The Miura operator at the M2-M5 intersection

with Davide Gaiotto [2012.04118] + work in progress

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1. Introduction

1.1. Geometry-representation correspondence

- Branes (sheaves) in toric Calabi-Yau 3-folds CY_3

  $\updownarrow$

  Representation theory of affine Yangians
- The famous AGT correspondence [Alday-Gaiotto-Tachikawa (2009)] relating the Virasoro algebra with $U(2)$ instantons on $\mathbb{C}^2 \subset \mathbb{C}^3$ and its various BPS/CFT generalizations can be thought of as special examples.
- Today only $CY_3 = \mathbb{C}^3$ or more generally $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_k$.

1. Introduction

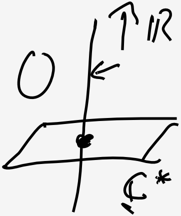
1.2. Geometry side

NON-COMPACT

CY_3 - COMPACT

- Consider M-theory on $\mathbb{C} \times \mathbb{C}^* \times \mathbb{R} \times \mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times \mathbb{C}_{\epsilon_3}$ in the presence of the Ω-background along $\mathbb{C}^3$.
- Brane configurations:

M-theory	$\mathbb{C}$	$\mathbb{C}^*$	$\mathbb{R}$	$\mathbb{C}_{\epsilon_1}$	$\mathbb{C}_{\epsilon_2}$	$\mathbb{C}_{\epsilon_3}$
N_1 M5		×		×	×	
N_2 M5		×		×		×
N_3 M5		×			×	×
n_1 M2			×	×		
n_2 M2			×		×	
n_3 M2			×			×



- More generally, one can exchange $\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times \mathbb{C}_{\epsilon_3}$ by $\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_k$ in Ω-background leading to the matrix generalization of the story.

1. Introduction

1.3. Representation theory side

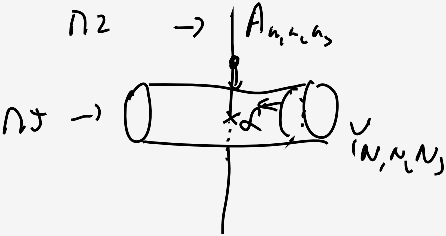
- Compactification on Ω-background leads to the algebra of operators on $\mathbb{R}$ for M2-branes and on $\mathbb{C}^*$ for M5-branes.
- Both the M2-brane algebra and modes of the M5-brane algebra need to satisfy relations of the universal algebra $\mathcal{Y}$ in order to admit non-anomalous coupling to the 5d theory on $\mathbb{C} \times \mathbb{C}^* \times \mathbb{R}$. [Costello (2016), Costello (2017)]
- In our examples of $CY_3 = \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_k$, the universal algebra $\mathcal{Y}$ is known as the affine Yangian of $\mathfrak{gl}_k$.
- The geometric picture predicts an existence of representations A_{n_1,n_2,n_3} and Y_{N_1,N_2,N_3} of the universal algebra $\mathcal{Y}$:

$$\mathcal{Y} \to A_{n_1,n_2,n_3} \qquad \mathcal{Y} \to Y_{N_1,N_2,N_3}$$

1. Introduction

1.4. Interaction of M2-M5 branes

- With Gaiotto, we investigated M2-M5 intersections and various aspects of the relation between A_{n_1,n_2,n_3} and Y_{N_1,N_2,N_3}:



2. Elementary representations

2.1. M5-brane algebra

- An M5-brane in Nekrasov's Ω-background leads to the $\widehat{gl}_1$ algebra living on $\mathbb{C}^*$ [Nakajima (1995), Gaiotto-Alday-Tachikawa (2009), Nekrasov-Witten (2010), Schiffmann-Vasserot (2012), Maulik-Okounkov (2012), Yagi (2012)]:

$$J(z)J(w) \sim -\frac{1}{\epsilon_1\epsilon_2}\frac{1}{(z-w)^2}, \quad \rightarrow \quad [J_m, J_n] = -\frac{1}{\epsilon_1\epsilon_2} m\delta_{m,-n}$$

- Different orientations (coming from permutations of $\epsilon_1, \epsilon_2, \epsilon_3$ satisfying $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$) give rise to $Y_{1,0,0}$, $Y_{0,1,0}$ and $Y_{0,0,1}$.

