on the various pathbreaking experimental discoveries and the deep mathematical structures which have been uncovered in the quest to understand those observations. In this way, a variety of physical phenomena have received their common explanation through a minimal set of theoretical tools. It is especially exhilarating for a theoretical physicist to find apparently unrelated phenomena unified this way under common insights. A pertinent example in this regard is the way the theory of quantum fields has been used to explore a wide range of phenomena cutting across the conventional borders which separate the various fields of physics.

In a similar vein, it has been realised that string theory, a set of ideas that had originally originated in the experimental studies of gauge theories, also provides a natural framework to address an entirely different question altogether - that of constructing quantum theories of gravity. This realisation has over the past few decades led string theorists to a theoretical structure which seamlessly interweaves some of the central ideas of theoretical physics - from quantum mechanics to gauge theories, from general relativity to supersymmetry apart from opening up many new avenues in mathematical physics.

Notable in this regard is the way string theory incorporates and extends the idea of duality in the space of theories. Since duality will be a crucial theme of the work that is reported in this synopsis, it is essential that we briefly review the idea of duality at this point. Duality is a phenomenon in quantum theories whereby two theories which are classically distinct lead to the same theory after quantisation. The simplest and oft-used example of duality is the fact that a quantum theory of a many-particle system is identical to a theory of quantum fields despite the fact that a classical many particle system and a classical field theory are very different as physical systems.

Over the last century, many such dualities have been discovered and they have served as a rich source of insights whenever they are applicable. A non-exhaustive list of such field theoretical dualities include Kramers-Wannier duality in the Lattice Ising theory, the phenomenon of bosonisation in various $1+1$d systems, Seiberg dualities of the $\mathcal{N}=1$ superconformal fixed points, various $\mathcal{N}=2$ dualities inspired by the pioneering solution of Seiberg and Witten and the Montonen-Olive duality in $\mathcal{N}=4$ supersymmetric Yang-Mills theory. String theory incorporates and extends various such dualities in it deep structure resulting in a rich web of string-string dualities connecting various string theories. More surprisingly, about a decade ago, it was realised that there are dualities which connect certain string theories on one side to field theories on the other side thus supporting the idea that the study of a particular corner in the space of all non-gravitational field theories leads one irrevocably into string theory and quantum gravity.

That one needed an apparently unrelated quest towards a quantum theory of gravity to lead us to such a statement about ordinary quantum field theories is itself revealing. More than anything else, it is humbling for a physicist to realise that our understanding of strongly coupled field theories is so rudimentary that some of the deepest statements about field theories in the last few decades has been discovered by pursuing an esoteric quest that according to naive intuition, seems far removed from such questions.

We live in very interesting times - despite the fact that physics has made enormous progress in understanding the many aspects of reality, there is still an enormous sea of challenging questions which are yet to be answered. These questions take various forms - what are the possible behaviours that are possible in a given system ? What is the most efficient way to understand such a behaviour ? What mathematical patterns underlie these behaviours ? It is fair to say that in many physical systems we are far away from answering these questions comprehensively. Our obstacles are many - there is no foolproof way to isolate the right degrees of freedom to describe a particular situation in physics except for symmetry considerations and prior experience. The most obvious variables are often the least useful because the phenomenon we are interested in arises out of strong interactions between the obvious degrees of freedom. Dualities, when present, give a rare window into these questions. In fact, it can be argued that almost everything that we analytically know about non-trivial quantum systems can be attributed to symmetries along with various exact and approximate dualities exhibited by the system (especially notable in this regard is the role of effective field theories which are essentially approximate dualities at the low-energy regime).

In this synopsis, we will describe a particular example of such a duality and its usefulness in answering questions about certain strongly coupled model field theories. These are questions regarding the transport phenomena in these systems which at the outset appear to be theoretically intractable, but we will take recourse to the symmetries and dualities to make progress. These dualities will lead us into the study of what naively appears to be an unrelated set of questions about black holes, black branes and their long-time evolution and at the end of it all, we will discover some very interesting transport phenomena which were discovered through the methods that are outlined in this synopsis.

2. Hydrodynamic description

How does a physicist proceed when confronted with a strongly coupled system ? Our first instinct is to ask for the basic symmetries that govern the system under study. Often strongly coupled field theories have spacetime symmetries of translations, rotations and boosts along with other global symmetries. Although exceptions to this statement (which involve spatially modulated and/or orientationally ordered phases of matter where these symmetries are spontaneously broken) are of great interest[1], we wish to focus on phases where these symmetries serve as good guides to the dynamics of the system[2]. In a relativistic theory, these symmetries guarantee the conservation of the energy-momentum tensor[3] $\nabla_\mu T^{\mu\nu} = 0$ and conservation of the charge current $\nabla_\mu J^\mu_i = 0$.

To begin with, we will look for a sector where the dynamics is essentially captured by the dynamics of the energy-momentum and charge flow. This in particular means that if there are other macroscopic degrees of freedom in the system, we are assuming that there is a sector where their dynamics can be decoupled and hence ignored. We do not know how generic the presence of such a sector is in strongly coupled systems[4] , but later on we will present examples where such a truncation to just the conserved charges can be justified at strong coupling.

Having said that, we should also add that the statics of many interacting systems have a thermodynamic limit where the conserved charges (along with entropy given as a function of these conserved charges) capture the essential physics. So, it is not entirely unnatural to expect that if we introduce a slow time-dependence with small spatial gradients such that local patches of the system are in thermodynamic equilibrium (and assuming we are able to introduce these spatial gradients without exciting the other macroscopic degrees of freedom), then the dynamics of the conserved quantities continues to be an effective way to describe the dynamics of the system.

Having convinced ourselves that this should be the case, we however face a conundrum - the energy momentum tensor $T^{\mu\nu}$ in d dimensions is a symmetric rank two tensor with $d(d+1)/2$ components and the d equations $\nabla_\mu T^{\mu\nu} = 0$ simply are not enough to determine

[1] After all, some of the most studied models in physics including solids, magnets and liquid crystals fall into this category.

[2] It would be an interesting exercise to generalise the methods detailed below to cover such ordered phases.There have been many attempts to apply holographic methods to spatially modulated phases but a general theory of the kind we describe below is yet to be developed.

[3] In a quantum field theory, a precise way to state the conservation laws is to invoke the Ward identities satisfied by arbitrary n-point functions of energy-momentum operator $\hat{T}^{\mu\nu}$ and the current operator $\hat{J}^\mu_i$. In our present discussion, we will focus mainly on the one-point functions $T^{\mu\nu}$ and J^μ_i ignoring admittedly interesting questions about the hydrodynamic fluctuations.

[4] What we do know is that phases with macroscopic degrees of freedom that are not conserved can exhibit qualitatively new features.

the dynamics of all these components[5] ! Similarly, for the ith charge we have a single equation $\nabla_\mu J_i^\mu = 0$ which is not enough to determine the dynamics of d components of J_i^μ. This in turn implies that either we should give up on treating each and every component of $T^{\mu\nu}$ and J_i^μ as independent or we should add more dynamical equations to the set of conservation laws.

We will again be guided by our thermodynamic intuition - a static thermodynamic equilibrium state in a field theory often has a rest frame (specified by a unit time-like vector u^μ) which we can think of as a frame where there is no energy flow - more precisely, u^μ can be thought of as the time-like eigenvector of the energy momentum tensor T^μ_ν. Such a state is characterised by its rest frame energy density $\varepsilon \equiv T^{\mu\nu} u_\mu u_\nu$ and the rest frame charge density $n_i \equiv -J_i^\mu u_\mu$ (where we have crucially used our assumption that all other macroscopic degrees of freedom can be ignored). This fact combined with our physical intuition derived from various real systems leads us to the basic idea that resolves our problem - if we treat u^μ, ε and n_i as our basic variables in terms of which T^μ_ν and J_i^μ are specified, then we have as many variables as there are conservation laws resulting in a consistent dynamical system. When the macroscopic behaviour of a quatum field theory - strongly coupled or otherwise reduces to such a system, we say that we have a hydrodynamic description emerging out of that particular quantum field theory.

Given the argument above, one can ask whether there are actually calculable model quantum field theories in which such an emergence of hydrodynamic description can actually be exhibited and the corresponding T^μ_ν and J_i^μ be calculated as functions of u^μ, ε and n_i. This question leads us into the field of transport phenomena in various field theories where for weakly coupled field theories, there are by now various well-developed perturbative formalisms in non-equilibrium statistical physics to answer this question. In case of strongly coupled field theories, there is no general approach except numerical studies and studying transport in quantum field theories is still a computationally expensive proposition. So, it would appear that to exhibit the emergence of hydrodynamic behaviour in a strongly coupled system is for all practical purposes an intractable problem. While this pessimistic assertion may be justified in general, as we have mentioned before, there are strongly coupled systems which can be made tractable by using dualities and string theory with its cornucopia of dualities is an apt place to look for such model systems.

3. Gauge-Gravity duality and Supersymmetric Yang-Mills

In this synopsis, we will be focusing on a particular duality called gauge-gravity duality in string theory and ask what can it tell us about transport in strongly coupled field theories. For reasons of tractability, we will focus on a strongly coupled but a conformal quantum field theory which has no dimensionful quantity in its microscopic description [6]. Further, all

[5] Unless $d(d+1)/2 = d$ whose only non-trivial solution is $d = 1$. We may try to save ourselves specialising to conformal field theories where the trace degree of freedom T^μ_μ is removed, in which case we have enough equations only when $d(d+1)/2 - 1 = d$ or $d = 2$.

[6] Note however that a conformal field theory at a finite temperature equilibrium state has a macroscopic scale - the temperature which is a characteristic of that state. Similar statements can be made about hydrodynamic states with a given energy density ε or charge density n.

the field theories we will consider have a parameter N characterising the number of degrees of freedom and we will be working in a large N limit. In higher dimensions, most known examples of strongly interacting conformal field theories happen to be supersymmetric, but supersymmetry will not play a direct role in our subsequent discussion.

