Integrable modules for full Toroidal Lie algebras associated with a quantum torus

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Rational quantum matrix q

We fix a positive integer $n \geq 2$.

Let $q = (q_{ij})_{(n+1)\times(n+1)}$ be a matrix such that $q_{ii} = 1$ and $q_{ij} = q_{ji}^{-1}$ and q_{ij} is a root of unity for

$0 \leq i, j \leq n$.

This is called a rational quantum matrix q.

Let $\mathbb{C}_q$ be the Laurent polynomial ring in $(n+1)$ non-commuting variables $t_0, t_1, \ldots t_n$

with $t_i t_j = q_{ij} t_j t_i$

for $0 \leq i, j \leq n$

$\mathbb{C}_q$ is $\mathbb{Z}^{n+1}$-graded with each graded component being one dimensional.

For $a = (a_0, a_1, a_2, \dots, a_n) \in \mathbb{Z}^{n+1}$,

let $t^a = t_0^{a_0} t_1^{a_1} \dots t_n^{a_n} \in \mathbb{C}_q$

We define maps $\sigma, f : \mathbb{Z}^{n+1} \times \mathbb{Z}^{n+1} \longrightarrow \mathbb{C}^*$ by

$$\sigma(a, b) = \prod_{0 \leq i \leq j \leq n} q_{ji}^{a_j b_i}$$

$$f(a, b) = \sigma(a, b)\, \sigma(b, a)^{-1}$$

We know $t^a t^b = \sigma(a, b)\, t^{a+b}$

$t^a t^b = f(a, b)\, t^b t^a$

The radical of f is defined by

$$\operatorname{rad} f = \{ a \in \mathbb{Z}^{n+1} \mid f(a, b) = 1 \ \forall\, b \in \mathbb{Z}^{n+1} \}$$

and $m \in \operatorname{rad} f$ iff $f(r, s) = 1$ $\forall$ $r, s \in \mathbb{Z}^{n+1}$ with $r + s = m$

$\mathbb{C}_q$, being an associative algebra, it has a natural Lie algebra structure.

Proposition (Berman, Gao, Krylyuk)

1.) Center $Z(\mathbb{C}_q)$ of $\mathbb{C}_q$ has a basis t^a, $a \in \mathrm{rad} f$.

2.) The Lie subalgebra $[\mathbb{C}_q, \mathbb{C}_q]$ of $\mathbb{C}_q$ has a basis t^a $a \in \mathbb{Z}^{n+1} - \mathrm{rad} f$.

3.) $\mathbb{C}_q = [\mathbb{C}_q, \mathbb{C}_q] \oplus Z(\mathbb{C}_q)$.

Let $M_d(\mathbb{C})$ be the associative algebra of $d \times d$ matrices.
Consider associative algebra

$$M_d(\mathbb{C}_q) = M_d(\mathbb{C}) \otimes \mathbb{C}_q$$

Corresponding Lie algebra is denoted by $gl_d(\mathbb{C}_q)$. Let $x(a) = x \otimes t^a$

Then Lie bracket is given by

$$[x(a), y(b)]_0 = \left(\sigma(a,b)xy - \sigma(b,a)yx\right)(a+b)$$

Universal Central Extension of $gl_d(\mathbb{C}_q)$

Let J be linear span of

$$x \otimes y + y \otimes x$$
$$xy \otimes z + yz \otimes x + zx \otimes y$$

inside $\mathbb{C}_q \otimes \mathbb{C}_q$ $\forall\, x, y, z \in \mathbb{C}_q$

Let $\langle x, y\rangle_0 = x \otimes y + J$ in $\mathbb{C}_q \otimes \mathbb{C}_q / J$.

Define $HC_1(\mathbb{C}_q) = \left\{ \sum_{i\in I} \langle x_i, y_i\rangle_0 \,\middle|\, \sum_{i \in I} [x_i, y_i]_0 = 0 \right\}$,

where I is any finite subset of $\mathbb{N}$.

Let $\tau = gl_d(\mathbb{C}_q) \oplus HC_1(\mathbb{C}_q)$

and Lie bracket is

$$[x(a), y(b)] = [x(a), y(b)]_0 + \operatorname{Tr}(xy) \langle t^a, t^b\rangle_0 \, \delta_{a+b, \operatorname{rad} f}$$

we write $\langle t^a, t^b\rangle = \langle t^a, t^b\rangle_0 \, \delta_{a+b, \operatorname{rad} f}$

$HC_1(\mathbb{C}_q)$ is $\mathbb{Z}^{n+1}$-graded.