2. Elementary representations

- All must be different representations of the $\mathfrak{gl}_1$ affine Yangian $\mathcal{Y}$ generated by $t_{2,0}, t_{0,n}$ for $n \in \mathbb{Z}$ subject to a relatively complicated system of relations. [Costello (2016)]
- Indeed, $\mathcal{Y}$ admits three Fock representations

$$\begin{aligned} Y_{0,0,1} : t_{0,n} &\rightarrow J_n \\ Y_{0,0,1} : t_{2,0} &\rightarrow \frac{\epsilon_1^2 \epsilon_2^3}{3} (J^3)_{-2} + \epsilon_1 \epsilon_2 \epsilon_3 \sum_{k=0}^{\infty} k J_{-k-1} J_{k-1} \end{aligned}$$

and analogously for $Y_{1,0,0}$ and $Y_{0,1,0}$. [Bershtein-Feigin-Merzon (2015), Gaiotto-MR (2017), Procházka-MR (2017), MR-Soibelman-Yang-Zhao (2018)]

2. Elementary representations

2.2. M2-brane algebra

- An M2-brane in the Nekrasov's Ω-background placed in the $\mathbb{C} \times \mathbb{C} \times \mathbb{R}$ geometry leads to the Weyl algebra living on $\mathbb{R}$ [Yagi (2014), Costello (2017), Kodera-Nakajima (2017), Gaiotto-Oh (2019)]:

$$[\epsilon_1 \partial, z] = \epsilon_1$$

- In order to get the algebra associated to $\mathbb{C} \times \mathbb{C}^* \times \mathbb{R}$, one needs to add generator $1/z$ satisfying obvious relations

$$[\epsilon_1 \partial, \frac{1}{z}] = -\epsilon_1 \frac{1}{z^2}, \qquad z\frac{1}{z} = 1$$

  and leading to the full algebra $A_{1,0,0}$.

- Different orientations give rise to $A_{1,0,0}$, $A_{0,1,0}$ and $A_{0,0,1}$.

2. Elementary representations

2.2. M2-brane algebra

- All must be different representations of the affine Yangian $\mathcal{Y}$. [Costello (2017)]
- Indeed, $\mathcal{Y}$ admits three polynomial representations

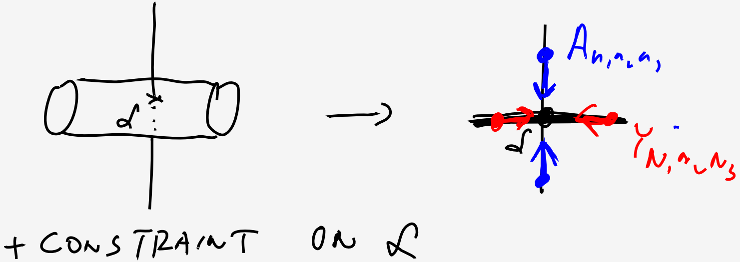
$$\begin{aligned} A_{0,0,1} : t_{0,n} &\rightarrow \frac{1}{\epsilon_3} z^n \\ A_{0,0,1} : t_{2,0} &\rightarrow \epsilon_3 \partial^2 \end{aligned}$$

and analogously for $A_{1,0,0}$ and $A_{0,1,0}$. [Gaiotto-MR (2020)]

3. Miura operator

3.1. M5-M2 intersections

- Operators at the intersection form a bi-module for both A_{n_1,n_2,n_3} and Y_{n_1,n_2,n_3}:



3. Miura operator

- Consistency of the coupling to the 5d theory is governed by the coproduct $\mathcal{Y} \to \mathcal{Y} \otimes \mathcal{Y}$ [Gaiotto-MR (2020)] given by

$$\Delta : t_{0,n} \quad \to \quad t_{0,n} \otimes \mathbb{1} + \mathbb{1} \otimes t_{0,n}$$

$$\Delta : t_{2,0} \quad \to \quad t_{2,0} \otimes \mathbb{1} + \mathbb{1} \otimes t_{2,0} + \epsilon_1\epsilon_2\epsilon_3 \sum_{k=0}^{\infty} k t_{0,-k-1} \otimes t_{k-1}$$

- In particular

$$[(Y_{N_1,N_2,N_3} \otimes A_{n_1,n_2,n_3}) \circ \Delta(t)]\mathcal{L} = \mathcal{L}[(A_{n_1,n_2,n_3} \otimes Y_{N_1,N_2,N_3}) \circ \Delta(t)]$$

for any $t \in \mathcal{Y}$.

- This was checked by an explicit calculation in the context of the twisted M-theory by [Oh-Zhou (2021)].

3. Miura operator

3.2. Appearance of the Miura operator

- What is $\mathcal{L}$ associated to the simplest configuration?

M-theory	$\mathbb{C}$	$\mathbb{C}^*$	$\mathbb{R}$	$\mathbb{C}_{\epsilon_1}$	$\mathbb{C}_{\epsilon_2}$	$\mathbb{C}_{\epsilon_3}$
M5		$\times$			$\times$	$\times$
M2			$\times$	$\times$		

3. Miura operator

- Composing

$$t_{0,n} \rightarrow t_{0,n} \otimes \mathbb{1} + \mathbb{1} \otimes t_{0,n}$$

$$t_{2,0} \rightarrow t_{2,0} \otimes \mathbb{1} + \mathbb{1} \otimes t_{2,0} + \epsilon_1\epsilon_2\epsilon_3 \sum_{k=0}^{\infty} k t_{0,-n-1} \otimes t_{n-1}$$

with representations $Y_{1,0,0} \otimes A_{1,0,0}$ gives

$$t_{0,n} \rightarrow J_n + \frac{z^n}{\epsilon_1}$$

$$t_{2,0} \rightarrow \frac{\epsilon_1^2\epsilon_2^3}{3}(J^3)_{-2} + \epsilon_1\epsilon_2\epsilon_3 \sum_{k=0}^{\infty} k J_{-k-1} J_{k-1}$$

$$+\epsilon_1 \partial^2 + \epsilon_2\epsilon_3 \sum_{k=0}^{\infty} k J_{-n-1} z^{n-1}$$

3. Miura operator

- The gauge-invariance condition looks like

$$\left(J_n + \frac{z^n}{\epsilon_1}\right)\mathcal{L} = \mathcal{L}\left(J_n + \frac{z^n}{\epsilon_1}\right)$$

$$\left(T_{2,0} + \epsilon_1 \partial^2 + \epsilon_2\epsilon_3 \sum_{k=0}^{\infty} k J_{-n-1} z^{n-1}\right)\mathcal{L}$$

$$= \mathcal{L}\left(T_{2,0} + \epsilon_1 \partial^2 + \epsilon_2\epsilon_3 \sum_{k=0}^{\infty} k J_{n-1} z^{-n-1}\right)$$

- Let us introduce the Miura operator [Gardner-Greene-Kruskal-Miura (1967), Drinfeld-Sokolov (1984), Fateev-Lukyanov (1988)]

$$\mathcal{L}^{1,0,0}_{1,0,0} = \epsilon_1 \partial - \epsilon_2 \epsilon_3 \sum_{m=-\infty}^{\infty} J_{-n} z^{n-1}$$

  thought of as an element of $Y_{1,0,0} \otimes A_{1,0,0}$.
- We have a natural bimodule for $A_{1,0,0}$ and $Y_{1,0,0}$!
- $\mathcal{L}$ satisfies the gauge invariance condition, leading to a new interpretation of the Miura operator as an operator at the junction of the simplest M2-M5 intersection!