A particularly good example of the kind of models under consideration is the $SU(N)$ supersymmetric gauge theory in 3+1 dimensions with a four-fold supersymmetry. Since we will later be discussing the transport properties in this model in some detail, we will briefly review the main features of this model. It has the following field content

- An $SU(N)$ gauge-field A_μ^a where $\mu = 0, 1, 2, 3$ denotes that it is a spacetime vector field and the index $a = 1, \ldots, N^2 - 1$ runs over the adjoint representation of the gauge group $SU(N)$.

- Four chiral fermions $\psi_a^{\alpha A}$ charged under the adjoint representation of $SU(N)$ where $\alpha = 1, 2$ denotes that it is a chiral fermion field and $A = 1, \ldots 4$ is the flavor index. The complex conjugate of $\psi_a^{\alpha A}$ is an anti-chiral spinor denoted as $\bar{\psi}_{Aa}^{\dot{\alpha}} = (\psi_a^{\alpha A})^*$

- Six real scalars $\Phi_a^{AB} = (\Phi_{aAB})^* = -\Phi_a^{BA}$ charged under the adjoint representation of $SU(N)$. It is convenient to define $\bar{\Phi}_{AB}$ via the relation

$$\Phi_a^{AB} \equiv \frac{1}{2}\epsilon^{ABCD}\bar{\Phi}_{CD}^a$$

The theory is defined by the Lagrangian density[1]

$$\begin{aligned}
-\mathcal{L} = &\frac{1}{4}F_{\mu\nu}^a F_a^{\mu\nu} - (D_\mu \bar{\Phi}_{AB})_a (D^\mu \Phi^{AB})_a + i\psi_a^{\alpha A}\sigma_{\alpha\dot{\alpha}}^\mu (D_\mu \bar{\psi}_A^{\dot{\alpha}})_a + \\
&+ g^2 f^{abc}\Phi_b^{AB}\Phi_c^{CD} f^{ade}\bar{\Phi}_{AB}^d \bar{\Phi}_{CD}^e + g\sqrt{2} f^{abc}\left[\psi_a^{\alpha A}\bar{\Phi}_{AB}^b \psi_\alpha^{cB} + \bar{\psi}_{\dot{\alpha}A}^a \Phi_b^{AB}\bar{\psi}_{cB}^{\dot{\alpha}}\right]
\end{aligned}$$

where f^{abc} are the structure constant of the gauge group $SU(N)$, g is the gauge-coupling and D_μ is the covariant derivative appropriate to the gauge group. This theory is among the very few interacting conformal field theories known in $3 + 1$ dimensions. In particular, the coupling g in the above lagrangian does not run with scale and the classical scale-invariance of the above lagrangian is continued to be preserved by the quantum fluctuations.

This theory is manifestly invariant under a $SU(4)$ flavor symmetry (often called the R-symmetry of this theory) under which the different fields transform as

$$\psi_\alpha^A \to U^A{}_B \psi_\alpha^B \qquad \bar{\psi}_{\dot{\alpha}A} \to (U^*)_A{}^B \bar{\psi}_{\dot{\alpha}B}$$

and

$$\Phi^{AB} \to U^A{}_C \Phi^{CD} (U^T)_D{}^B \qquad \bar{\Phi}_{AB} \to (U^*)_A{}^C \bar{\Phi}_{CD} (U^\dagger)^D{}_B$$

where U is a 4×4 unitary matrix with ($UU^\dagger = 1$). This classical global symmetry is continued to be respected by the quantum fluctuations and hence we have quantum global $SU(4)$ symmetry in our theory with the corresponding Noether currents. For reasons that will become clearer later, it is especially convenient to focus on the transport of the Noether charge associated with a $U(1)$ subgroup of $SU(4)$ made up of Us of the form

$$U = \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & e^{-3i\theta} \end{pmatrix} \in SU(4)$$

One of the main results of the work presented here would be the claim(that we hope to substantiate by the end of this synopsis) that in an appropriate limit, using gauge-gravity duality, we can precisely argue for the existence of certain novel phenomena in the transport of this charge and compute the corresponding transport coefficients.

The study of this conformal field theory can be simplified further by taking the number of colors N in this theory to be large. Following 't Hooft, to get a sensible limit we should keep the 't Hooft coupling $\lambda \equiv g^2 N$ fixed as N is taken to infinity. Such a limit gives us an interacting theory whose interactions are governed by the parameter λ. This then is a precise model system, where the questions of the previous section can be asked - what are the transport properties of this model when λ is very large ? Are they even calculable ? Can one see the emergence of hydrodynamic behaviour in the macroscopic dynamics of this model ? The answer to these apparently intractable questions about an interacting theory is surprisingly in affirmative because of the gauge-gravity duality. A variety of transport phenomenon in this strongly interacting theory using the gauge-gravity duality have been by now well-studied by various authors. In the course of this synopsis, we will describe only the studies to which I have contributed and briefly review those parts of other works which provide the necessary background.

4. Weyl Covariance and Hydrodynamics

Before applying gauge-gravity duality to derive the hydrodynamics emerging out of an underlying conformal theory, it is worthwhile to understand the structure of such a hydrodynamic description[7]. The fact that the microscopic theory is devoid of scales imposes strict constraints on the kind of transport that can occur in the macroscopic description. We will summarise in this section, a formalism developed in [2] to naturally incorporate these constraints into the hydrodynamic description.

A conformal field theory living on a d dimensional spacetime $\mathcal{M}$ with a metric $g_{\mu\nu}$ is by definition a theory which is covariant (upto quantum Weyl anomalies) under the Weyl transformation which replaces the old metric $g_{\mu\nu}$ with $\tilde{g}_{\mu\nu}$ given by

$$g_{\mu\nu} = \mathrm{e}^{2\phi(x)}\tilde{g}_{\mu\nu}; \qquad g^{\mu\nu} = \mathrm{e}^{-2\phi(x)}\tilde{g}^{\mu\nu} \tag{4.1}$$

In some sense, these theories are naturally thought of (again upto Weyl anomalies) as living on a spacetime $\mathcal{M}$ which has a class of metrics $\mathcal{C}_g$ which are related to our original metric

[7]This section is based on the work that was done by the author and presented in the form of the paper [2].

by a Weyl transformation. We will henceforth refer to the fluid which emerges out of such a conformal field theory at a finite temperature as a conformal fluid.

The conformal fluid which emerges out of a conformal field theory inherits Weyl-covariance from its microscopic theory. The question that we wish to answer is how this Weyl-covariance manifests itself at the level of hydrodynamics and how does one go about constructing Weyl-covariant quantities in hydrodynamics. Let u^μ be the unit time-like vector describing the fluid motion. Using $g_{\mu\nu}u^\mu u^\nu = \tilde{g}_{\mu\nu}\tilde{u}^\mu\tilde{u}^\nu = -1$, we get the Weyl transformation of the velocity $u^\mu = \mathrm{e}^{-\phi}\tilde{u}^\mu$. Now, we can go ahead and construct various derivatives of this velocity field, provided we have a way of differentiating fields in a Weyl-covariant way.

To see how this might be possible, we begin with the Weyl-transformation of the Christoffel symbols

$$\Gamma_{\lambda\mu}{}^{\nu} = \widetilde{\Gamma}_{\lambda\mu}{}^{\nu} + \delta^\nu_\lambda\partial_\mu\phi + \delta^\nu_\mu\partial_\lambda\phi - \tilde{g}_{\lambda\mu}\tilde{g}^{\nu\sigma}\partial_\sigma\phi \tag{4.2}$$

from which it follows that

$$\mathcal{A}_\nu \equiv u^\mu\nabla_\mu u^\nu - \frac{\nabla_\mu u^\mu}{d-1}u_\nu = \tilde{\mathcal{A}}_\nu + \partial_\nu\phi. \tag{4.3}$$

This transformation means that hydrodynamics provides a natural 'gauge field' for Weyl transformations which can be used to construct a Weyl-covariant derivative. We define a Weyl covariant derivative $\mathcal{D}$ such that, if a tensorial quantity $Q^{\mu\ldots}_{\nu\ldots}$ obeys $Q^{\mu\ldots}_{\nu\ldots} = e^{-w\phi}\widetilde{Q}^{\mu\ldots}_{\nu\ldots}$, then $\mathcal{D}_\lambda\ Q^{\mu\ldots}_{\nu\ldots} = e^{-w\phi}\widetilde{\mathcal{D}}_\lambda\widetilde{Q}^{\mu\ldots}_{\nu\ldots}$ where

$$\begin{aligned}\mathcal{D}_\lambda\ Q^{\mu\ldots}_{\nu\ldots} \equiv&\ \nabla_\lambda\ Q^{\mu\ldots}_{\nu\ldots} + w\ \mathcal{A}_\lambda Q^{\mu\ldots}_{\nu\ldots}\\ &+ \left[g_{\lambda\alpha}\mathcal{A}^\mu - \delta^\mu_\lambda\mathcal{A}_\alpha - \delta^\mu_\alpha\mathcal{A}_\lambda\right] Q^{\alpha\ldots}_{\nu\ldots} + \ldots\\ &- \left[g_{\lambda\nu}\mathcal{A}^\alpha - \delta^\alpha_\lambda\mathcal{A}_\nu - \delta^\alpha_\nu\mathcal{A}_\lambda\right] Q^{\mu\ldots}_{\alpha\ldots} - \ldots\end{aligned} \tag{4.4}$$

Note that the above covariant derivative is metric compatible ($\mathcal{D}_\lambda g_{\mu\nu} = 0$). In mathematical terms, what we have done is to use the additional mathematical structure provided by a fluid background (namely a unit time-like vector field with conformal weight $w = 1$) to define what is known as a *Weyl connection* over $(\mathcal{M}, \mathcal{C}_g)$ where $\mathcal{M}$ is the spacetime manifold with the conformal class of metrics $\mathcal{C}_g$.

