Today we will discuss the work of Etinghof, Frenkel, and Kazhdan (EFK) that I mentioned at the start on Friday. There were earlier developments by Teschner and perhaps Teschner has already spoken about this subject this morning. My lecture is based on the paper arXiv: 2107.01732 with D. Gaiotto.

EFK considered a Hilbert space $\mathcal{H}$ of L^2 functions (or better, half-densities) on $\mathcal{M}(G)$ (the moduli space of stable holomorphic G-bundles over C). They constructed operators on $\mathcal{H}$ that are related to the usual constructions of geometric Langlands, and found interesting duality theorems and conjectures concerning the action of these operators. As we proceed with the physical setup, we will see what those statements might be.

In geometric quantization, one would understand the L^2 functions on $\mathcal{M}(G)$ in terms of quantization of the cotangent bundle $T^*\mathcal{M}(G) \cong \mathcal{M}_H(G)$. In other words, the Hilbert space of EFK is what one gets if takes the Higgs bundle moduli space $\mathcal{M}_H(G)$ with real symplectic structure $\omega = \mathrm{Im}\,\Omega$, and quantizes it via geometric quantization, using the fact that it is a cotangent bundle, $\mathcal{M}_H(G) \cong T^*\mathcal{M}(G)$.

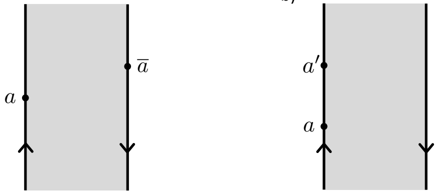
On the other hand, if we are going to get anywhere in terms of predictions from duality, we need to understand quantization of $\mathcal{M}_H(G)$ via branes. For this, the first step is to pick a complexification of $\mathcal{M}_H(G)$. Any complex manifold Y, viewed as a real manifold, has a canonical complexification, namely $\widehat{Y} = Y_1 \times Y_2$, where Y_1 and Y_2 are two copies of Y, with opposite complex structures I and $-I$. Therefore, the involution τ of $\widehat{Y}$ that exchanges the two factors of Y is antiholomorphic. Its fixed point set is the diagonal $Y \subset Y_1 \times Y_2$. Since the complex structures on the two factors are opposite, we can take the holomorphic symplectic form of $\widehat{Y}$ to be $\widehat{\Omega} = \frac{1}{2}\Omega \boxplus \frac{1}{2}\overline{\Omega}$, i.e. $\frac{1}{2}\Omega$ on Y_1 and $\frac{1}{2}\overline{\Omega}$ on Y_2. Then the restriction of $\widehat{\Omega}$ to the diagonal Y is $\mathrm{Re}\,\Omega$, in other words Y is Lagrangian for $\mathrm{Im}\,\Omega$ and symplectic for $\mathrm{Re}\,\Omega$. So this is the situation in which quantization of Y, using a Lagrangian brane $\mathcal{B}$ supported on Y, makes sense. Moreover, the real polarization of Y that leads to the Hilbert space studied by EFK does analytically continue to a holomorphic polarization of $\widehat{Y}$. Hence the Hilbert space they study is the one that arises in brane quantization.

However, we have to ask what are the observables in brane quantization. Let us recall Hitchin's integrable system. The Higgs bundle moduli space has a Hitchin fibration $\pi : \mathcal{M}_H \to B$, where $B \cong \mathbb{C}^n$ for some n. The linear functions on B are Hitchin's commuting Hamiltonians. Classically the global holomorphic functions on $\mathcal{M}_H$ are just the pullbacks of functions on B, i.e. the algebra $\mathcal{A}_0$ of holomorphic functions on $\mathcal{M}_H$ is the algebra of polymomial functions of the Hitchin Hamiltonians.