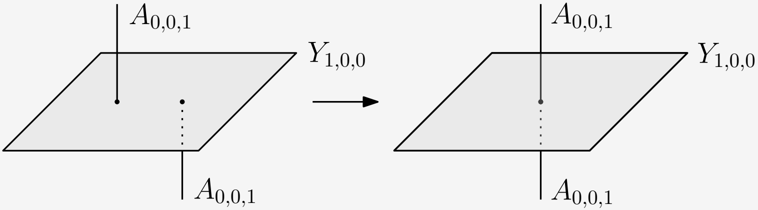
3. Miura operator

3.3. Miura operators with different orientation

- What is $\mathcal{L}$ associated to the following configuration?

M-theory	$\mathbb{C}$	$\mathbb{C}^*$	$\mathbb{R}$	$\mathbb{C}_{\epsilon_1}$	$\mathbb{C}_{\epsilon_2}$	$\mathbb{C}_{\epsilon_3}$
M5		$\times$			$\times$	$\times$
M2			$\times$			$\times$

- We can now split into:



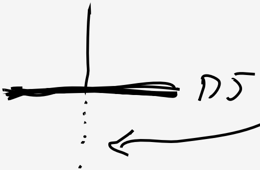
3. Miura operator

- To the endpoints, we can associate vertex operators

$$\exp[\epsilon_2\phi(z)], \qquad \exp[-\epsilon_2\phi(z)]$$

that also satisfy a version of the gauge-invariance condition

$$\begin{aligned}[(Y_{1,0,0}\otimes A_{0,0,1})\circ\Delta(t)]\mathcal{L} &= \mathcal{L}[Y_{1,0,0}(t)] \\ [Y_{1,0,0}(t)]\mathcal{L} &= \mathcal{L}[(A_{0,0,1}\otimes Y_{1,0,0})\circ\Delta(t)]\end{aligned}$$



3. Miura operator

- Let me sketch a cute way to re-derive the pseudo-differential Miura operators from [Procházka-MR (2018)]:

$$
\begin{aligned}
\mathcal{L}^{0,0,1}_{1,0,0} &= \exp[\epsilon_2\phi(z)]\exp[-\epsilon_2\phi(\tilde{z})] \\
&= :\exp[\epsilon_2\phi(z)](z-\tilde{z})^{\frac{\epsilon_2}{\epsilon_3}}\exp[-\epsilon_2\phi(\tilde{z})]: \\
&\propto :\exp[\epsilon_2\phi(z)](\epsilon_3\partial)^{\frac{\epsilon_1}{\epsilon_3}}\exp[-\epsilon_2\phi(z)]: \\
&= (1-\epsilon_1\epsilon_2 J(z)(\epsilon_3\partial)^{-1}+\dots)(\epsilon_3\partial)^{\frac{\epsilon_1}{\epsilon_3}}
\end{aligned}
$$

where we used the fact that

$$
z^n\partial^\alpha z^m \leftrightarrow \frac{\alpha! z^n\tilde{z}^m}{(z-\tilde{z})^{\alpha+1}}
$$

gives an isomorphism of $A_{1,0,0}$ bimodules.

3. Miura operator

- One can check that this operator satisfies

$$[(Y_{1,0,0} \otimes A_{0,0,1}) \circ \Delta(t)]\mathcal{L} \;=\; \mathcal{L}[(A_{0,0,1} \otimes Y_{1,0,0}) \circ \Delta(t)]$$

- Permuting $\epsilon_1, \epsilon_2, \epsilon_3$, we get 9 elementary Miura operators

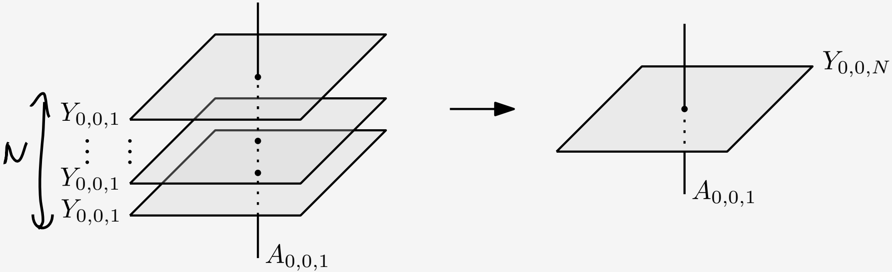
$$\begin{matrix} \mathcal{L}^{1,0,0}_{1,0,0} & \mathcal{L}^{1,0,0}_{0,1,0} & \mathcal{L}^{1,0,0}_{0,0,1} \\ \mathcal{L}^{0,1,0}_{1,0,0} & \mathcal{L}^{0,1,0}_{0,1,0} & \mathcal{L}^{0,1,0}_{0,0,1} \\ \mathcal{L}^{0,0,1}_{1,0,0} & \mathcal{L}^{0,0,1}_{0,1,0} & \mathcal{L}^{0,0,1}_{0,0,1} \end{matrix}$$

  that satisfy the gauge-invariance condition with $\mathcal{A}$ and $\mathcal{Y}$ in one of the elementary representations.

4. Composing elementary junctions

4.1. Corner vertex operator algebra

- Having defined elementary Miura operators, we can compose them. Let us now look at



4. Composing elementary junctions

- The original purpose of the Miura operator is the Miura transformation (in the context of N-KdV hierarchies and $\mathcal{W}_N$ vertex operator algebras):

$$\mathcal{L}^{0,0,1}_{0,0,1}\tilde{\mathcal{L}}^{0,0,1}_{0,0,1} = (\epsilon_3\partial - \epsilon_1\epsilon_2 J(z))(\epsilon_3\partial - \epsilon_1\epsilon_2\tilde{J}(z))$$

$$= (\epsilon_3\partial)^2 - \epsilon_1\epsilon_2(J(z)+\tilde{J}(z))\epsilon_3\partial + \epsilon_1^2\epsilon_2^2(J\tilde{J})(z) - \epsilon_1\epsilon_2\epsilon_3\partial\tilde{J}(z)$$

z, ∂ ↔ J_n z, ∂ ↔ $\tilde{J}_n$

→ SUBALGEBRA in $(\widehat{gl}_1)^{\oplus 2}$

→ VIRASORO ⊗ $\widehat{gl}_1$

4. Composing elementary junctions

- The fused junction satisfies

$$[(Y_{0,0,N} \otimes A_{0,0,1}) \circ \Delta(t)]\mathcal{L} \quad = \quad \mathcal{L}[(A_{0,0,1} \otimes Y_{0,0,N}) \circ \Delta(t)]$$

for representation

$$t \quad \to \quad [\underbrace{Y_{0,0,1} \otimes Y_{0,0,1} \otimes \cdots \otimes Y_{0,0,1}}_{N}] \circ \underbrace{\Delta(\Delta(\ldots\Delta}_{N-1}(t)\ldots))$$

known as the $\mathcal{W}_N$ vertex operator algebra. [Schiffmann-Vasserot (2012), Maulik-Okounkov (2012)]

4. Composing elementary junctions

- Analogously, one can fuse junctions

$$\underbrace{\mathcal{L}^{0,0,1}_{1,0,0}\dots\mathcal{L}^{0,0,1}_{1,0,0}}_{N_1}\underbrace{\mathcal{L}^{0,0,1}_{0,1,0}\dots\mathcal{L}^{0,0,1}_{0,1,0}}_{N_2}\underbrace{\mathcal{L}^{0,0,1}_{0,0,1}\dots\mathcal{L}^{0,0,1}_{0,0,1}}_{N_3}$$

  i.e. perform a generalized Miura transformation [Procházka-MR (2018)].

- This satisfies the gauge invariance for the general corner vertex operator algebra Y_{N_1,N_2,N_3} given by

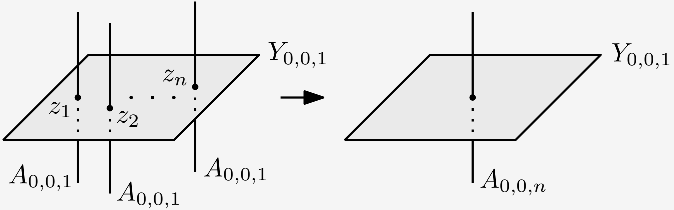
$$t \quad \rightarrow \quad [\underbrace{Y_{1,0,0}\otimes\dots}_{N_1}\otimes\underbrace{Y_{0,1,0}\otimes\dots}_{N_2}\otimes\underbrace{Y_{0,0,1}\otimes\dots}_{N_3}]\circ\underbrace{\Delta(\dots\Delta}_{N_1+N_2+N_3-1}(t)\dots)$$

  [MR-Soibelman-Yang-Zhao (2018)]

4. Composing elementary junctions

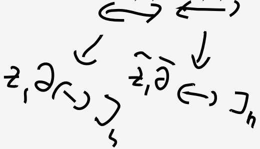
4.2. Cherednik-algebras

- What if we fuse M2-branes instead? What is A_{n_1,n_2,n_3}?



4. Composing elementary junctions

- Let us look at a motivating example from [Procházka (2019)]:



$$
\begin{aligned}
\mathcal{L}^{0,0,1}_{0,0,1}\tilde{\mathcal{L}}^{0,0,1}_{0,0,1} &= (\epsilon_3\partial - \epsilon_1\epsilon_2 J(z))(\epsilon_3\tilde{\partial} - \epsilon_1\epsilon_2 J(\tilde{z})) \\
&= \epsilon_3^2\partial\tilde{\partial} - \frac{\epsilon_1\epsilon_2}{(z-\tilde{z})^2} \\
&\quad -\epsilon_1\epsilon_2\epsilon_3 \sum_{m=-\infty}^{\infty} J_{-n}(z^{n-1}\tilde{\partial} + \tilde{z}^{n-1}\partial) \\
&\quad +\epsilon_1^2\epsilon_2^2 \sum_{m,n=-\infty}^{\infty} :J_{-m}J_{-n}: z^{m-1}\tilde{z}^{n-1}
\end{aligned}
$$

who pointed out a connection to Calogero-Moser Hamiltonians [Calogero (1972), Sutherland (1973), Moser (1975)].

- In turn Calogero-Moser Hamiltonians are known to be generators of so-called spherical Cherednik algebra (truncations of DAHA).

4. Composing elementary junctions

- Can we make the connection more precise? Is there a three-parametric generalization of spherical Cherednik algebras?
- Let us look at the construction of $A_{0,0,2}$ from the coproduct

$$
\begin{aligned}
t_{0,n} &\to \frac{1}{\epsilon_3} z^n + \frac{1}{\epsilon_3} \tilde{z}^n \\
t_{2,0} &\to \epsilon_3 \partial^2 + \epsilon_3 \tilde{\partial}^2 + \epsilon_1\epsilon_2\epsilon_3 \sum_{m=0}^{\infty} m \frac{z^{-m-1}}{\epsilon_1} \frac{\tilde{z}^{m-1}}{\epsilon_1} \\
&\to \epsilon_3 \partial^2 + \epsilon_3 \tilde{\partial}^2 + \frac{\epsilon_2\epsilon_3}{\epsilon_1} \frac{2}{(z-\tilde{z})^2}
\end{aligned}
$$

recovering the Dunkl realization of the Cherednik algebra $A_{0,0,2}$ from the same coproduct!

4. Composing elementary junctions

- Nothing stops us here and we can define its three-parametric generalization A_{n_1,n_2,n_3} as

$$t \to [\underbrace{A_{1,0,0} \otimes \dots}_{n_1} \otimes \underbrace{A_{0,1,0} \otimes \dots}_{n_2} \otimes \underbrace{A_{0,0,1} \otimes \dots}_{n_3}] \circ \underbrace{\Delta(\dots \Delta}_{n_1+n_2+n_3-1} (t) \dots)$$

generalizing the two-parametric family of [Sergeev-Veselov (2003)].

$$t_{0,d} = \epsilon_1^{-1} \sum_{i=1}^{n_1} z_i^d + \epsilon_2^{-1} \sum_{i=1}^{n_2} (z_i')^d + \epsilon_3^{-1} \sum_{i=1}^{n_3} (z_i'')^d$$

$$\begin{aligned} t_{2,0} = {} & \epsilon_1 \sum_{i=1}^{n_1} \partial_{z_i}^2 + \frac{\epsilon_2\epsilon_3}{\epsilon_1} \sum_{i<j} \frac{2}{(z_i - z_j)^2} + \epsilon_1 \sum_{i,j} \frac{2}{(z_i' - z_j'')^2} + \\ & + \epsilon_2 \sum_{i=1}^{n_2} \partial_{z_i'}^2 + \frac{\epsilon_1\epsilon_3}{\epsilon_2} \sum_{i<j} \frac{2}{(z_i' - z_j')^2} + \epsilon_2 \sum_{i,j} \frac{2}{(z_i - z_j'')^2} + \\ & + \epsilon_3 \sum_{i=1}^{n_3} \partial_{z_i''}^2 + \frac{\epsilon_1\epsilon_2}{\epsilon_3} \sum_{i<j} \frac{2}{(z_i'' - z_j'')^2} + \epsilon_3 \sum_{i,j} \frac{2}{(z_i - z_j')^2} \end{aligned}$$

4. Composing elementary junctions

- It is straightforward to check that the composed Miura operator

$$\underbrace{\mathcal{L}^{1,0,0}_{0,0,1}\dots\mathcal{L}^{1,0,0}_{0,0,1}}_{n_1}\underbrace{\mathcal{L}^{0,1,0}_{0,0,1}\dots\mathcal{L}^{0,1,0}_{0,0,1}}_{n_2}\underbrace{\mathcal{L}^{0,0,1}_{0,0,1}\dots\mathcal{L}^{0,0,1}_{0,0,1}}_{n_3}$$

  satisfies the gauge invariance condition for A_{n_1,n_2,n_3} intersecting $Y_{0,0,1}$.

5. Matrix generalization

- The above story has a matrix generalization associated to the geometry $\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_k$.
- The universal algebra $\mathcal{Y}$ universal algebra generated by $t_{2,0}, t_{0,n}$ becomes the $\mathfrak{gl}_k$ affine Yangian generated by $e^i_{j;0,n}, t_{2,0}$ with $i,j=1,\dots,k$.
- The algebra $\mathcal{Y}$ again admits three elementary M2-brane representations [Gaiotto-MR (in preparation)]

$$
\begin{aligned}
e^i_{j;0,n} &\rightarrow E^i_j z^n \\
t_{2,0} &\rightarrow \epsilon_\alpha \partial^2
\end{aligned}
$$

for $\alpha=1,2,3$ and where $\epsilon_3=-\epsilon_2-k\epsilon_1$ and E^i_j are in a particular representation of the $\mathfrak{gl}_k$ Lie algebra depending on α.

5. Matrix generalization

- Analogously, the algebra $\mathcal{Y}$ again admits three elementary M5-brane representations. [MR (2019), Prochazka-Eberhardt (2019)]
- Furthermore, the coproduct formulas naturally generalize to this case leading to three-parametric families of M2-brane and M5-brane algebras including spin Calogero-Moser system and its various generalizations.
- The two families of algebras interact via matrix generalization of the Miura operator and vertex operators.

6.Conclusion

- Miura operators are operators at the junction of M2-M5 branes in twisted M-theory. They satisfy R-matrix-like gauge-invariance condition.
- The coproduct of the affine $\mathfrak{gl}_1$ Yangian can be used to define a three-parametric family of corner algebras Y_{N_1,N_2,N_3} but also a three-parametric family of Cherednik algebras A_{n_1,n_2,n_3}.
- Miura operators can be fused to describe junctions between more complicated M2-M5 intersections that relate such more complicated algebras.
- Screening charges, pseudo-differential Miura operators and the complicated structure of degenerate representations emerge naturally in the current picture.