A torsionless connection ∇^{weyl} is called a Weyl connection (see for example, [3] and references therein) if for every metric in the conformal class $\mathcal{C}_g$ there exists a one form $\mathcal{A}_\mu$ such that $\nabla^{weyl}_\mu g_{\nu\lambda} = 2\mathcal{A}_\mu g_{\nu\lambda}$. Having a fluid over the manifold provides us a natural one form $\mathcal{A}_\mu$ (see below), which can in turn be used to define a Weyl connection. The 'prolonged' covariant derivative $\mathcal{D}$ is related to this Weyl connection via the relation $\mathcal{D}_\mu = \nabla^{weyl}_\mu + w\mathcal{A}_\mu$. In terms of this covariant derivative, the condition for Weyl connection is just the statement of metric compatibility ($\mathcal{D}_\lambda g_{\mu\nu} = 0$) and the one-form $\mathcal{A}_\mu$ is uniquely determined by requiring that the covariant derivative of u^μ be transverse ($u^\lambda\mathcal{D}_\lambda u^\mu = 0$) and traceless ($\mathcal{D}_\lambda u^\lambda = 0$).

We can define a curvature associated with the Weyl-covariant derivative by the usual procedure of evaluating the commutator between two covariant derivatives. For a covariant vector field $V_\mu = \mathrm{e}^{-w\phi}\widetilde{V}_\mu$, we get

$$
\begin{aligned}
[\mathcal{D}_\mu, \mathcal{D}_\nu]V_\lambda &= w\ \mathcal{F}_{\mu\nu}\ V_\lambda + \mathcal{R}_{\mu\nu\lambda}{}^{\alpha}\ V_\alpha \qquad \text{with} \\
\mathcal{F}_{\mu\nu} &= \nabla_\mu \mathcal{A}_\nu - \nabla_\nu \mathcal{A}_\mu \\
\mathcal{R}_{\mu\nu\lambda\sigma} &= R_{\mu\nu\lambda\sigma} - \delta^\alpha_{[\mu} g_{\nu][\lambda}\delta^\beta_{\sigma]}\left(\nabla_\alpha \mathcal{A}_\beta + \mathcal{A}_\alpha \mathcal{A}_\beta - \frac{\mathcal{A}^2}{2} g_{\alpha\beta}\right) + \mathcal{F}_{\mu\nu} g_{\lambda\sigma}
\end{aligned}
\tag{4.5}
$$

where we have introduced two new Weyl-invariant tensors $\mathcal{F}_{\mu\nu} = \widetilde{\mathcal{F}}_{\mu\nu}$ and $\mathcal{R}_{\mu\nu\lambda}{}^{\alpha} = \widetilde{\mathcal{R}}_{\mu\nu\lambda}{}^{\alpha}$ and $B_{[\mu\nu]} \equiv B_{\mu\nu} - B_{\nu\mu}$ indicates antisymmetrisation[8]. Since these Weyl-covariant counterparts of curvature tensors will play a useful role in the formulation of conformal hydrodynamics, we will briefly describe their properties[9].

We can write down similar expressions involving Ricci tensor, Ricci scalar and Einstein tensor.

$$
\begin{aligned}
\mathcal{R}_{\mu\nu} &\equiv \mathcal{R}_{\mu\alpha\nu}{}^{\alpha} = R_{\mu\nu} + (d-2)\left(\nabla_\mu \mathcal{A}_\nu + \mathcal{A}_\mu \mathcal{A}_\nu - \mathcal{A}^2 g_{\mu\nu}\right) + g_{\mu\nu}\nabla_\lambda \mathcal{A}^\lambda + \mathcal{F}_{\mu\nu} = \widetilde{\mathcal{R}}_{\mu\nu} \\
\mathcal{R} &\equiv \mathcal{R}_{\alpha}{}^{\alpha} = R + 2(d-1)\nabla_\lambda \mathcal{A}^\lambda - (d-2)(d-1)\mathcal{A}^2 = \mathrm{e}^{-2\phi}\widetilde{\mathcal{R}} \\
\mathcal{G}_{\mu\nu} &\equiv \mathcal{R}_{\mu\nu} - \frac{\mathcal{R}}{2} g_{\mu\nu} = G_{\mu\nu} + (d-2)\left[\nabla_\mu \mathcal{A}_\nu + \mathcal{A}_\mu \mathcal{A}_\nu - \left(\nabla_\lambda \mathcal{A}^\lambda - \frac{d-3}{2}\mathcal{A}^2\right) g_{\mu\nu}\right] + \mathcal{F}_{\mu\nu}
\end{aligned}
\tag{4.7}
$$

These curvature tensors obey various Bianchi identities [10]

$$
\begin{aligned}
\mathcal{R}_{\mu\nu\lambda}{}^{\alpha} + \mathcal{R}_{\lambda[\mu\nu]}{}^{\alpha} &= 0 \\
\mathcal{D}_\lambda \mathcal{F}_{\mu\nu} + \mathcal{D}_{[\mu}\mathcal{F}_{\nu]\lambda} &= 0 \\
\mathcal{D}_\lambda \mathcal{R}_{\mu\nu\alpha}{}^{\beta} + \mathcal{D}_{[\mu}\mathcal{R}_{\nu]\lambda\alpha}{}^{\beta} &= 0
\end{aligned}
\tag{4.8}
$$

and various reduced Bianchi identities[11]

$$
\begin{aligned}
\mathcal{R}_{[\mu\nu]} = \mathcal{R}_{\mu\nu\alpha}{}^{\alpha} &= d\ \mathcal{F}_{\mu\nu} \\
\mathcal{D}_{[\mu}\mathcal{R}_{\nu]\lambda} + \mathcal{D}_\sigma \mathcal{R}_{\mu\nu\lambda}{}^{\sigma} &= 0 \\
\mathcal{D}_\lambda\left(\mathcal{G}^{\mu\lambda} - \mathcal{F}^{\mu\lambda}\right) &= 0
\end{aligned}
\tag{4.9}
$$

The tensor $\mathcal{R}_{\mu\nu\lambda\sigma}$ does not have the same symmetry properties as that of the usual Riemann tensor. For example,

$$
\begin{aligned}
\mathcal{R}_{\mu\nu\lambda\sigma} + \mathcal{R}_{\mu\nu\sigma\lambda} &= 2\ \mathcal{F}_{\mu\nu} g_{\lambda\sigma} \\
\mathcal{R}_{\mu\nu\lambda\sigma} - \mathcal{R}_{\lambda\sigma\mu\nu} &= -\delta^\alpha_{[\mu} g_{\nu][\lambda}\delta^\beta_{\sigma]}\mathcal{F}_{\alpha\beta} + \mathcal{F}_{\mu\nu} g_{\lambda\sigma} - \mathcal{F}_{\lambda\sigma} g_{\mu\nu} \\
\mathcal{R}_{\mu\alpha\nu\beta} V^\alpha V^\beta - \mathcal{R}_{\nu\alpha\mu\beta} V^\alpha V^\beta &= \mathcal{F}_{\mu\nu}\ V^\alpha V_\alpha
\end{aligned}
\tag{4.10}
$$

[8] As is evident from the notation above, we use calligraphic alphabets to denote the Weyl-covariant counterparts of the usual curvature tensors. Our notation for the usual Riemann tensor is defined by the relation

$$
[\nabla_\mu, \nabla_\nu]V_\lambda = R_{\mu\nu\lambda}{}^{\sigma} V_\sigma. \tag{4.6}
$$

Note that the curvature tensors used in this synopsis are negative of those employed in [2].

[9] Note that these curvature tensors are essential even if one is in a flat spacetime, since most of these Weyl-covariant curvatures do not vanish for a general velocity configuration in flat spacetime.

[10] These identities can be derived from the Jacobi identity for the covariant derivative - $[\mathcal{D}_{[\mu}, [\mathcal{D}_{\nu]}, \mathcal{D}_\lambda] + [\mathcal{D}_\lambda, [\mathcal{D}_\mu, \mathcal{D}_\nu]] = 0$

[11] These identities are obtained from the Bianchi identities by contractions.

The conformal tensors of the underlying spacetime manifold appear in the above formalism as a subset of conformal observeables in hydrodynamics. These conformal tensors are the Weyl-covariant tensors that are independent of the background fluid velocity. The Weyl curvature $C_{\mu\nu\lambda\sigma}$ is a well-known example of a conformal tensor. We have

$$\mathcal{C}_{\mu\nu\lambda\sigma} \equiv \mathcal{R}_{\mu\nu\lambda\sigma} + \delta^{\alpha}_{[\mu} g_{\nu][\lambda} \delta^{\beta}_{\sigma]} \mathcal{S}_{\alpha\beta} = C_{\mu\nu\lambda\sigma} + \mathcal{F}_{\mu\nu} g_{\lambda\sigma} = \mathrm{e}^{2\phi} \widetilde{\mathcal{C}}_{\mu\nu\lambda\sigma} \tag{4.11}$$

where the Schouten tensor $\mathcal{S}_{\mu\nu}$ is defined as

$$\mathcal{S}_{\mu\nu} \equiv \frac{1}{d-2}\left(\mathcal{R}_{\mu\nu} - \frac{\mathcal{R} g_{\mu\nu}}{2(d-1)}\right) = S_{\mu\nu} + \left(\nabla_\mu \mathcal{A}_\nu + \mathcal{A}_\mu \mathcal{A}_\nu - \frac{\mathcal{A}^2}{2} g_{\mu\nu}\right) + \frac{\mathcal{F}_{\mu\nu}}{d-2} = \widetilde{\mathcal{S}}_{\mu\nu} \tag{4.12}$$

From equation (4.11), it is clear that $C_{\mu\nu\lambda\sigma} = \mathcal{C}_{\mu\nu\lambda\sigma} - \mathcal{F}_{\mu\nu} g_{\lambda\sigma}$ is clearly a conformal tensor. Such an analysis can in principle be repeated for the other known conformal tensors in arbitrary dimensions.

The Weyl Tensor $C_{\mu\nu\lambda\sigma}$ has the same symmetry properties as that of Riemann Tensor $R_{\mu\nu\lambda\sigma}$.

$$\begin{aligned} C_{\mu\nu\lambda\sigma} &= -C_{\nu\mu\lambda\sigma} = -C_{\mu\nu\sigma\lambda} = C_{\lambda\sigma\mu\nu} \\ \text{and} \quad C_{\mu\alpha\lambda}{}^{\alpha} &= 0 \end{aligned} \tag{4.13}$$

From which it follows that $C_{\mu\alpha\nu\beta} u^\alpha u^\beta$ is a symmetric traceless and transverse tensor - a fact which will turn out to be important later in our discussion of conformal hydrodynamics.