What are the quantum observables in brane quantization of Y? For brane quantization, we define the canonical coisotropic brane $\widehat{\mathcal{B}}_{\rm cc}$ of $\widehat{Y} = Y_1 \times Y_2$. It is just a product $\widehat{\mathcal{B}}_{\rm cc} = \mathcal{B}_{{\rm cc},1} \times \mathcal{B}_{{\rm cc},2}$ of canonical coisotropic branes on the two factors Y_1 and Y_2. So the algebra that acts on the quantization of Y is $\mathrm{Hom}(\widehat{\mathcal{B}}_{\rm cc}, \widehat{\mathcal{B}}_{\rm cc}) = \mathcal{A} \otimes \overline{\mathcal{A}}$, where $\mathcal{A} = \mathrm{Hom}(\mathcal{B}_{{\rm cc},1}, \mathcal{B}_{{\rm cc},1})$ and $\overline{\mathcal{A}} = \mathrm{Hom}(\mathcal{B}_{{\rm cc},2}, \mathcal{B}_{{\rm cc},2})$ are quantum-deformed versions of $\mathcal{A}_0$ and $\overline{\mathcal{A}}_0$, the holomorphic and antiholomorphic functions on Y. However, for $Y = \mathcal{M}_H(G)$, one can show that the quantum-deformed rings $\mathcal{A}$ and $\overline{\mathcal{A}}$ are still commutative. This is due to Hitchin for SL_2 and to Beilinson and Drinfeld in general. The quantum deformed objects are the commuting differential operators that are the quantization of Hitchin's classical commuting Hamiltonians.

The four-dimensional gauge theory picture gives another explanation of the fact that the quantum-deformed ring is still commutative, similar to the explanation I gave of the fact that Hecke functors at distinct points $p, p' \in C$ commute.

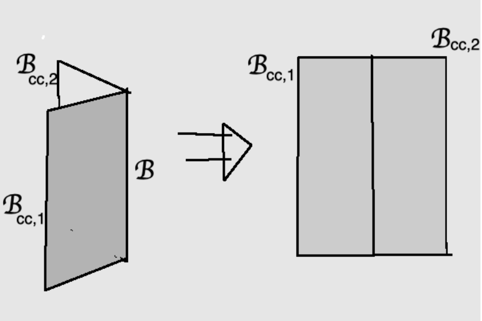
a) b)



We want to understand the action of $\mathrm{Hom}(\widehat{\mathcal{B}}_{\mathrm{cc}}, \widehat{\mathcal{B}}_{\mathrm{cc}}) = \mathcal{A} \otimes \overline{\mathcal{A}}$ on $\mathcal{H} = \mathrm{Hom}(\mathcal{B}, \widehat{\mathcal{B}}_{\mathrm{cc}}) = \mathsf{L}^2(\mathcal{M}(G))$. First of all, there is no mystery about the fact that $\mathcal{A} \otimes \overline{\mathcal{A}}$ does act on $\mathsf{L}^2(\mathcal{M}(G))$. This is just the statement that holomorphic and antiholomorphic differential operators on $\mathcal{M}(G)$ can act on "functions" (more accurately half-densities) on $\mathcal{M}(G)$. Holomorphic operators trivially commute with antiholomorphic ones, and the algebras of holomorphic or antiholomorphic differential operators are separately commutative. But what are the joint eigenvalues of Hitchin's Hamiltonians and their complex conjugates? To answer this question, we want to apply duality.

To apply duality to $\mathcal{H} = \mathrm{Hom}(\mathcal{B}, \widehat{\mathcal{B}}_{\mathrm{cc}})$ where $\widehat{\mathcal{B}}_{\mathrm{cc}} = \mathcal{B}_{\mathrm{cc},1} \times \mathcal{B}_{\mathrm{cc},2}$, naively we need to understand the duals of the three branes involved, namely $\mathcal{B}_{\mathrm{cc},1}$, $\mathcal{B}_{\mathrm{cc},2}$, and $\mathcal{B}$. $\mathcal{B}_{\mathrm{cc},1}$ is the brane associated to deformation quantization of the ring of holomorphic functions, and as I explained before, its dual is the structure sheaf of the variety of opers, $L_{\mathrm{op}} \subset \mathcal{M}_H(G^\vee)$. An oper is a flat bundle whose holomorphic structure obeys a certain condition. $\mathcal{B}_{\mathrm{cc},2}$ is the brane associate to deformation quantization of the ring of antiholomorphic functions. So its dual is the structure sheaf of $L_{\overline{\mathrm{op}}}$, the Lagrangian submanifold that parametrizes flat bundles whose antiholomorphic structure obeys the oper condition. What about the brane $\mathcal{B}$ that is supported on the diagonal?