Note $\boxed{\langle t^a, t^b\rangle = \sigma(a,b) \sum_{i=0}^{n} a_i \langle t_i, t^{a+b} t_i^{-1}\rangle}$

Zero degree elements of $HC_1(\mathbb{C}_q)$ are $c_i = \langle t_i, t_i^{-1} \rangle$ for $i = 0, \ldots, n$.

Let $\mathrm{Der}(\mathbb{C}_q)$ be the space of all derivations of $\mathbb{C}_q$. Then we have

Lemma (Berman, Gao, Krylyuk)

1.) $$\mathrm{Der}(\mathbb{C}_q) = \bigoplus_{a \in \mathbb{Z}^{n+1}} (\mathrm{Der}(\mathbb{C}_q))_a$$

2.) $$\mathrm{Der}(\mathbb{C}_q)_a = \begin{cases} \mathbb{C}\, \mathrm{ad}\, t^a & a \notin \mathrm{rad} f \\ \bigoplus_{i=0}^{n} \mathbb{C}\, t^a \partial_i & a \in \mathrm{rad} f \end{cases}$$

Denote $$D(u, r) = \sum_{i=0}^{n} u_i t^r \partial_i$$ for $u \in \mathbb{C}^n$, $r \in \mathrm{rad} f$.

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$\mathrm{Der}(\mathbb{C}_q)$ is a Lie algebra with

1) $[\mathrm{ad}\, t^r, \mathrm{ad}\, t^s] = (\sigma(r,s) - \sigma(s,r))\, \mathrm{ad}\, t^{r+s}$

$\forall\, r, s \notin \mathrm{rad} f$

2) $[D(u,r), \mathrm{ad}\, t^s] = (u,s)\, \sigma(r,s)\, \mathrm{ad}\, t^{r+s}$

$\forall\, r \in \mathrm{rad} f$

$s \notin \mathrm{rad} f, \; u \in \mathbb{C}^{n+1}$

3) $[D(u,r), D(u',r')] = D(w, r+r')$

$\forall\, r, r' \in \mathrm{rad} f$

$u, u' \in \mathbb{C}^{n+1}$

$w = \sigma(r,r')\left((u,r')u' - (u',r)u\right)$

$\mathrm{Der}(\mathbb{C}_q)$ naturally acts on $gl_d(\mathbb{C}_q)$

$Der(\mathbb{C}_q)$ acts on $HC_1(\mathbb{C}_q)$ as

$$D(u,r)\langle t^a, t^b\rangle = (u,a)\,\sigma(r,a)\langle t^{a+r}, t^b\rangle$$
$$+ (u,b)\,\sigma(r,b)\langle t^a, t^{b+r}\rangle$$

$$\mathrm{ad}\, t^s \langle t^a, t^b\rangle = 0$$

We define Lie algebra

$$\breve{\tau} = gl_d(\mathbb{C}_q) \oplus HC_1(\mathbb{C}_q) \oplus Der(\mathbb{C}_q)$$

brackets are

1.) $[D(u,r),\ x(a)] = (u,a)\,\sigma(r,a)\,x(r+a)$

$\forall\, r \in \mathrm{rad}\, f,\ a \in \mathbb{Z}^{n+1}$

$u \in \mathbb{C}^{n+1}$

2.) $[\mathrm{ad}\, t^s,\ x(a)] = \big(\sigma(s,a) - \sigma(a,s)\big)\, x(a+s)$

$\forall\, s \notin \mathrm{rad}\, f,\ a \in \mathbb{Z}^{n+1}$

3) $[\operatorname{ad} t^s, \operatorname{ad} t^r] = \sigma(s, r) - \sigma(r, s) \operatorname{ad} t^{s+r}$

$\forall\, r, s \notin \operatorname{rad} f$

4) $[D(u, r), \operatorname{ad} t^s] = (u, s)\, \sigma(r, s)\, \operatorname{ad} t^{r+s}$

$\forall\, r \in \operatorname{rad}(f)$

$s \notin \operatorname{rad}(f)$

$u \in \mathbb{C}^{n+1}$

5) $[D(u, r), D(u', r') = D(w, r+r')$

$w = \sigma(r, r')\big((u, r')u' - (u', r)u\big)$

$\forall\, r, r' \in \operatorname{rad} f, \quad u, u' \in \mathbb{C}^{n+1}$

6) $[D(u, r), \langle t^a, t^b\rangle] = (u, a)\, \sigma(r, a) \langle t^{a+r}, t^b\rangle + (u, b)\, \sigma(r, b) \langle t^a, t^{b+r}\rangle$

7) $[\operatorname{ad} t^s, \langle t^a, t^b\rangle] = 0 \quad a, b \in \mathbb{Z}^{n+1}$

$s \notin \operatorname{rad} f$.

(11)

Cartan subalgebra $\tilde{h}$ of $\tilde{\tau}$

$\tilde{h} = h \oplus \mathbb{C} I \oplus \sum_{i=0}^{n} \mathbb{C}\, c_i \oplus D$ is the Cartan sub. of $\tilde{\tau}$.

where h is the Cartan subalgebra of $sl_d(\mathbb{C})$ spanned by elements

$E_{ii} - E_{i+1,i+1}$ for $1 \leq i \leq d-1$

& I is the identity matrix of $gl_d(\mathbb{C})$

let $D = \mathbb{C}$ linear span of $\{\partial_i \mid 0 \leq i \leq n\}$

$c_i = \langle t_0, t_i^{-1} \rangle \quad i = 0, \ldots, n$

We define $I^*, \delta_i, \omega_i \in \tilde{h}^*$

for $0 \leq i \leq n$

$\delta_i(h) = 0 \quad \delta_i(I) = 0 \quad \delta_i(c_j) = 0$

$\delta_i(\partial_j) = \delta_{ij}$

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$w_i(h) = 0$ $\quad w_i(\partial_j) = 0$

$w_i(I) = 0$ $\quad w_i(c_j) = \delta_{ij}$

and $I^*(h) = 0 = I^*(c_i) = I^*(\partial_i) = 0$

$I^*(I) = 1$

Let $\alpha_1, \alpha_2, \ldots \alpha_{d-1}$ be standard Simple roots of h

Let Δ_0 be the root system for $sl_d(\mathbb{C})$.

Let $\gamma = \alpha + \delta_m$ for $\alpha \in \Delta_0$

Then γ is called a real root and δ_m, a null root.

Let $\Delta = \{\alpha + \delta_m, \ \delta_m \mid \alpha + \delta_m \in \Delta^{re}, \ \delta_m \text{ null root}\}$

is a root system of $\check{\tau}$.

We define

$$\check{\tau}_{\alpha+\delta_m} = (sl_d(\mathbb{C}))_\alpha \otimes \mathbb{C}t^m$$

where $sl_d(\mathbb{C}) = \bigoplus_{\alpha \in \Delta_0 \cup \{0\}} (sl_d(\mathbb{C}))_\alpha$

is the root space decomp.

of $sl_d(\mathbb{C})$.

and $$\check{\tau}_{\delta_m} = \begin{cases} h \otimes \mathbb{C}t^m \oplus I \otimes \mathbb{C}t^m \oplus \mathbb{C}adt^m & m \notin rad f \\ h \otimes \mathbb{C}t^m \oplus I \otimes \mathbb{C}t^m \oplus (HG(\mathbb{C}_q))_m \oplus \left(\overset{n}{\oplus} \mathbb{C}t^m \partial_i\right) & m \in rad f \setminus \{0\} \end{cases}$$

where $(HC_1(\mathbb{C}_q))_m = \mathbb{C}\langle t^a, t^b \rangle$
with $a+b=m$.

We define a non-degenerate symmetric, bilinear form on $\mathfrak{h}^*$

For a real root γ, we define reflection

$$r_\gamma(\lambda) = \lambda - \lambda(\gamma^\vee)\gamma$$
$$\lambda \in \mathfrak{h}^*$$

The gp. generated by r_γ, γ real roots, is called Weyl group W.

$\mathfrak{h}$ is W- invariant.