Now, we turn to the study of how various quantities of relevance to hydrodynamics can be constructed in this formalism. The Weyl-covariant derivative of the velocity field naturally breaks up into a symmetric part and an antisymmetric part

$$\begin{aligned} P_{\mu\nu} &\equiv g^{\mu\nu} + u^\mu u^\nu, \\ \mathcal{D}_\mu u^\nu &= \nabla_\mu u^\nu + u_\mu (u.\nabla) u^\nu - \frac{\nabla.u}{d-1} P_\mu{}^\nu = \sigma_\mu{}^\nu + \omega_\mu{}^\nu = \mathrm{e}^{-\phi} \widetilde{\mathcal{D}}_\mu \tilde{u}^\nu, \\ \sigma^{\mu\nu} &\equiv \frac{1}{2}\left(P^{\mu\lambda} \nabla_\lambda u^\nu + P^{\nu\lambda} \nabla_\lambda u^\mu\right) - \frac{\nabla.u}{d-1} P^{\mu\nu} = \frac{1}{2}\left(\mathcal{D}^\mu u^\nu + \mathcal{D}^\nu u^\mu\right) = \mathrm{e}^{-3\phi} \tilde{\sigma}^{\mu\nu}, \\ \omega^{\mu\nu} &\equiv \frac{1}{2}\left(P^{\mu\lambda} \nabla_\lambda u^\nu - P^{\nu\lambda} \nabla_\lambda u^\mu\right) = \frac{1}{2}\left(\mathcal{D}^\mu u^\nu - \mathcal{D}^\nu u^\mu\right) = \mathrm{e}^{-3\phi} \widetilde{\omega}^{\mu\nu}. \end{aligned} \tag{4.14}$$

The shear strain rate $\sigma_{\mu\nu}$ is a symmetric traceless tensor which tells us the rate at which the fluid element around a point is sheared whereas the vorticity $\omega_{\mu\nu}$ is an antisymmetric tensor that roughly tells us how fast the fluid is swirled around a point. Other important quantities in hydrodynamics are the energy density ε, pressure p, entropy density s, temperature T, conserved charge density n_i and corresponding chemical potentials μ_i. The scaling properties of these thermodynamic quantities is directly determined by the naive dimensional analysis

$$\begin{aligned} &\varepsilon = e^{-d\phi} \tilde{\varepsilon} \ , \quad p = e^{-d\phi} \tilde{p} \ , \quad s = e^{-(d-1)\phi} \tilde{s} \ , \\ &T = e^{-\phi} \tilde{T} \ , \quad n_i = e^{-(d-1)\phi} \tilde{n}_i \ , \quad \mu_i = e^{-\phi} \tilde{\mu}_i \ . \end{aligned} \tag{4.15}$$

We now turn to the Weyl-transformation of the conserved currents $T_{\mu\nu}$ and J_i^μ. The scaling of J_i^μ is again fixed by naive dimensional analysis $J_i^\mu = e^{-d\phi}\tilde{J}_i^\mu$. It is easily shown that for such a J_i^μ, we have $\mathcal{D}_\mu J_i^\mu = \nabla_\mu J_i^\mu$ so that statement that it is conserved is consistent with Weyl-invariance.

We now turn to $T_{\mu\nu}$ - here we face a subtlety due to quantum anomalies in the Weyl transformation. If we ignore it, and use naive dimensional analysis to fix its scaling as $T^{\mu\nu}_{\text{classical}} = \mathrm{e}^{-(d+2)\phi}\widetilde{T}^{\mu\nu}_{\text{classical}}$ then we get

$$\mathcal{D}_\mu T^{\mu\nu}_{\text{classical}} = \nabla_\mu T^{\mu\nu}_{\text{classical}} + \mathcal{A}^\nu T^{\mu}_{\mu\,\text{classical}}$$

so we recover the familiar statement that for the conservation of classical energy momentum to be consistent with the Weyl-covariance, we should also impose that the classical energy momentum tensor be traceless. Now, we turn to the actual quantum theory - the Weyl transformation of $T_{\mu\nu}$ in quantum theories is non-trivial because of the presence of Weyl anomaly - a quantum obstruction to the existence of a tensor which is both conserved and Weyl-covariant. This in turn means that $T^\lambda{}_\lambda = \mathcal{W}[g]$ where $\mathcal{W}[g]$ is the trace contribution due to the Weyl anomaly and it depends only on the microscopic field content and the ambient spacetime in which the conformal fluid lives and is independent of the state under consideration.

Often there exists another symmetric traceless tensor $T^{\mu\nu}_{\text{conf}}$ which is not conserved, but is Weyl-covariant.

$$T^{\mu\nu}_{\text{conf}} \equiv T^{\mu\nu} - T^{\mu\nu}_{anom}[g] = \mathrm{e}^{-(d+2)\phi}\widetilde{T}^{\mu\nu}_{\text{conf}}$$

where $T^{\mu\nu}_{anom}[g]$ characterises the contribution due to Weyl anomaly which depends only on the background spacetime and the field content. We will assume from now on that this indeed the case for the models under consideration. In this case, one can show that the transformation of the energy-momentum tensor is such that the quantity

$$\mathcal{D}_\mu T^{\mu\nu} \equiv \nabla_\mu T^{\mu\nu} + \mathcal{A}^\nu(T^\mu{}_\mu - \mathcal{W}[g]) = \mathrm{e}^{-(d+2)\phi}\tilde{\mathcal{D}}_\mu\widetilde{T}^{\mu\nu}$$

is Weyl-covariant. This in turn implies that the two statements of the conservation of $T_{\mu\nu}$ and freezing the trace of $T_{\mu\nu}$ to the Weyl-anomaly of the microscopic theory together can be packaged into a Weyl-covariant statement $\mathcal{D}_\mu T^{\mu\nu} = 0$. Since we have shown that the basic equations of conformal hydrodynamics are Weyl-covariant, the only constraint on the hydrodynamics coming from the underlying conformal field theory is that $T^{\mu\nu}_{\text{conf}}$ and J_i^μ be Weyl-covariant functionals of the basic hydrodynamic variables. This constraint is easily implemented by working within the manifestly Weyl-covariant formalism that we have just outlined. This then completes our programme to formulate hydrodynamics in a manifestly Weyl-covariant way.

We can now use this Weyl-covariant derivative to enumerate all the Weyl-covariant scalars, transverse vectors (i.e, vectors that are everywhere orthogonal to the fluid velocity field u^μ) and the transverse traceless tensors in the charged hydrodynamics that involve no more than second order derivatives. We will do this enumeration 'on-shell', i.e., we will enumerate those quantities which remain linearly independent even after the equations of motion are taken into account.

The basic fields in the charged hydrodynamics are the fluid velocity u^μ with weight unity, the fluid temperature T with with weight unity and the chemical potentials μ_i with weight unity. This implies that an arbitrary function of μ_i/T is Weyl-invariant and hence one could always multiply a Weyl-covariant tensor by such a function to get another Weyl-covariant tensor. Hence, in the following list only linearly independent fields appear. To make contact with the conventional literature on hydrodynamics we will work with the charge densities n_i (with weight $d-1$) rather than the chemical potentials μ_i. For simplicity, we will confine ourselves to the case where there is only one charge.

At one derivative level,

- Weyl invariant scalars : None
- Weyl invariant transverse vector : $n^{-1}P_\mu^\nu \mathcal{D}_\nu n$.
  In d=4 , we also have $l^\alpha \equiv \epsilon^{\mu\nu\lambda\alpha}u_\mu \nabla_\nu u_\lambda$.
- Weyl-invariant symmetric traceless transverse tensors : $T\sigma_{\mu\nu}$

At the two derivative level,

- Weyl invariant scalars :

$$\begin{gathered} T^{-2}\sigma_{\mu\nu}\sigma^{\mu\nu}, \quad T^{-2}\omega_{\mu\nu}\omega^{\mu\nu}, \quad T^{-2}\mathcal{R}, \\ T^{-2}n^{-1}P^{\mu\nu}\mathcal{D}_\mu\mathcal{D}_\nu n \quad \text{and} \quad T^{-2}n^{-2}P^{\mu\nu}\mathcal{D}_\mu n\mathcal{D}_\nu n \end{gathered} \tag{4.16}$$

  In d=4 ,

$$T^{-2}n^{-1}l^\mu\mathcal{D}_\mu n$$

- Weyl-invariant transverse vectors :

$$\begin{gathered} T^{-1}P_\mu^\nu \mathcal{D}_\lambda \sigma_\nu{}^\lambda, \qquad T^{-1}P_\mu^\nu \mathcal{D}_\lambda \omega_\nu{}^\lambda, \\ T^{-1}n^{-1}\sigma_\mu{}^\lambda\mathcal{D}_\lambda n \qquad \text{and} \qquad T^{-1}n^{-1}\omega_\mu{}^\lambda\mathcal{D}_\lambda n \end{gathered} \tag{4.17}$$

  In d=4 ,

$$T^{-1}\sigma_{\mu\nu}\, l^\nu$$

  .