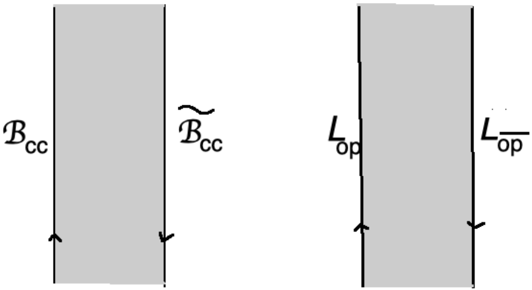
There is an "unfolding trick" that shows we do not have to worry about dualizing the diagonal:



In the unfolded version of the problem, there is just a single copy of $\mathcal{M}_H$ and the Hilbert space is $\mathcal{H} = \mathrm{Hom}(\mathcal{B}_{\rm cc}, \widetilde{\mathcal{B}}_{\rm cc})$ where $\widetilde{\mathcal{B}}_{\rm cc}$ is a conjugate version of $\mathcal{B}_{\rm cc}$ adapted to the opposite complex structure on $\mathcal{M}_H$.

So we just dualize $\mathcal{B}_{\mathrm{cc}}$ and $\widetilde{\mathcal{B}}_{\mathrm{cc}}$ to get L_{op} and $L_{\overline{\mathrm{op}}}$, that is the varieties in $\mathcal{M}_H(G^\vee)$ that parametrize flat $G^\vee$ bundles over C whose holomorphic and antiholomorphic structures, respectively, are opers. $\mathcal{A}$ becomes $\mathrm{Hom}(L_{\mathrm{op}}, L_{\mathrm{op}})$, which is just the algebra of holomorphic functions on L_{op}, and $\overline{\mathcal{A}}$ becomes $\mathrm{Hom}(L_{\overline{\mathrm{op}}}, L_{\overline{\mathrm{op}}})$, which is the algebra of holomorphic functions on $L_{\overline{\mathrm{op}}}$.

The two dual pictures are here:



The spectrum of $\mathcal{A} \otimes \overline{\mathcal{A}}$ is $\mathrm{Hom}(\widetilde{\mathcal{B}}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}}) = \mathrm{Hom}(L_{\overline{\mathrm{op}}}, L_{\mathrm{op}})$. L_{op} and $L_{\overline{\mathrm{op}}}$ are B-branes, so $\mathrm{Hom}(L_{\overline{\mathrm{op}}}, L_{\mathrm{op}})$ is an intersection or Hom space in the B-model.

A flat bundle E (for SL_2, say) is an oper holomorphically if holomorphically it is a nontrivial extension

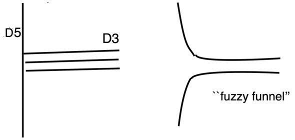
$$0 \to K^{1/2} \to E \to K^{-1/2} \to 0$$

and it is an oper antiholomorphically if antiholomorphically it is a nontrivial extension

$$0 \to \overline{K}^{1/2} \to E \to \overline{K}^{-1/2} \to 0.$$

Both of these conditions are holomorphic in complex structure J, in which $\mathcal{M}_H$ parametrizes flat bundles with complex structure group.

These conditions on E sound rather mysterious when stated without any explanation. Beilinson and Drinfeld found them by studying the current algebra of G (i.e. the conformal blocks of the Kac-Moody algebra of G) at the critical level $k = -h$ where it develops a large center. Gaiotto and I were able to recover these conditions from studying the D3-D5 system. As I remarked the other day, the brane $\mathcal{B}_{\mathrm{cc}}$ descends from a (slightly deformed) D3-NS5 boundary condition in string theory, so its dual comes from the D3-D5 system



The fuzzy funnel is related to Nahm's equations and the singular boundary condition that Nahm (1979) used in studying magnetic monopoles. Analysis of the singularity from a standpoint of complex geometry leads to the conditions defining an oper (see Gaiotto and EW, arXiv:1106.4789).

By the method I explained in the previous lecture using an antiholomorphic involution τ, $\mathrm{Hom}(\widetilde{\mathcal{B}}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$ carries a hermitian inner product, which we expect to be positive as it corresponds to the quantization of a cotangent bundle. The dual $\mathrm{Hom}(L_{\overline{\mathrm{op}}}, L_{\mathrm{op}})$ likewise carries a nondegenerate inner product which is defined in a similar way using an antiholomorphic involution $\tau^\vee$ of $\mathcal{M}_H(G^\vee)$. This nondegenerate inner product can be defined for any B-branes $L_{\overline{\mathrm{op}}}$, L_{op} that are exchanged by $\tau^\vee$. However, for generic B-branes this inner product is not positive-definite.