Integrable modules A module V for $\hat{\tau}$ is called integrable if

1.) $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$

where $V_\lambda = \{ v \in V \mid h.v = \lambda(h) v \ \forall h \in \mathfrak{h} \}$

2) $\dim V_\lambda < \infty \quad \forall \ \lambda \in \mathfrak{h}^*$

3.) $\forall \ \alpha \in \Delta^{re}$, $x \in \hat{\tau}_\alpha$, x acts locally nilpotently on V.

~~i.e. $\exists$ $v \in V$~~

i.e. $\forall v \in V$, $\exists$ $n = n(x, v) \in \mathbb{N}$ such that $x^n . v = 0$

Let $P(V) = \{\lambda \in \mathfrak{h}^* \mid V_\lambda \neq (0)\}$

Lemma If V is irreducible integrable module for $\tilde{\tau}$ ~~Now~~ then $\lambda(c_i)$ is a constant integer $\forall \lambda \in P(V)$.

Now we take a natural triangular decomposition of $\tilde{\tau}$.

Define

$$HG_1^+ = \bigoplus_{\substack{0 \le i \le n \\ 0 < s_0 \in \mathbb{Z} \\ s \in \mathbb{Z}^n}} \mathbb{C}\langle t_i, t_0^{s_0} t^s t_i^{-1} \rangle$$

For $r \in \mathrm{rad} f$, $s \notin \mathrm{rad} f$

$$D^+ = \{D(u, r), \ \mathrm{ad}\, t^s \mid r_0, s_0 > 0\}$$

$$HG^0 = \bigoplus_{\substack{0 \le i \le n \\ s \in \mathbb{Z}^n}} \mathbb{C}\langle t_i, t^s t_i^{-1} \rangle$$

$$D^0 = \{ D(u,r), \ \mathrm{ad}\, t^s \mid r_0, \ s_0 = 0 \}$$

$$HC_1^- = \bigoplus_{\substack{0 \le i \le n \\ 0 > s_0 \in \mathbb{Z} \\ s \in \mathbb{Z}^n}} \mathbb{C} \langle t_i, t_0^{s_0} t^s t_i^{-1} \rangle$$

$$D^- = \{ D(u,r), \ \mathrm{ad}\, t^s \mid r_0, \ s_0 < 0 \}$$

$$\tilde{\tau}^+ = (sl_d(\mathbb{C}))_+ \otimes \mathbb{C}_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}] \oplus$$

$$\oplus\, gl_d(\mathbb{C}) \otimes t_0 \mathbb{C}_q[t_0, t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

$$\oplus\, HC_1^+ \oplus D^+,$$

$$\tilde{\tau}^- = (sl_d(\mathbb{C}))_- \otimes \mathbb{C}_q[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

$$\oplus\, gl_d(\mathbb{C}) \otimes t_0^{-1} \mathbb{C}_q[t_0^{-1}, t_1^{\pm 1}, \dots, t_n^{\pm 1}]$$

$$\oplus\, HC_1^- \oplus D^-$$

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$$\hat{\tau}^0 = \mathfrak{h} \otimes \mathbb{C}_q[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \oplus I \otimes \mathbb{C}_q[t_1^{\pm 1}, \ldots, t_n^{\pm 1}] \oplus HC_1^0 \oplus D^0.$$

We take $\Delta^+ = \{\alpha + \delta_m, \delta_{m'} \mid m_0, m_0' > 0$ or $m_0 = 0$ and $\alpha > 0\}$

Now by above Lemma

$\lambda(C_i) = c_i$ for $0 \leq i \leq n$
$\forall\ \lambda \in P(V)$
and $c_i \in \mathbb{Z}$

<u>Theorem</u>: Let V be an irreducible, integrable module for $\hat{\tau}$ with finite dimensional weight spaces.
If $c_0 > 0$ and $c_1 = c_2 = \ldots c_n = 0$, then there exists a nonzero element $v \in V$ such that $\hat{\tau}^+ . v = 0$

Let $V = \{v \in V \mid \hat{\tau}^+ . v = 0\}$, then $\exists$ a $\lambda \in P(V)$ such that $h.v = \lambda(h)v$ for all $v \in V^+$

and $h \in \mathfrak{h} \oplus \mathbb{C} C_0 \oplus \mathbb{C} d_0$.

Theorem 2, Let V be irreducible, integrable module for $\check{\tau}$ with finite dimensional weight spaces. If C_i acts non-trivially for some $0 \leq i \leq n$. Then upto an automorphism, $\exists$ a highest weight λ and using this highest weight we assume that $c_0 > 0$ & $c_1 = c_2 = \cdots$

$c_n = 0$

& $V^+ \neq 0$

This V^+ is a $\check{\tau}^0$-submodule of V and irreducible for $\hat{\tau}^0$ and $V = U(\check{\tau}^-)V^+$.

Action of central extension part on highest weight space V^+

$$HC_1^0 = \underset{\substack{s \in \text{rad} f \\ s_0 = 0}}{\oplus} \mathbb{C}\langle t_0, t^s t_0^{-1} \rangle \oplus \underset{\substack{1 \leq i \leq n \\ s \in \text{rad} f,\, s_0 = 0}}{\oplus} \mathbb{C}\langle t_i, t^s t_i^{-1} \rangle$$

acts associatively on V^+

acts trivially on V^+

Complete reducibility

We prove that quotient of highest weight space is completely reducible for a subalgebra of $\hat{\tilde{\tau}}^{0}$

Let $\mathbb{C}_q' = \mathbb{C}_q[t_1^{\pm 1}, \ldots\ldots, t_n^{\pm 1}]$

Consider $D(u,r) = \sum_{i=1}^{n} u_i t^r \partial_i$

and let $I(u,r) = D(u,r) - D(u,0)$

for $u \in \mathbb{C}^n$, $r \in \operatorname{rad} f'$

We note
$\operatorname{span}\{I(u,r), \operatorname{ad} t^s, u \in \mathbb{C}^n, r \in \operatorname{rad} f', s \notin \operatorname{rad} f'\}$

is a Lie subalgebra of $\operatorname{Der}(\mathbb{C}_q')$ with brackets

$$[I(u,r), I(v,s)] = I(w, r+s) + (v,r)\, I(u,r) - (u,s)\, I(v,s)$$

$$[I(u,r),\ ad\, t^{a}] = (u,a)\,(ad\, t^{r+a} - ad\, t^{a})$$

$u, v \in \mathbb{C}^n$, $r, s \in rad f'$, $a \notin rad f'$.

Now $L' = \left(h \otimes \mathbb{C}_{q'} \oplus I \otimes \mathbb{C}_{q'} \oplus HC_1^0 \oplus \underset{r \in rad f'}{\mathbb{C} t^r \partial_0}\right)$

$\rtimes \operatorname{span}\{I(u,r),\ ad\, t^s,\ u \in \mathbb{C}^n, r \in rad f', s \notin rad f'\}$

is a Lie subalgebra of $\hat{\tau}^0$.

Consider subspace $W = \operatorname{span}\{\langle t_0, t^m t_0^{-1}\rangle . v - v \mid v \in V^+, m \in rad f'\}$

which is a submodule of V^+ for Lie subalgebra L'.

Consider quotient $\hat{V}^+ = V^+/W$

$\hat{V}^+$ is a finite dimensional module for the above Lie algebra.

$\langle t_0, t^m t_0^{-1} \rangle$ and $t^r \partial_0$ act scalarly on $\hat{V}^+$, $m, r \in \mathrm{rad} f'$.

So we identify L' with

$$\mathcal{L} = (\mathfrak{h} \otimes \mathbb{C}_{q'} \oplus I \otimes \mathbb{C}_{q'}) \rtimes \mathrm{span}\{I(u, r), \\ \mathrm{ad}\, t^s, u \in \mathbb{C}^n, r \in \mathrm{rad} f', s \notin \mathrm{rad} f\}$$

So $\hat{V}^+$ is a module for $\mathcal{L}$.

Now we take

$L(\hat{V}^+) = \hat{V}^+ \otimes \mathbb{C}_q$

and define a $\tilde{\tau}^0$-module structure on

and prove that

$$V^+ \cong L(\hat{V}^+)(\bar{0}) \text{ as } \tilde{\tau}^0\text{-module.}$$

<u>Theorem</u> (Santanu, —) Let V be an irreducible integrable $\breve{\tau}$-module with finite dimensional weight spaces with $\langle t_0, t_0^{-1} \rangle$ acting as c_0 and $\langle t_i, t_i^{-1} \rangle$ acting trivially on V for $1 \leq i \leq n$.

Then $V \cong U(\breve{\tau})M / M^{Rad}$.

~~where M is an irreducible~~