- Weyl-invariant symmetric traceless transverse tensors :

$$\begin{gathered} C_{\mu\alpha\nu\beta}u^\alpha u^\beta, \quad u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu}, \\ \omega_\mu{}^\lambda\sigma_{\lambda\nu} + \omega_\nu{}^\lambda\sigma_{\lambda\mu}, \quad \sigma_\mu{}^\lambda\sigma_{\lambda\nu} - \frac{P_{\mu\nu}}{d-1}\,\sigma_{\alpha\beta}\sigma^{\alpha\beta}, \quad \omega_\mu{}^\lambda\omega_{\lambda\nu} + \frac{P_{\mu\nu}}{d-1}\,\omega_{\alpha\beta}\omega^{\alpha\beta}, \\ n^{-1}\,\Pi_{\mu\nu}^{\alpha\beta}\,\mathcal{D}_\alpha\mathcal{D}_\beta n, \quad n^{-2}\,\Pi_{\mu\nu}^{\alpha\beta}\,\mathcal{D}_\alpha n\,\mathcal{D}_\beta n. \end{gathered} \tag{4.18}$$

  In d=4,

$$\begin{gathered} \frac{1}{4}\,\epsilon^{\alpha\beta\lambda}{}_\mu\,\epsilon^{\gamma\theta\sigma}{}_\nu C_{\alpha\beta\gamma\theta}\,u_\lambda u_\sigma, \qquad \frac{1}{2}\epsilon_{\alpha\beta\lambda(\mu}C^{\alpha\beta}{}_{\nu)\sigma}u^\lambda u^\sigma, \quad \mathcal{D}_{(\mu}l_{\nu)}, \\ n^{-1}\Pi_{\mu\nu}^{\alpha\beta}l_\alpha\mathcal{D}_\beta n, \quad n^{-1}\epsilon^{\alpha\beta\lambda}{}_{(\mu}\sigma_{\nu)\lambda}u_\alpha\mathcal{D}_\beta n. \end{gathered} \tag{4.19}$$

where we have introduced the projection tensor $\Pi_{\mu\nu}^{\alpha\beta}$ which projects out the transverse traceless symmetric part of second rank tensors

$$\Pi_{\mu\nu}^{\alpha\beta} \equiv \frac{1}{2}\left[P_\mu^\alpha P_\nu^\beta + P_\nu^\alpha P_\mu^\beta - \frac{2}{d-1}P^{\alpha\beta}P_{\mu\nu}\right]$$

These invariants can now be used to write down the most general $T_{\mu\nu}$ and J_i^μ consistent with Weyl-covariance. The energy-momentum tensor and the charged currents of the fluid are usually divided into a zero-derivative part and a part involving at least one derivative

$$\begin{aligned} T^{\mu\nu}_{conf} &= \ p\left(g^{\mu\nu} + d\ u^\mu u^\nu\right) + \pi^{\mu\nu} \\ J_i^\mu &= \rho_i u^\mu + \nu_i^\mu \end{aligned} \tag{4.20}$$

where we can take the visco-elastic stress $\pi^{\mu\nu}$ to be transverse ($u_\mu \pi^{\mu\nu} = 0$) and traceless ($\pi^\mu{}_\mu = 0$) and the diffusion current ν_i^μ to be transverse ($u_\lambda \nu_i^\lambda = 0$). Hence, $\pi^{\mu\nu}$ and ν_i^μ are linear combination of transverse traceless Weyl-covariant tensors and transverse Weyl-covariant vectors of appropriate weight.

5. Derivative expansion in gauge-gravity duality

In order to obtain the explicit forms of $T_{\mu\nu}$ and J_i^μ , we have to 'solve' for the transport phenomenon in the strongly coupled field theory. As advertised before, this can be achieved by using gauge-gravity duality to reformulate this into a tractable problem. In this section, we will very quickly review the basic ideas behind this reformulation.

We begin with the interesting idea that large-N interacting field theories can often be rewritten in terms of a 'Master field' which is not really a single field in the same spacetime as the field theory but actually a collection of single-trace degrees of freedom that dominate the dynamics in the large-N limit. Often it is convenient to think of these degrees of freedom as living in an abstract space, their dynamics being governed in the Large-N limit by classical equations in that space. It is in this sense that large-N limit can be thought of as a classical limit. In practice however , with most field theories, it is almost impossible to guess the correct way to reformulate the large-N degrees of freedom and check whether this idea holds true for the model under consideration. This is one of the main reasons why large-N QCD remains an unsolved problem.

In a now celebrated work[4, 5], Juan Maldacena argued that for many quantum field theories, string theory gives a way out by providing us with a duality called gauge-gravity duality using which the large-N degrees of freedom and their dynamics are easily identified and tackled. In particular, he conjectured that the the $3+1$ dimensional $SU(N)$ gauge theory with fourfold supersymmetry is dual to a particular kind of a $9+1$ dimensional string theory called the Type IIB string theory quantised with an $\mathrm{AdS}_5\times\ \mathrm{S}^5$ boundary condition.

In this dual description, taking the large-N limit is same as taking the classical limit of the string theory and the relevant degrees of freedom and their dynamics in this limit are completely captured by the classical IIB string theory. Further taking the 't Hooft coupling large is equivalent in the dual description to making the typical mass of a massive string excitation very large so that only the massless degrees of freedom of the IIB string survive.

The massless degrees of freedom of IIB string theory contains a dynamical metric interacting with the other massless matter fields. Hence, we get the remarkable statement that under the gauge-gravity duality, large-N dynamics of the strongly coupled field theory gets mapped to a classical gravitational theory with an appropriate matter fields.

The fact that we get a metric in the gravity description indicates that the total energy-momentum survives as a good degree of freedom in the large-N dynamics of the strongly coupled field theory. In a similar way, the survival of the charge as a degree of freedom in the gauge theory side is encoded by the presence of a massless vector field on the gravity side. In particular, according to the gauge-gravity duality, the dynamics of these degrees of freedom are well-captured by the Einstein and Maxwell equations with the appropriate source terms. In a similar vein, one can now read off the other relevant degrees of freedom at strong coupling and their interactions directly from the low-energy lagrangian of the type IIB theory. Thus, Einstein equations in particular and gravitational dynamics in general emerge out of the large-N limit of an ordinary gauge theory and they serve as an effective way to encode the dynamics of the 'master field'.

Conversely, the classical gravitational dynamics in any gravity theory in $d+1$ dimensions with matter when studied with AdS_{d+1} boundary conditions mimics the dynamics of the 'master field' for a large-N conformal field theory in d dimensions. Given this fact, we will in the rest of the synopsis (except for the last few sections) work in a general gravity theory in $d+1$ dimensions with the understanding that putting $d = 3 + 1 = 4$ will give us the answers for the $3 + 1$ dimensional $SU(N)$ gauge theory with the fourfold supersymmetry.

It is a fact in classical gravity that any two derivative gravity theory having AdS as a solution and having no particles with spin> 2 has a consistent truncation to a sector with just gravity and a negative cosmological constant. Such a sector has an action

$$S = \frac{1}{16\pi G_{AdS}} \int d^{d+1}x \sqrt{g_{d+1}}\, [R + d(d-1)]$$

This implies that at least within the class of strongly coupled field theories whose large-N description takes the form of a gravitational theory, it is quite generic to have a sector where we can focus on the dynamics of energy and momentum alone without worrying about exciting other modes. This statement justifies our assumption at the beginning of the synopsis where we chose to focus on a sector with only conserved charges excited.

There are no results of similar generality once we try to truncate to a sector keeping only the metric and a massless vector field in the bulk. Generically, there exists other excitations which do not decouple, but in special cases, we can achieve a consistent truncation. To be concrete we will look at the IIB gravity compactified on an S^5. In particular, there exists a consistent truncation consisting of just the metric and a $U(1)$ gauge field - this happens in the sector which is equally charged under all the three cartans of $SO(6)$. We can write down a truncated action of the form

$$
\begin{aligned}
S &= \frac{1}{16\pi G_{\mathrm{AdS}}} \int \left[\sqrt{-g_5}(R+12) - \frac{1}{2}\mathbf{F} \wedge *_{\mathbf{5}}\mathbf{F} + \frac{2\kappa}{3}\mathbf{A} \wedge \mathbf{F} \wedge \mathbf{F} \right] \\
&= \frac{1}{16\pi G_{\mathrm{AdS}}} \int \sqrt{-g_5} \left[R + 12 - \frac{1}{4}F_{AB}F^{AB} + \frac{\kappa}{6}\epsilon^{PQRST}A_P F_{QR} F_{ST} \right]
\end{aligned}
\tag{5.1}
$$

with $\kappa = (2\sqrt{3})^{-1}$ for the IIB gravity. The $SO(6)$ massless vector fields in the gravity theory encode the $SU(4) \cong SO(6)$ charge current degrees of freedom and working through the duality in detail, one concludes that there exists a sector in the hydrodynamics of the gauge theory where we can decouple all excitations except the energy-momentum and a $U(1)$ charge associated with the subgroup

$$
U = \begin{pmatrix} e^{i\theta} & 0 & 0 & 0 \\ 0 & e^{i\theta} & 0 & 0 \\ 0 & 0 & e^{i\theta} & 0 \\ 0 & 0 & 0 & e^{-3i\theta} \end{pmatrix} \in SU(4)
$$

that we had introduced before. As we will see by the end of this synopsis, this particular sector exhibits some very interesting transport phenomena.

We will now go on to describe how we can use the duality to study the non-equilibrium dynamics of the field theory under consideration. We begin with the equilibrium configurations in the field theory at a given temperature and chemical potentials which are dual to equilibrium black-brane configurations in gravity at the same temperature and chemical potentials. These black-brane configurations are gravitational configurations with a planar horizon and by various semiclassical arguments due to Bekenstein and Hawking, they are known to exhibit thermodynamic properties like entropy and temperature.

For example, when all the chemical potentials are turned off, the black-brane solution at rest takes the form

$$
ds^2 = 2dvdr - r^2[1-(br)^{-d}]dv^2 + r^2 dx_{d-1}^2 \tag{5.2}
$$

in the ingoing Eddington-Finkelstein co-ordinates. No matter fields other than the metric are excited in these solutions. In general, a uniformly moving black-brane solution with a constant velocity u^μ is given by

$$
ds^2 = -2u_\mu dx^\mu dr + r^2\eta_{\mu\nu}dx^\mu dx^\nu + r^2(br)^{-d}u_\mu u_\nu dx^\mu dx^\nu \tag{5.3}
$$

The thermodynamic properties of this solution are

$$
\begin{aligned}
&\varepsilon = \frac{(d-1)}{16\pi G_{AdS} b^d}\ , \quad p = \frac{1}{16\pi G_{AdS} b^d}\ , \quad s = \frac{1}{4G_{AdS} b^d}\ , \quad T = \frac{d}{4\pi b}\ , \\
&\quad n_i = 0\ , \qquad \mu_i = 0\ .
\end{aligned}
\tag{5.4}
$$

Hence, this solution that we have written down describes thermal state in the corresponding conformal field theory. In the same vein, one can construct thermal states with chemical potentials turned on by looking for charged black-brane solutions solving Einstein-Maxwell equations with the appropriate sources.