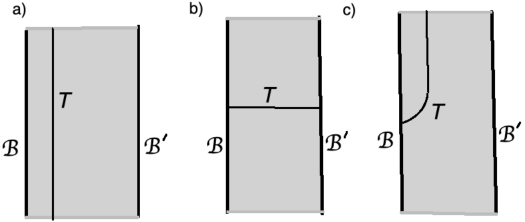
Consider a point $x \in L_{\mathrm{op}} \cap L_{\overline{\mathrm{op}}}$, corresponding to a flat bundle E that is an oper both holomorphically and antiholomorphically. The antiholomorphic involution $\tau^\vee$ that is used to define the hermitian structure on the $G^\vee$ side will map E to the complex conjugate flat bundle $\overline{E}$, which is also an oper both holomorphically and antiholomorphically. If E is not isomorphic to $\overline{E}$, then they are both null vectors for the hermitian form on $\mathrm{Hom}(L_{\overline{\mathrm{op}}}, L_{\mathrm{op}})$, which in that case is not positive-definite. But the duality predicts that it should be positive-definite, so we expect that, as conjectured by EFK and proved in some cases, E is always isomorphic to $\overline{E}$. Thus, the claim is that the flat $G_{\mathbb{C}}$ bundles that are opers both holomorphically and antiholomorphically are actually real, that is their structure group reduces to a real form of G. (I don't know if it is always the split real form.)

Assuming the intersection points are all real in that sense, one can show as a general statement about the B-model that $\mathrm{Hom}(L_{\overline{\mathrm{op}}}, L_{\mathrm{op}})$ is positive-definite only if all intersection points of $L_{\overline{\mathrm{op}}}$ and L_{op} are always transverse. (There is actually an interesting further necessary condition, which is unproved.) This statement is also among the results/conjectures of EFK. Note that if a flat bundle E is an oper holomorphically and is also real, then it is also an oper antiholomorphically. Bundles with this property are what EFK call real opers.

Thus assuming the duality is true and the hermitian form is positive-definite, the joint spectrum of $\mathcal{A}$ and $\overline{\mathcal{A}}$ corresponds to real opers. Let me recall that according to Beilinson and Drinfeld, an element x of $\mathcal{A}$ (1) is a holomorphic differential operator on Bun_G and (2) corresponds to a holomorphic function f_x on L_{op}. Similarly an element x' of $\overline{\mathcal{A}}$ (1) is an antiholomorphic differential operator on Bun_G and (2) corresponds to a holomorphic function $f_{x'}$ on $L_{\overline{\mathrm{op}}}$. So an intersection point p of L_{op} and $L_{\overline{\mathrm{op}}}$ determines a pair of eigenvalues of the differential operators x, x', namely $f_x(p)$ and $f_{x'}(p)$. This is the proposal for the joint spectrum of Hitchin's quantized Hamiltonians.

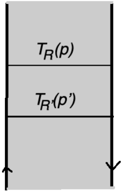
EFK also introduced Hecke operators as operators on the Hilbert space $\mathcal{H}$ that arises in quantization of $\mathcal{M}_H(G)$ as a real symplectic manifold. I explained already that line operators of the four-dimensional gauge theory can be used to define functors on the category of branes (boundary conditions). Moreover 't Hooft and Wilson line operators correspond to dual pairs of functors on the A and B model categories.

The same line operators represent operators on the quantum Hilbert space. I've tried to explain this with the following picture:



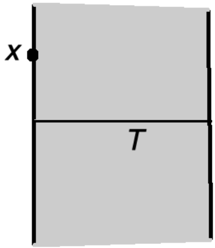
In a), I show the old story of a line operator T as a functor on the category of branes. In b), the same line operator is used to make an operator $T : \mathcal{H} = \mathcal{H}$ where $\mathcal{H} = \mathrm{Hom}(\mathcal{B}', \mathcal{B})$ (for some branes $\mathcal{B}, B'$). The picture in b) raises the question of what is happening at the corner where T ends on a brane $\mathcal{B}$. The point of c) is to explain that the corner represents an element of $\mathrm{Hom}(\mathcal{B}, T\mathcal{B})$. So actually to define the operator $T : \mathrm{Hom}(B', \mathcal{B})$ in a), one needs "junctions" $\alpha \in \mathrm{Hom}(\mathcal{B}, T\mathcal{B})$ and $\beta \in \mathrm{Hom}(T\mathcal{B}', \mathcal{B}')$.

The argument that showed commutativity of the functors $T_R(p)$, $T_{R'}(p')$ corresponding to different points $p, p' \in C$ (and representations R, R' of the dual group) goes over immediately to show that the operators $T_R(p)$, $T_{R'}(p')$ for different points $p, p' \in C$ commute. Because they live at different points in C, they can be moved up and down past each other in this picture without singularity:



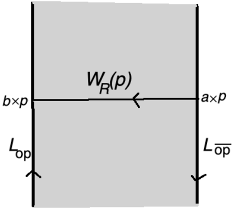
Moreover as in the discussion of line operators as functors, we can set $p = p'$ and retain commutativity.

A similar argument shows that Hecke operators $T_R(p)$ commute with the quantized version of Hitchin's holomorphic (or antiholomorphic) Hamiltonians. A two-dimensional picture makes it look like there could be a problem in moving a Hamiltonian $x \in \mathcal{A}$ past a Hecke operator $T_R(p)$, but in four dimensions it is obvious that there is no problem:



Finally, we can ask what are the predictions of the duality for the eigenvalues of the Hecke operators. To answer this, we start on the $G^\vee$ side. We are going to find the eigenvalues of a Wilson operator $W_R(p)$ acting on $\mathrm{Hom}(L_{\overline{\mathrm{op}}}, L_{\mathrm{op}})$, and then the duality will predict that the eigenvalues of of the Hecke operator $T_R(p)$ are the same. Also, we will find what kind of data is needed for the "corners" that make $W_R(p)$ into an operator; again, the duality predicts that the same data is needed to define the Hecke operator $T_R(p)$.

Given a $G^\vee$ bundle E with connection over $\Sigma \times C$, and a representation R of $G^\vee$, we form the associated bundle $E_R = E \times_{G^\vee_{\mathbb{C}}} R$, which also comes with a connection. $W_R(\gamma)$, for a path γ in $\Sigma \times C$, is defined by parallel transport of the connection on E_R along γ. For the present application, we fix a point $p \in C$ and two points a, b on the right and left boundaries of Σ, and a path in $\Sigma \times p$ from $a \times p$ to $b \times p$:



If $E_{R,a\times p}$ and $E_{R,b\times p}$ are the fibers of E_R at $a \times p$ and $b \times p$, then parallel transport defines, for each connection, a linear transformation $W_R(p) : E_{R,a\times p} \to E_{R,b\times p}$ or equivalently an element

$$W_{R,p} :\in \mathrm{Hom}(E_{R,a\times p} \otimes E_{R',b\times p}, \mathbb{C}),$$

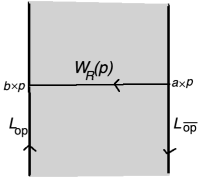
where R' is the dual representation to R.

A quantum operator is going to come from something that is a complex valued function of connnections. We do not have that yet; what we have is that for each connection we have defined

$$W_{R,p} \in \mathrm{Hom}(E_{R,a\times p} \otimes E_{R',b\times p}, \mathbb{C}).$$

To get an operator, we need to supply elements $v \in E_{R,a\times p}$, $w \in E_{R',b\times p}$ and then $W_{R,p}(v \otimes w)$ is the function of connections whose quantization will be an operator (an easy one to diagonalize and evaluate, as we will see). In order to avoid unnecessary details, I will state the following for the case that $G^\vee = SL_2$ and $R = R'$ is the two-dimensional representation, but there is a straightforward generalization to any $G^\vee$ and R.

In this picture,

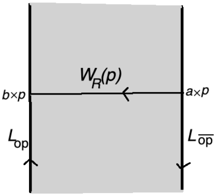


the boundary condition on the right boundary is that the $G^\vee$ bundle E, restricted to the right boundary, is an antiholomorphic oper. This means that there is a nonsplit exact sequence

$$0 \to \overline{K}^{1/2} \xrightarrow{\bar{j}} E_R \to \overline{K}^{-1/2} \to 0$$

where $\overline{K}$ is the canonical bundle of $\overline{C}$ (C with opposite complex structure) and $\overline{K}^{1/2}$ is a square root of it. (We pick here a square root, but as noted by EFK on the dual side, the choice cancels out in a moment.) So if we pick a vector $v \in \overline{K}_p^{1/2}$ (the fiber of $\overline{K}^{1/2}$ at p) then this gives a vector $\bar{j}(v) \in E_{R,a\times p}$.