Having understood the equilibrium configurations, we can then set up a systematic expansion where we allow the the thermodynamic quantities to slowly vary and find the corresponding time-dependent gravitational configurations. We will just review the setting up of this expansion in the zero chemical potential case - this method has been generalised to the finite chemical potential case, which while technically intricate is nevertheless straightforward. The logic behind - and the method of implementation of - this perturbative procedure have been described in detail in [6]. Further it was described in [6], how this perturbative procedure establishes a map between solutions of fluid dynamics and regular long wavelength solutions of Einstein gravity with a negative cosmological constant.

We start with the ansatz

$$g_{MN} = g^{(0)}_{MN} + \epsilon g^{(1)}_{MN} + \epsilon^2 g^{(2)}_{MN} + \dots$$

Here $g^{(0)}_{MN}$ is given by (5.3), ϵ is the small parameter of the derivative expansion, and $g^{(k)}_{MN}$ are the corrections to the bulk metric that we will determine with the aid of the bulk Einstein equation. In implementing our perturbative procedure we adopt a choice of gauge. As in all the metrics described above, we use the coordinates r, x^μ for our bulk spaces. We use x^μ as coordinates that parameterise the boundary and r is a radial coordinate. In order to give precise meaning to our coordinates we need to adopt a choice of gauge-we choose the gauge $g_{rr} = 0$ together with $g_{r\mu} = -u_\mu$.

The Bulk Einstein equations decompose into 'constraints' on the boundary hydrodynamic data and 'dynamical equations' for the bulk metric along the tubes which are solved order by order in the derivative expansion. The dynamical equations determine the corrections that should be added to our initial metric to make it a solution of the Einstein equations. At each order, we get inhomogeneous linear equations -but, with the same homogeneous parts. These inhomogeneous linear equations obtained from Einstein equations can be solved order by order by imposing regularity at the zeroth order future horizon and appropriate asymptotic fall off at the boundary.These boundary conditions - together with a clear definition of velocity, which fixes the ambiguity of adding zero modes - give a unique solution for the metric, as a function of the original boundary velocity and temperature profile inputted into the metric $g^{(0)}_{MN}$ - order by order in the boundary derivative expansion.

Now, we turn to the 'constraints'. The 'constraints' on the boundary data can be shown to be equivalent to the requirement of the conservation of the boundary stress tensor. Recall that we have already used the dynamical Einstein equations to determine the full bulk metric - and hence the boundary stress tensor - as a function of the input velocity and temperature fields. It follows that the constraint Einstein equations reduce simply to the equations of fluid dynamics, i.e. the requirement of a conserved stress tensor which, in turn, is a given function of temperature and velocity fields. In the next few sections, we will detail the results obtained by performing such a boundary derivative expansion.

6. Hydrodynamics from gravity in arbitrary dimensions

Before presenting the metric that is dual to a given hydrodynamic state, we will pause to ask how the Weyl covariance of the hydrodynamic state gets encoded in the metric[12]. Let us begin by performing the boundary Weyl-transformation in the boosted black-brane metric (5.3)

$$u_\mu = e^{\phi}\tilde{u}_\mu, \quad \eta_{\mu\nu} = e^{2\phi}\tilde{g}_{\mu\nu} \quad \text{and} \quad b = e^{\phi}\tilde{b}$$

where the transformation of b can be fixed by its relation to the thermodynamic quantities. Further, since for a constant velocity u^μ, the Weyl connection $\mathcal{A}_\mu = 0$ we also have $0 = \tilde{\mathcal{A}}_\mu + \partial_\mu\phi$

Under such a transformation, the metric becomes

$$ds^2 = -2\tilde{u}_\mu dx^\mu(d\tilde{r} + \tilde{r}\tilde{\mathcal{A}}_\nu dx^\nu) + \tilde{r}^2\tilde{g}_{\mu\nu}dx^\mu dx^\nu + \tilde{r}^2(\tilde{b}\tilde{r})^{-d}\tilde{u}_\mu\tilde{u}_\nu dx^\mu dx^\nu \tag{6.1}$$

where we have defined $\tilde{r}$ via $r = e^{-\phi}\tilde{r}$. We note that written in this form the black-brane metric is form-invariant under the boundary Weyl-transformations ! This is a particular example of the general principle that the boundary Weyl-transformations get realised in the gravity theory as a specific set of diffeomorphisms that redefine the radial slicing. The fact that different slicings of the same gravity solution may in fact be interpreted as states of the same theory in distinct though Weyl equivalent background metrics, reflects the Weyl invariance of the dual field theory.

To make this more precise, any metric that obeys that gauge choice $g_r^{(d+1)}r = 0$ and $g_r^{(d+1)}\mu = -u\mu$ can be put in the form

$$ds^2 = -2u_\mu(x)dx^\mu(dr + \mathcal{V}_\nu(r,x)dx^\nu) + \mathfrak{G}_{\mu\nu}(r,x)dx^\mu dx^\nu \tag{6.2}$$

where $\mathfrak{G}_{\mu\nu}$ is transverse, i.e., $u^\mu\mathfrak{G}_{\mu\nu} = 0$. [13]

Consider now a bulk-diffeomorphism of the form $r = e^{-\phi}\tilde{r}$ along with a scaling in the temperature of the form $b = e^{\phi}\tilde{b}$ where we assume that $\phi = \phi(x)$ is a function only of the boundary co-ordinates. The metric components transform as

$$\mathcal{V}_\mu = e^{-\phi}\left[\tilde{\mathcal{V}}_\mu + \tilde{r}\ \partial_\mu\phi\right], \quad u_\mu = e^{\phi}\tilde{u}_\mu \quad \text{and} \quad \mathfrak{G}_{\mu\nu} = \tilde{\mathfrak{G}}_{\mu\nu} \tag{6.4}$$

Recall however that, within our procedure, the quantities $\mathfrak{G}_{\mu\nu}$ and $\mathcal{V}_\mu$ are each functions of u^μ and b. Now u^μ and b each pick up a factor of e^ϕ under the same diffeomorphism. We conclude that consistency demands that $\mathcal{V}_\mu$ and $\mathfrak{G}_{\mu\nu}$ are functions of b and u^μ that transform like a connection and remain invariant under boundary Weyl transformation respectively. It follows immediately that, for instance $\mathfrak{G}_{\mu\nu}$ is a linear sum of the Weyl invariant forms listed before, with coefficients that are arbitrary functions of br. Similarly,

[12] In this section, we summarise the work that was first presented in the paper[7]

[13] All the Greek indices are raised and lowered using the boundary metric $g_{\mu\nu}$ defined by

$$g_{\mu\nu} = \lim_{r\to\infty} r^{-2}\left[\mathfrak{G}_{\mu\nu} - u_{(\mu}\mathcal{V}_{\nu)}\right] \tag{6.3}$$

and u_μ is the unit time-like velocity field in the boundary, i.e., $g^{\mu\nu}u_\mu u_\nu = -1$.

$\mathcal{V}_\mu - r\mathcal{A}_\mu$ is a linear sum of Weyl-covariant vectors (both transverse and non-transverse) with weight unity. Symmetry requirements do not constrain the form of these coefficients, which have to be determined via direct calculation.

Using a Weyl-covariant form of the procedure outlined in [6], we find that the final metric for the zero chemical potential case in arbitrary dimensions can be written in a Weyl-covariant form (with only terms involving no more than two derivatives shown)

$$
\begin{split}
ds^2 = & -2u_\mu dx^\mu \left(dr + r\, A_\nu dx^\nu\right) + \left[r^2 g_{\mu\nu} + u_{(\mu}\mathcal{S}_{\nu)\lambda}u^\lambda - \omega_\mu{}^\lambda \omega_{\lambda\nu}\right] dx^\mu dx^\nu \\
& + \frac{1}{(br)^d}(r^2 - \frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta})u_\mu u_\nu dx^\mu dx^\nu + 2(br)^2 F(br)\left[\frac{1}{b}\sigma_{\mu\nu} + F(br)\sigma_\mu{}^\lambda\sigma_{\lambda\nu}\right] dx^\mu dx^\nu \\
& - 2(br)^2\left[K_1(br)\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu} + K_2(br)\frac{u_\mu u_\nu}{(br)^d}\frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{2(d-1)} - \frac{L(br)}{(br)^d}u_{(\mu}P_{\nu)}^\lambda \mathcal{D}_\alpha \sigma^\alpha{}_\lambda\right] dx^\mu dx^\nu \\
& - 2(br)^2 H_1(br)\left[u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + \sigma_\mu{}^\lambda\sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu} + C_{\mu\alpha\nu\beta}u^\alpha u^\beta\right] dx^\mu dx^\nu \\
& + 2(br)^2 H_2(br)\left[u^\lambda\mathcal{D}_\lambda\sigma_{\mu\nu} + \omega_\mu{}^\lambda\sigma_{\lambda\nu} + \omega_\nu{}^\lambda\sigma_{\mu\lambda}\right] dx^\mu dx^\nu + \ldots
\end{split}
\tag{6.5}
$$

The various functions appearing in the metric are defined by the integrals

$$
\begin{split}
F(br) &\equiv \int_{br}^\infty \frac{y^{d-1}-1}{y(y^d-1)}dy \\
H_1(br) &\equiv \int_{br}^\infty \frac{y^{d-2}-1}{y(y^d-1)}dy \\
H_2(br) &\equiv \int_{br}^\infty \frac{d\xi}{\xi(\xi^d-1)}\int_1^\xi y^{d-3}dy\left[1+(d-1)yF(y)+2y^2F'(y)\right] \\
&= \frac{1}{2}F(br)^2 - \int_{br}^\infty \frac{d\xi}{\xi(\xi^d-1)}\int_1^\xi \frac{y^{d-2}-1}{y(y^d-1)}dy \\
K_1(br) &\equiv \int_{br}^\infty \frac{d\xi}{\xi^2}\int_\xi^\infty dy\; y^2F'(y)^2 \\
K_2(br) &\equiv \int_{br}^\infty \frac{d\xi}{\xi^2}\Big[1-\xi(\xi-1)F'(\xi) - 2(d-1)\xi^{d-1} \\
&\qquad + \Big(2(d-1)\xi^d-(d-2)\Big)\int_\xi^\infty dy\; y^2F'(y)^2\Big] \\
L(br) &\equiv \int_{br}^\infty \xi^{d-1}d\xi\int_\xi^\infty dy\;\frac{y-1}{y^3(y^d-1)}
\end{split}
$$

We have checked using Mathematica upto $d = 10$ that the above metric solves Einstein equations with negative cosmological constant provided the velocity u^μ satisfies the hydrodynamic equations. The above formula hence gives a map from the solutions of hydrodynamic equations to the solutions of Einstein equations thus connecting two widely different fields of classical physics !

Value of τ_ω/b for various dimensions		
d	Value of $\tau_\omega/b = \int_1^\infty \frac{y^{d-2}-1}{y(y^d-1)}dy$	τ_ω/b (Numerical)
3	$\frac{1}{2}\left(\text{Log } 3 - \frac{\pi}{3\sqrt{3}}\right)$	0.247006...
4	$\frac{1}{2}\text{Log } 2$	0.346574...
5	$\frac{1}{4}\left(\text{Log } 5 + \frac{2\pi}{5}\sqrt{1-\frac{2}{\sqrt{5}}} - \frac{2}{\sqrt{5}}\text{ArcCoth } \sqrt{5}\right)$	0.396834...
6	$\frac{1}{4}\left(\text{Log } 3 + \frac{\pi}{3\sqrt{3}}\right)$	0.425803...

The dual stress tensor corresponding to this metric is given by

$$\begin{aligned} T^{\text{conf}}_{\mu\nu} &= p\left(g_{\mu\nu} + d u_\mu u_\nu\right) - 2\eta\sigma_{\mu\nu} \\ &\quad - 2\eta\tau_\omega\left[u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu} + \omega_\mu{}^\lambda \sigma_{\lambda\nu} + \omega_\nu{}^\lambda \sigma_{\mu\lambda}\right] \\ &\quad + 2\eta b\left[u^\lambda \mathcal{D}_\lambda \sigma_{\mu\nu} + \sigma_\mu{}^\lambda \sigma_{\lambda\nu} - \frac{\sigma_{\alpha\beta}\sigma^{\alpha\beta}}{d-1}P_{\mu\nu} + C_{\mu\alpha\nu\beta}u^\alpha u^\beta\right] \end{aligned} \tag{6.6}$$

with

$$p = \frac{1}{16\pi G_{\text{AdS}} b^d}\ , \quad \eta = \frac{s}{4\pi} = \frac{1}{16\pi G_{\text{AdS}} b^{d-1}} \quad \text{and} \quad \tau_\omega = b\int_1^\infty \frac{y^{d-2}-1}{y(y^d-1)}dy \tag{6.7}$$

This result is a generalisation to the fluid dynamical stress tensor on an arbitrary curved manifold in general dimension d reported in [8, 6, 2, 9, 10, 11] for special values of d and most recently by [12] for flat space in arbitrary dimensions. The values of τ_ω for some of the lower dimensions is shown[14] in the table 6. The way the various Weyl-covariant terms combine together in the energy-momentum tensor is intriguing - in fact, it can be shown[14] that there are universal relations between the second-order transport coefficients that come out of gravity.

These perturbative solutions of Einstein equations can be compared against a class of exact solutions of Einstein's equations. This class of solutions is the set of rotating black holes in the global AdS spaces. The dual boundary stress tensor to these solutions varies on the length scale unity (if we choose our boundary sphere to be of unit radius). On the other hand the temperature of these black holes may be taken to be arbitrarily large. It follows that, in the large temperature limit, these black holes are dual to 'slowly varying' field theory configurations that should be well described by fluid dynamics. All of these remarks, together with nontrivial evidence for this expectation was described in [15]. In fact, we can go further[7] and complete the programme initiated in [15] for uncharged blackholes by demonstrating that the full bulk metric of these high temperature rotating

[14]More generally, the integral appearing in the expression for τ_ω can be evaluated in terms of the derivative of the Gamma function or more directly in terms of 'the harmonic number function' with the fractional argument (as was noted in [13])

$$\tau_\omega = -\frac{b}{d}\left[\gamma_E + \frac{d}{dz}\text{Log }\Gamma(z)\right]_{z=2/d} = -\frac{b}{d}\text{Harmonic}[2/d-1]$$

For large d, τ_ω has an expansion of the form $\tau_\omega/b = 1/2 - \pi^2/(3d^2) + \ldots$.

black holes agrees in detail with the 2nd order bulk metric determined by our analysis earlier . This exercise was already carried out in [11] for the special case $d = 4$.

Consider the AdS-Kerr BHs in arbitrary dimensions - exact solution for the rotating blackholes in general AdS_{d+1} in different coordinates is derived in reference [16]. Following [16], we begin by defining two integers n and ϵ via $d = 2n + \epsilon$ with $\epsilon = d \bmod 2$. We can then parametrise the $d+1$ dimensional AdS Kerr solution by a radial co-ordinate r, a time co-ordinate $\hat{t}$ along with $d-1 = 2n+\epsilon-1$ spheroidal co-ordinates on S^{d-1}. We will choose these spheroidal co-ordinates to be $n+\epsilon$ number of direction cosines $\hat{\mu}_i$ (obeying $\sum_{k=1}^{n+\epsilon} \hat{\mu}_k^2 = 1$) and $n+\epsilon$ azimuthal angles $\hat{\varphi}_i$ with $\hat{\varphi}_{n+1} = 0$ identically. The angular velocities along the different $\hat{\varphi}_i$s are denoted by a_i ($a_{n+1}$is taken to be zero identically).

In this 'altered' Boyer-Lindquist co-ordinates, AdS Kerr metric assumes the form (See equation (E.3) of the [16])

$$\begin{aligned} ds^2 = &-W(1+r^2)d\hat{t}^2 + \frac{\mathfrak{F} dr^2}{1-2M/V} + \frac{2M}{V\mathfrak{F}}\left(W d\hat{t} - \sum_{i=1}^{n}\frac{a_i \hat{\mu}_i^2 d\hat{\varphi}_i}{1-a_i^2}\right)^2 \\ &+ \sum_{i=1}^{n+\epsilon}\frac{r^2+a_i^2}{1-a_i^2}\left[d\hat{\mu}_i^2 + \hat{\mu}_i^2 d\hat{\varphi}_i^2\right] - \frac{1}{W(1+r^2)}\left(\sum_{i=1}^{n+\epsilon}\frac{r^2+a_i^2}{1-a_i^2}\hat{\mu}_i d\hat{\mu}_i\right)^2 \end{aligned} \tag{6.8}$$

where

$$W \equiv \sum_{i=1}^{n+\epsilon}\frac{\hat{\mu}_i^2}{1-a_i^2} \quad ; \quad V \equiv r^d\,(1+\frac{1}{r^2})\prod_{i=1}^{n}(1+\frac{a_i^2}{r^2}) \quad \text{and} \quad \mathfrak{F} \equiv \frac{1}{1+r^2}\sum_{i=1}^{n+\epsilon}\frac{r^2\hat{\mu}_i^2}{r^2+a_i^2} \tag{6.9}$$

This expression can be further simplified[7] and the AdS Kerr metric in arbitrary dimensions can be rewritten in the form

$$\begin{aligned} ds^2 = &-2u_\mu dx^\mu\left(dr + r\ \mathcal{A}_\nu dx^\nu\right) + \left[r^2 g_{\mu\nu} + u_{(\mu}\mathcal{S}_{\nu)\lambda}u^\lambda - \omega_\mu{}^\lambda\omega_{\lambda\nu}\right]dx^\mu dx^\nu \\ &+ \frac{r^2 u_\mu u_\nu}{b^d \det\left[r\ \delta^\mu_\nu - \omega^\mu{}_\nu\right]}dx^\mu dx^\nu \end{aligned} \tag{6.10}$$

where the determinant of a tensor M^λ_σ is defined by

$$\epsilon_{\mu\nu\ldots}M^\mu_\alpha M^\nu_\beta\ldots = \det\left[M^\lambda_\sigma\right]\epsilon_{\alpha\beta\ldots}$$

We have checked this form explicitly using Mathematica upto $d = 8$. It is easily checked that this metric agrees (upto second order in boundary derivative expansion) with the metric presented in (6.5) upon inserting the velocity and temperature fields as

$$\begin{aligned} &u^\mu\partial_\mu \equiv \partial_t + a_i\partial_{\varphi_i} \quad , \quad \mathcal{A}_\mu = 0 \quad , \quad b \equiv (2M)^{-1/d} \\ &g_{\mu\nu} \equiv W\left[-dt^2 + \sum_i\left(d\mu_i^2 + \mu_i^2 d\varphi_i^2\right)\right] \end{aligned} \tag{6.11}$$

The exact energy momentum tensor for the AdS Kerr Black Hole described can be computed

$$T^{\mu\nu}_{conf} = p(g^{\mu\nu} + du^\mu u^\nu) \quad \text{with} \quad p = \frac{1}{16\pi G_{AdS}b^d} \tag{6.12}$$

which is consistent with (6.6) if we take into account the fact that $\sigma_{\mu\nu} = 0$ in these configurations.

7. Global Charge transport in $\mathcal{N} = 4$ SYM

In the previous section, we presented our calculation of various transport coefficients in the simplest sector of hydrodynamics where the entire dynamics is that of the energy-momentum transport. In this section, we will turn to a slightly more complicated sector where there is an interplay between charge transport and energy transport. This in the gravity language translates to the study of Einstein-Maxwell system with appropriate sources.

As we had emphasised before, given a generic gravitational theory describing the large N behaviour of a strongly coupled CFT, we are not guaranteed to find a truncation to just the Einstein-Maxwell system. In the field theory language, it implies that at least in these class of systems it is very difficult to disentangle the charge dynamics from the dynamics of the other long wavelength modes. We had already mentioned that there do exist theories where such a truncation is achievable and we gave an example of a sector in the supersymmetric gauge theory where this happens- this is IIB gravity compactified on an S^5 with a consistent truncation consisting of just the metric and a $U(1)$ gauge field in the sector which is equally charged under all the three cartans of $SO(6)$.

We can repeat the derivative expansion to solve the Einstein-Maxwell system with a Chern-Simons term order by order for the metric and the gauge field. This was done in [17] and the expressions that we finally get for the metric and the gauge field are complicated and we will refer the reader to [17] for the full expressions. In turn, the energy momentum and charge transport upto second order in the boundary derivative expansion could be computed from that metric and gauge field and this gives us many new transport coefficients in the supersymmetric gauge theory. We again refer the reader to [17] (and [18] which appeared independently with similar results) for the detailed expressions . But in this synopsis, we will try to focus on an interesting result that comes out of these expressions.

To see this very clearly, it is useful to focus our attention on a known exact solution of the Einstein-Maxwell-Chern-Simons system which describes a charged rotating blackhole. We begin with a truncated action of the form

$$\begin{aligned} S &= \frac{1}{16\pi G_{\text{AdS}}} \int \left[\sqrt{-g_5}(R+12) - \frac{1}{2}\mathbf{F}\wedge *_{\mathbf{5}}\mathbf{F} + \frac{2\kappa}{3}\mathbf{A}\wedge\mathbf{F}\wedge\mathbf{F} \right] \\ &= \frac{1}{16\pi G_{\text{AdS}}} \int \sqrt{-g_5} \left[R + 12 - \frac{1}{4}F_{AB}F^{AB} + \frac{\kappa}{6}\epsilon^{PQRST}A_P F_{QR} F_{ST} \right] \end{aligned} \tag{7.1}$$

with $\kappa = (2\sqrt{3})^{-1}$ for the IIB gravity.

This action has many known blackhole solutions. The general blackhole solutions which solves the equations coming out of this action was found in [19]. Their solution is given by[15]

$$
\begin{aligned}
ds^2 = &-\frac{\left(r^2+1\right)\Delta_\Theta dt_1{}^2}{\left(1-\omega_1{}^2\right)\left(1-\omega_2{}^2\right)} + \frac{2(m-q\omega_1\omega_2)}{\rho^2} - \frac{q^2}{\rho^4} \\
&+ \frac{(\mathrm{d}\psi_1+dt_1\omega_2)^2\left(r^2+\omega_2{}^2\right)\cos^2\Theta}{1-\omega_2{}^2} + \frac{(\mathrm{d}\phi_1+dt_1\omega_1)^2\left(r^2+\omega_1{}^2\right)\sin^2\Theta}{1-\omega_1{}^2} \\
&+ \frac{\rho^2\mathrm{d}r^2r^2}{q^2-2\omega_1\omega_2 q-2mr^2+\left(r^2+1\right)\left(r^2+\omega_1{}^2\right)\left(r^2+\omega_2{}^2\right)} \\
&+ \frac{\rho^2\mathrm{d}\Theta^2}{\Delta_\Theta} + \frac{2q}{\rho^2}\left(\omega_1(\mathrm{d}\psi_1+dt_1\omega_2)\cos^2\Theta + (\mathrm{d}\phi_1+dt_1\omega_1)\omega_2\sin^2\Theta\right) \\
&\times\left[\frac{\Delta_\Theta dt_1}{\left(1-\omega_1{}^2\right)\left(1-\omega_2{}^2\right)} - \frac{\omega_2(\mathrm{d}\psi_1+dt_1\omega_2)\cos^2\Theta}{1-\omega_2{}^2} - \frac{\omega_1(\mathrm{d}\phi_1+dt_1\omega_1)\sin^2\Theta}{1-\omega_1{}^2}\right] \\
\mathbf{A} = &-\frac{\sqrt{3}q}{\rho^2}\left[\frac{\Delta_\Theta dt_1}{\left(1-\omega_1{}^2\right)\left(1-\omega_2{}^2\right)} - \frac{\omega_2(\mathrm{d}\psi_1+dt_1\omega_2)\cos^2\Theta}{1-\omega_2{}^2} - \frac{\omega_1(\mathrm{d}\phi_1+dt_1\omega_1)\sin^2\Theta}{1-\omega_1{}^2}\right]
\end{aligned}
\tag{7.2}
$$

where we use the definitions

$$
\begin{aligned}
\rho^2 &\equiv r^2+\omega_1{}^2\cos^2\Theta+\omega_2{}^2\sin^2\Theta \\
\Delta_\Theta &\equiv 1-\omega_1{}^2\cos^2\Theta-\omega_2{}^2\sin^2\Theta
\end{aligned}
\tag{7.3}
$$

After some manipulations (which closely follow the methods outlined in [11, 7]) , we find that the final metric and the gauge field can be written in a manifestly Weyl-covariant form

$$
\begin{aligned}
ds^2 &= -2u_\mu dx^\mu\left(dr+r\;\mathcal{A}_\nu dx^\nu\right) + \left[r^2 g_{\mu\nu} + u_{(\mu}\mathcal{S}_{\nu)\lambda}u^\lambda - \omega_\mu{}^\lambda\omega_{\lambda\nu}\right]dx^\mu dx^\nu \\
&\quad + \left[\left(\frac{2m}{\rho^2}-\frac{q^2}{\rho^4}\right)u_\mu u_\nu + \frac{q}{2\rho^2}u_{(\mu}l_{\nu)}\right]dx^\mu dx^\nu \\
\mathbf{A} &= \frac{\sqrt{3}q}{\rho^2}u_\mu dx^\mu \quad ; \quad \rho^2\equiv r^2+\frac{1}{2}\omega_{\alpha\beta}\omega^{\alpha\beta} \quad ; \quad l_\mu\equiv\epsilon_{\mu\nu\lambda\sigma}u^\nu\omega^{\lambda\sigma}
\end{aligned}
\tag{7.4}
$$

with

$$
\begin{aligned}
T_{\mu\nu} &= p(g_{\mu\nu}+4u_\mu u_\nu) + 2\kappa l_{(\mu}J_{\nu)} + \frac{1}{64\pi G_{\mathrm{AdS}}}\left(R^{\alpha\beta}R_{\alpha\mu\beta\nu} - \frac{R^2}{12}g_{\mu\nu}\right) \\
J_\mu &= n u_\mu \qquad \text{where} \quad l_\mu\equiv\epsilon_{\mu\nu\lambda\sigma}u^\nu\omega^{\lambda\sigma} \quad ; \quad p\equiv\frac{m}{8\pi G_{\mathrm{AdS}}} \quad \text{and} \quad n\equiv\frac{\sqrt{3}q}{8\pi G_{\mathrm{AdS}}}
\end{aligned}
\tag{7.5}
$$

We note that when the bulk Chern-Simons coupling κ is non-zero, apart from the conventional diffusive transport, there is an additional non-dissipative contribution to the energy current which is proportional to the vorticity of the fluid. To the extent we know of, this is a hitherto unknown effect in the hydrodynamics which is exhibited by the conformal fluid made of $\mathcal{N}=4$ SYM matter. This new non-dissipative transport can be traced back to the Chern-Simons term in the gravity theory which according to the gauge-gravity duality,

[15] Note that the parameter q here is the negative of the one used in [19].

encodes the information about the global anomalies in the field theory. This suggests that this transport closely related to the $U(1)^3$ global anomaly in the field theory.

It would be interesting to find a direct boundary reasoning that would lead to the presence of such a term - however, as of yet, we do not have such an explanation. However, an indirect explanation was provided by the authors of [20], where they give a clever entropic argument which relates this coefficient to the anomaly. This suggests the possibility that such a transport is universal, i.e., it is present in any field theory which has global anomalies and it would be useful to explicitly check whether this is the case by calculating this transport coefficient in a calculable model - say a spin model.

The presence of such an effect was indirectly observed by the authors of [15] where they noted a discrepancy between the thermodynamics of charged rotating AdS blackholes and the fluid dynamical prediction with the third term in the charge current absent. We have verified that this discrepancy is resolved once we take into account the effect of the third term in the thermodynamics of the rotating $\mathcal{N} = 4$ SYM fluid. In fact, one could go further and compare the first order metric obtained in [18, 17] with the rotating blackhole metric written in an appropriate gauge. We have done this comparison up to first order and we find that the metrics agree up to that order.

8. Conclusion

In this synopsis, we have tried to summarise some of our work on the hydrodynamics that emerges out of an underlying strongly coupled conformal field theory at a finite temperature and chemical potentials. The hydrodynamic transport in such theories can be studied using the methods of gauge-gravity duality which reformulate the questions about transport into questions in a classical gravity theory interacting with matter. Further, we have seen that the underlying Weyl covariance of the conformal field theory finds a natural expression both in hydrodynamics and in gravity.

Our study has thrown up very interesting questions about the hydrodynamic transport - we have already remarked on the apparent universality of second order transport coefficients from gravity which is puzzling. We can also enquire as to whether there is a clever way by which such universal relations can be derived at all orders in derivative expansion.

A more intriguing phenomenon is the novel kind of non-dissipative transport in the hydrodynamics which seems to be driven by an underlying anomaly. As we had already remarked, it will be worthwhile look for a solvable model where this kind of transport is present. Given the possibility that it is a universal effect, there is a slim hope that there might be experimental systems where such a transport can be measured.

We have in this synopsis derived an expression which provides a metric solution to Einstein equations for a give hydrodynamic state. This immediately raises the possibility that a transition to turbulence in the hydrodynamics is accompanied by a turbulence like phenomenon in gravity. It would be interesting to see whether we can develop a detailed phenomenology of such a transition in gravity.

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"Where (or of what) one cannot speak, one must pass over in silence."

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Publications by the author summarised in this synopsis

[P1] S. Bhattacharyya, R. Loganayagam, I. Mandal, S. Minwalla and A. Sharma, "Conformal Nonlinear Fluid Dynamics from Gravity in Arbitrary Dimensions," JHEP **0812**, 116 (2008), [arXiv:0809.4272 [hep-th]].

[P2] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam and P. Surowka, "Hydrodynamics from charged black branes," arXiv:0809.2596 [hep-th].

[P3] R. Loganayagam,"Entropy Current in Conformal Hydrodynamics," JHEP **0805**, 087 (2008), [arXiv:0801.3701 [hep-th]].

[P4] S. Bhattacharyya, S. Lahiri, R. Loganayagam and S. Minwalla, "Large rotating AdS black holes from fluid mechanics," JHEP **0809**, 054 (2008), [arXiv:0708.1770 [hep-th]].

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