Similarly on the left boundary



the boundary condition says that E is a holomorphic oper, implying that there is a nonsplit exact sequence

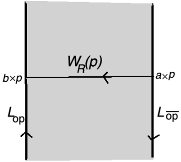
$$0 \to K^{1/2} \xrightarrow{j} E_{R'} \to K^{-1/2}.$$

So we pick $w \in K_p^{1/2}$ and this gives us $j(w) \in E_{R',b\times p}$. In other words, the "corners" correspond to $v \in \overline{K}^{1/2}$ and $w \in K^{1/2}$. Once they are picked, we define

$$\widehat{W}_{R,p} = W_{R,p}(\bar{j}(v) \otimes j(w))$$

and this is the complex-valued function of connections that we will interpret as a quantum operator.

This step is trivial, because in the B-model,



one can actually assume that the connection is pulled back from C, up to a possible "twist" by an element of the center of $G^\vee$, which for $G^\vee = SL(2,\mathbb{R})$ is $\{\pm 1\}$. So up to sign, $\widehat{W}_{R,p}$ is just the natural dual pairing $(\ ,\ ) : E_{R,p} \otimes E_{R',p} \to \mathbb{C}$, and therefore its value at a given real oper is simply

$$\pm(\bar{j}(v), j(w)).$$

It is convenient to write this formula in this way in terms of v, w, but it actually only depends on $v \otimes w \in K^{1/2} \otimes \overline{K}^{1/2} = |K|$, and thus in particular as observed by EFK there is no dependence on a choice of spin structure.

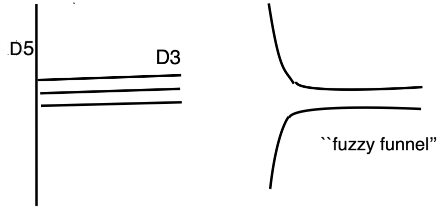
The duality then predicts that also on the dual side, the definition of $T(p)$ as an operator requires "corners" $v, w \in \overline{K}_p^{1/2},\ K_p^{1/2}$. This is true, as shown by EFK using an algebro-geometric formula. The duality further predicts that the eigenvalues of the resulting operator are what we found on the $G^\vee$ side: they are

$$\pm(\bar{j}(v), j(w))$$

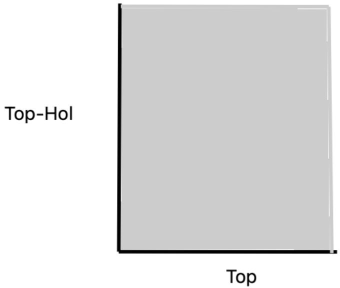
for all the possible real opers. Both signs do occur; this is related on the G side to the fact that minimal Hecke modification of an $SO(3)$ bundle $E \to C$ acts nontrivially on $w_2(E) \in H^2(C, \mathbb{Z}_2) \cong \mathbb{Z}_2$.

So far I have roughly explained the main points of sections 2-5 of my paper with Gaiotto. I will skip over section 6 (which is on an analogous story for real forms, which isn't as well understood), and say a few words about sections 7-9.

The twisted *A*-model that we've been talking about is a 4d topological field theory, but it is not true that all *A*-branes that we've discussed preserve this symmetry. Plenty of "topological" branes exist, but the *deformed* D3-NS5 and D3-D5 branes are only topological-holomorphic

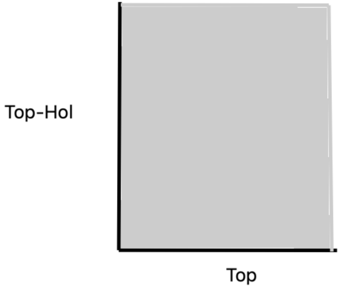


At a junction between a topological-holomorphic boundary condition and a topological one, there is typically a 2d chiral algebra:



In particular, at $\theta = \pi$, if the holomorphic-topological boundary condition is the deformed D3-NS5 boundary condition related to $\mathcal{B}_{\mathrm{cc}}$, the chiral algebra is the Kac-Moody algebra (current algebra) of G at the "critical level" $k = -h$. Current algebra at the critical level was a primary tool in the work of Beilinson and Drinfeld, but its relation to the 4d gauge theory was obscure when Kapustin and I were working on that subject in 2005-6. The occurrence of chiral algebras at the "corner" was explained in a series of papers by Gaiotto (with co-authors on some papers) of which an early one, with M. Rapcak, was arXiv:1703.00982.

Any well-defined path integral with such a “corner” will generate a conformal block for the chiral algebra that lives at the corner:



To make contact with the construction in today's lecture, we want a topological-holomorphic junction and a topological-antiholomorphic one: