# Quantization by Branes and Geometric Langlands

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Lectures at QFT-GRT 2021, ICTS

Today and in the last lecture on Tuesday, I will explain work with Davide Gaiotto in which we aim to understand in terms of quantum field theory the picture developed in

P. Etinghof E. Frenkel, and D. Kazhdan, "An Analytic Version Of The Langlands Correspondence For Complex Curves," arXiv:1908.09677

Some earlier developments: J. Teschner, "Quantization Conditions Of The Quantum Hitchin System and the Real Geometric Langlands Correspondence," arXiv:1707.07873, among others.

In the gauge theory approach (A. Kapustin and EW, arXiv:hep-th/0604151), the starting point for geometric Langlands is $\mathcal{N} = 4$ super Yang-Mills theory in four dimensions with gauge group $G$, or Langlands-GNO dual group $G^\vee$. The theory based on $G$ has a "gauge coupling constant" $e$ and topological angle $\theta$, which combine to a complex parameter

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi \mathrm{i}}{e^2}.$$

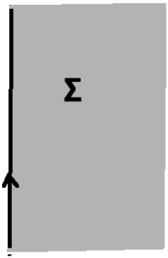
There is an elementary symmetry $\tau \cong \tau + 1$. The theory based on $G^\vee$ has an analogous parameter $\tau^\vee$, also with an elementary symmetry $\tau^\vee \cong \tau^\vee + 1$. The claim of "electric-magnetic duality," whose earliest version goes back to C. Montonen and D. Olive (1977), is that the theories based on $G$ and on $G^\vee$ are equivalent under

$$\tau^\vee = \frac{-1}{n_{\mathfrak{g}} \tau}.$$

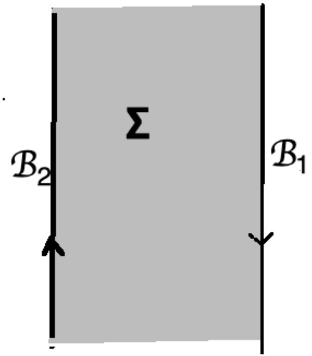
($n_{\mathfrak{g}} = 1, 2$, or 3 depending on $G$.)

This isn't particularly a statement about geometric Langlands, and people studying it are usually studying questions that have no obvious relation to geometric Langlands. However, we can specialize to the situation that leads to geometric Langlands. First, we consider a "topological twist" that leads to a $\mathbb{CP}^1$ family of topological field theories, parametrized say by complex parameters $\Psi$ or $\Psi^\vee$. "Twisting" means picking a linear combination $Q$ of the supersymmetry generators such that $Q^2 = 0$, and passing everywhere to the cohomology of $Q$ – one considers only operators, states, and boundary conditions that are $Q$-invariant, modulo $Q$-exact operators and states. $\mathbb{CP}^1$ comes in because there are a pair of supercharges $Q_1, Q_2$ that come in and one can define $Q$ to be any nonzero linear combination $Q = u_1 Q_1 + u_2 Q_2$, with $\Psi = u_1/u_2$. If we study the whole family, then the equivalence under $\Psi^\vee = -1/n_{\mathfrak{g}}\Psi$ (and $\Psi \to \Psi + 1$, $\Psi^\vee \to \Psi^\vee + 1$) becomes the duality of "quantum geometric Langlands." However, today we will consider the basic geometric Langlands duality betwen $\Psi = 0$ for $G$ ("the $A$-model") and $\Psi^\vee = \infty$ for $G^\vee$ ("the $B$-model").

In general, quantum field theory in dimension $d$ associates a number to a $d$-manifold, a vector space ("the space of physical states") to a $d-1$-manifold, and a category ("the category of boundary conditions") to a $d-2$-manifold. For geometric Langlands, we restrict ourselves to four-manifolds of the form $\Sigma \times C$, where $C$ is an oriented two-manifold that we keep fixed and only vary $\Sigma$ (another two-manifold). So we define a category of boundary conditions that depends on $C$. It is convenient to draw two-dimensional pictures in which we only exhibit $\Sigma$. If $\Sigma$ has a boundary, then to formulate quantum field theory on $\Sigma$ we need to specify a boundary condition

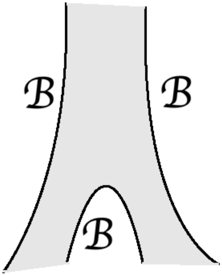


Boundary conditions make a category – given two boundary conditions $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathrm{Hom}(B_1, \mathcal{B}_2)$ is the vector space that the quantum field theory assigns to this picture

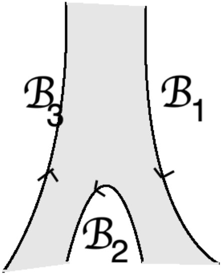



(Physicists usually don't use this terminology.)

With this definition, $\mathrm{Hom}(\mathcal{B}, \mathcal{B})$ is an algebra for every $\mathcal{B}$

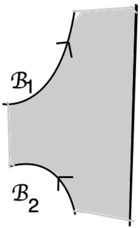


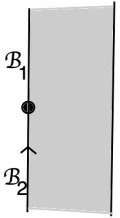
Similarly




and moreover $\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)$ is a left $\mathrm{Hom}(\mathcal{B}_2, \mathcal{B}_3)$ module, etc.

An element of $\mathrm{Hom}(\mathcal{B}_2, \mathcal{B}_1)$ can be represented as an open string joining on to the rest of the picture or by a local operator insertion at which the boundary condition jumps from $\mathcal{B}_2$ to $\mathcal{B}_1$:



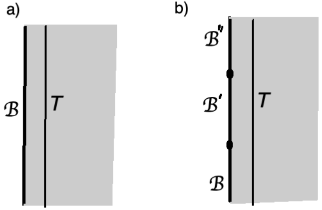




For the equivalence of these pictures, one does not need topological invariance; two-dimensional conformal invariance would be enough. In conformal field theory, one would think of the open strings in the picture as going off to infinity, and one would say that there is a conformal mapping from the picture on the left to the one on the right.

What I've been explaining is that two-dimensional topological field theories correspond to "categories."

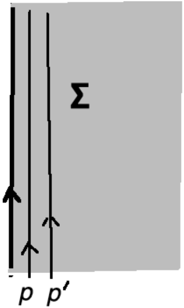
So electric-magnetic duality will give an isomorphism between the category $\mathcal{C}$ associated to $C$ in the $G$ theory at $\Psi = 0$ and the category $\mathcal{C}^\vee$ associated to $C$ in the $G^\vee$ theory at $\Psi^\vee = \infty$. In addition, there is a natural mapping between certain natural functors on $\mathcal{C}$ and on $\mathcal{C}^\vee$. These functors come from "line operators" of quantum field theory. In a quantum field theory defined on a manifold $M$, a line operator is some sort of modification of the definition of the theory along an embedded one-manifold $K \subset M$. The natural line operators at $\Psi = \infty$ are "Wilson operators" (the holonomy of a connection, interpreted as an operator in quantum field theory) and the natural line operators at $\Psi = 0$ are "'t Hooft operators." In physics, Wilson and 't Hooft operators are usually used in analyzing the confinement of quarks in atomic nuclei – and other subtleties involving the "universality classes" of quantum field theories.

But in comparing to the usual language in mathematical discussions of geometric Langlands, a line operator is a functor from the category of boundary conditions to itself. We can understand that statement from this picture:




The point in a) is just that a line operator $T$ that runs along a boundary with some boundary condition $\mathcal{B}$ makes a new boundary condition $T\mathcal{B}$. This explains how $T$ acts on objects of the category. Its action on morphisms in the category is shown in b).

I've been drawing pictures in two dimensions, but there are two more dimensions not drawn and $T$ really depends on the choice of a point $p \in C$.
Such pictures make it obvious that the line operators $T(p)$, $T(p')$, for $p, p' \in C$ commute



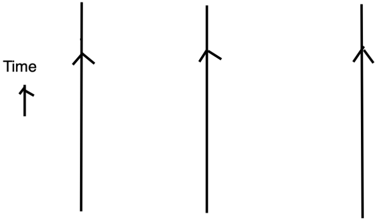

The picture describes $T(p')T(p)\mathcal{B}$, but for $p \neq p'$, we can move the two line operators through each other without any singularity, so $T(p)T(p')\mathcal{B} = T(p')T(p)\mathcal{B}$.

So the two dual categories $\mathcal{C}$ and $\mathcal{C}^\vee$ are equipped with dual families $T(p)$ and $W(p)$ of functors, parametrized by the choice of $p \in C$ (and some other data) and commuting at distinct points. $T(p)$ corresponds to the usual Hecke functors of geometric Langlands, while $W(p)$ corresponds to its dual in the usual geometric Langlands duality. Since these functors depend on more data (a representation $R$ of $G^\vee$), one can also consider the composition $T_R(p)T_{R'}(p)$ of Hecke functors at the same point $p$ but associated to possibly different representations of $G^\vee$. They commute, but a little less obviously. The duality says that the algebra of compositions $T_R(p)T_{R'}(p)$ of Hecke operators at the same point is the same as the corresponding algebra of Wilson operators $W_R(p)W_{R'}(p)$, which (on elementary grounds) is the tensor algebra of representations of $G^\vee$:

$$W_R(p)W_{R'}(p) = W_{R\otimes R'}(p),$$

where $W_{R\otimes R'}(p)$ can be expanded as a sum of irreducibles. (The fact that the decomposition of the $T$'s matches that of the $W$'s is known as the geometric Satake correspondence.)

As I have said, Wilson operators are just constructed from the holonomy of a connection, but 't Hooft operators are more subtle. Let me briefly explain how 't Hooft operators are defined and how they are related to the "geometric Hecke transformations" that mathematicians study in geometric Langlands. The *A*-model, analogously to the 2d *A*-models that are related to the Fukaya category and mirror symmetry, localizes on the solutions of some elliptic partial differential equations. If we consider a time-independent situation with 't Hooft operators that run in the time direction




then we can reduce to three-dimensional PDE's, and these are the Bogomolny equations.

The Bogomolny equations are equations for a pair $A, \phi$, where $A$ is a connection on a $G$-bundle $E \to W_3$, with $W_3$ an oriented 3-dimensional Riemannian manifold, and $\phi$ is a section of $\mathrm{ad}(E) \to W_3$ (i.e. an adjoint-valued 0-form). If $F = \mathrm{d}A + A \wedge A$ is the curvature of $A$, then the Bogomolny equations are

$$F = \star \mathrm{d}_A \phi.$$

($\star$ is the Hodge star and $\mathrm{d}_A$ is the gauge-covariant extension of the exterior derivative.)

The Bogomolny equations have many remarkable properties and we will focus on just one aspect. We consider the Bogomolny equations on $W_3 = \mathbb{R} \times C$ with $C$ a Riemann surface. Any connection $A$ on a $G$-bundle $E \to C$ determines a holomorphic structure on $E$ (or more exactly on its complexification): one simply writes $\mathrm{d}_A = \overline{\partial}_A + \partial_A$ and uses $\overline{\partial}_A$ to define the complex structure. (In complex dimension 1, there is no integrability condition that must be obeyed by a $\overline{\partial}$ operator.) The fact that every $A$ defines a holomorphic structure on the bundle is an important fact that, for example, will also come up on Monday when we discuss the Narasimhan-Seshadri theorem. So for any $y \in \mathbb{R}$, by restricting $E \to \mathbb{R} \times C$ to $E \to \{y\} \times C$, we get a holomorphic bundle $E_y \to C$. However, if the Bogomolny equations are satisfied, $E_y$ is canonically independent of $y$. Indeed, a consequence of the Bogomolny equations is that $\overline{\partial}_A$ is independent of $r$ up to conjugation. If we parametrize $\mathbb{R}$ by $y$ then

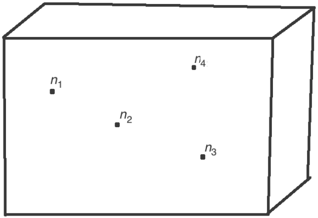
$$\left[\frac{D}{Dy} - \mathrm{i}\phi, \overline{\partial}_A\right] = 0.$$

The Bogomolny equations admit solutions with a singularity at isolated points in $\mathbb{R}^3$ (or in a more general 3-manifold $W_3$). Let me first describe the picture for $U(1)$. One fixes an integer $n$ and one observes that the Bogomolny equation has an exact solution for any $x_0 \in \mathbb{R}^3$:

$$\phi = \frac{n}{2|\vec{x} - \vec{x}_0|}, \quad F = \star \mathrm{d}\phi.$$

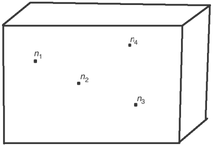
I have only defined $F$ and not the connection $A$ whose curvature is $F$ or the line bundle $\mathcal{L}$ on which $A$ is connection, but such an $\mathcal{L}$ and $A$ exist (and are essentially unique) if $n \in \mathbb{Z}$.

For $G = U(1)$, since the Bogomolny equations are linear, they have a unique solution with singularities labeled by specified integers $n_1, n_2, \ldots$ at specified points in $\mathbb{R}^3$:




In the context of the Langlands correspondence, the integers $n_i$ should be understood as labeling representations $R_i$ of the dual group, which for$G = U(1)$ is simply $G^\vee = U(1)$, so an irreducible representation is indeed labeled by an integer, the "charge."

We pick a decomposition $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$ where we identify $\mathbb{R}^2$ as $\mathcal{C}$. Suppose that the singularities are at $y_i \times p_i$, with $y_i \in \mathbb{R}$, $p_i \in \mathcal{C}$:




For each $y \notin \{y_1, \ldots, y_n\}$, the indicated solution of the Bogomolny equations determines a holomorphic line bundle $\mathcal{L}_y \to \mathcal{C}$, and this naturally extends to $\mathcal{L}_y \to \mathbb{CP}^1$. $\mathcal{L}_y$ is constant up to isomorphism for $y$ not equal to one of the $y_i$, but even when $y$ crosses one of the $y_i$, $\mathcal{L}_y$ is constant when restricted to $\mathbb{CP}^1 \backslash p_i$. In crossing $y = y_i$, $\mathcal{L}_y$ undergoes a Hecke modification

$$\mathcal{L}_y \to \mathcal{L}_y \otimes \mathcal{O}(p_i)^{n_i}.$$

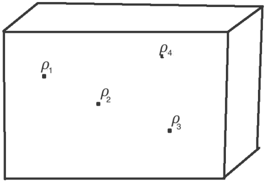
The solution describes a sequence of Hecke modifications of a line bundle over $\mathcal{C}$.

We can do something similar for any simple Lie group $G$. pause Let $T$ be the maximal torus of $G$ and let $\mathfrak{t}$ be its Lie algebra. Pick a homomorphism $\rho : \mathfrak{u}(1) \to \mathfrak{t}$. Up to a Weyl transformation, such a $\rho$ is equivalent to a dominant weight of the dual group $G^\vee$, so it corresponds to a representation $R^\vee$ of $G^\vee$. We turn the singular solution of the $U(1)$ Bogomolny equations that we already used (with $n = 1$) into a singular solution for $G$ simply by

$$(A, \phi) \to (\rho(A), \rho(\phi)).$$

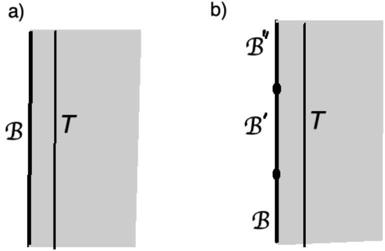
Then we look for solutions of the Bogomolny equations for $G$ with singularities of this type at specified points $y_i \times p_i \in \mathbb{R}^3$.

The picture is the same as before




except that now the points $y_i \times p_i$ are labeled by homomorphisms $\rho_i : \mathfrak{u}(1) \to \mathfrak{t}$, or in other words by representations $R_i^\vee$ of the dual group $G^\vee$, rather than by integers $n_i$. Such a solution describes a sequence of Hecke modifications at $p_i$ of type $\rho_i$, mapping a trivial $G$-bundle $E \to \mathbb{CP}^1$ to itself. Kronheimer showed in the 1980's that the space of solutions of the Bogomolny equations on $\mathbb{R}^3$, with these singularities, and trivial at infinity is a hyper-Kahler manifold. (a space of Hecke modifications from the trivial bundle to itself).

Now if we look again at the picture that told us to interpret a line operator $T$ as a “functor” on the category of boundary conditions




we see that $T$ acts on the bundle that is defined by the gauge theory on the boundary via a Hecke modification.

To go into a little more detail, I want to introduce a useful language for formulating a simplified version of geometric Langlands duality. Let $\mathcal{M}_H(G, C)$ or just $\mathcal{M}_H(G)$ be the moduli space of $G$-Higgs bundles on $C$. As shown by Hitchin, it is a hyper-Kahler manifold. In one complex structure, $I$, it parametrizes Higgs bundles over $C$. In another complex structure, namely $J$, it parametrizes flat $G_{\mathbb{C}}$ bundles over $C$. $I$, $J$, and $K = IJ$ act as the usual unit quaterions ($IJ + JI = 0$, etc.). There are also the corresponding three Kahler forms $\omega_I$, $\omega_J$, and $\omega_K$.

Geometric Langlands duality can be understood for many purposes as a mirror symmetry between the $A$-model of $\mathcal{M}_H(G)$ in symplectic structure $\omega_K$ and the $B$-model of $\mathcal{M}_H(G^\vee)$ in complex structure $J$. (This instance of mirror symmetry was first studied mathematically by Hausel and Thaddeus (2002).) One can definitely ask questions for which the two-dimensional picture is inadequate. Mathematicians describe this by saying that the theory should be formulated on the stack of $G$-bundles, $\mathrm{Bun}_G$, not on a finite-dimensional moduli space. Physicists describe it by saying that the correct formulation is as a duality of four-dimensional theories. (Four is the minimum: there is a six-dimensional formulation that explains some things better, but that is not for today.) These two descriptions are not as different as you may think, since Atiyah and Bott showed in 1981 that a model of $\mathrm{Bun}_G$ is the space of all connections on a fixed smooth $G$-bundle $E \to C$; so four-dimensional gauge theory is a two-dimensional theory with target $\mathrm{Bun}_G$.

I want to use the two-dimensional description of geometric Langlands duality as mirror symmetry between $\mathcal{M}_H(G)$ and $\mathcal{M}_H(G^\vee)$ to explain why the category that appears on the "automorphic" side of the duality is a category of $\mathcal{D}$-modules on $\mathrm{Bun}_G$. The most familiar branes in the $A$-model or Fukaya category of a real symplectic manifold $Y$ are "Lagrangian branes," supported on a Lagrangian submanifold $L$. However, Kapustin and Orlov discovered (2001) that in general one can also define "coisotropic $A$-branes" that are supported on a coisotropic submanifold $R \subset Y$ that is above the middle dimension. The construction of coistropic $A$-branes is delicate in general, but the simplest case is the case relevant to geometric Langlands.

Kapustin and Orlov had a simple condition for a coisotropic $A$-brane of rank 1 whose support is all of $Y$. Suppose we are studying the $A$-model of $Y$ equipped with a symplectic form $\omega$. Suppose also that the model has a "$B$-field," which I will just call $B$. For our purposes, this is a closed two-form that appears as a form in the action of a two-dimensional $\sigma$ model:

$$\Delta I = -\mathrm{i} \int_{\Sigma} B.$$

Consider a rank 1 $A$-brane $\mathcal{B}$, endowed with a "Chan-Paton" line bundle $\mathcal{L} \to Y$, with a unitary connection of curvature $F$. The Kapustin-Orlov condition for this data to define a coisotropic $A$-brane is that

$$I = \omega^{-1}(F + B),$$

which can be viewed as a linear transformation of the tangent bundle $TY$ of $Y$, satisfies $I^2 = -1$ and defines an integrable complex structure.

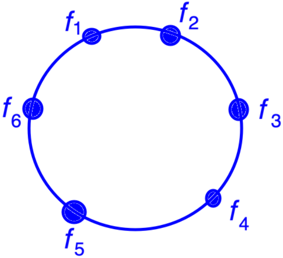
This is a difficult condition to satisfy in general, but there is an easy way to do so in the case relevant for geometric Langlands. Suppose that $Y$ is actually a complex symplectic manifold with complex structure $I$ and holomorphic symplectic form $\Omega = \omega_J + \mathrm{i}\omega_K$. (We are interested in the case that $Y$ is the Higgs bundle moduli space, which satisfies these conditions. Suppose further that we are interested in the $A$-model of $Y$ with the real symplectic structure $\omega = \omega_K$. Then we can solve the Kapustin-Orlov conditions by taking

$$F + B = \omega_J,$$

which leads to $\omega^{-1}(F + B) = \omega_K^{-1}\omega_J = I$, an integrable complex structure. I will call the $A$-brane constructed this way the "canonical coisotropic $A$-brane," $\mathcal{B}_{\mathrm{cc}}$. (On Tuesday, it will be important that we could also have taken $F + B = -\omega_J$, leading to $\omega^{-1}(F + B) = -I$, another integrable complex structure.) The various ways to satisfy $F + B = \omega_J$ differ from each other by "$B$-field gauge transformations." For today I will choose $F = 0$, $B = \omega_J$.

It turns out that, additively, $\mathcal{A} = \mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$ is the algebra of holomorphic functions on $Y$ in complex structure $I$. But the algebra structure is different; $\mathcal{A}$ is a "deformation quantization" of the commutative ring $\mathcal{A}_0$ of holomorphic functions on $Y$, in the sense described yesterday. I would like to explain a little of how that comes about.

Remember that ordinary deformation quantization of the ring of smooth functions on a real symplectic manifold $M$ is given by a path integral on a circle




The integral is over maps $X : S^1 \to M$. The action is taken to be

$$I = -\frac{\mathrm{i}}{\hbar}\int_{S^1} X^*(T),$$

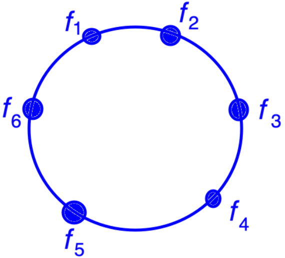
where $T$, with curvature $\omega = \mathrm{d}T$, is a connection on a prequantum line bundle.

In the context of deformation quantization, one studies this path integral perturbatively in $\hbar$. In perturbation theory, one can assume that the map $X : S^1 \to M$ is almost constant, and hence extends in a natural way (up to homotopy) to a map $X : D \to M$. Hence we could have written the action as

$$I = -\frac{\mathrm{i}}{\hbar} \int_D X^*(\omega).$$

This, incidentally, shows that deformation quantization, as opposed to quantization, depends only on $\omega$ and not on the choice of a prequantum line bundle.

In the path integral, we also make cyclically ordered insertions along the circle




of functions $f_1, f_2, \cdots, f_n$, and we compute

$$\langle f_1 f_2 \cdots f_n \rangle = \int DX \, \exp\left(\frac{\mathrm{i}}{\hbar}\int_D (X^*(\omega)\right) \, f_1(X(t_1) f_2(X(t_2)) \cdots f_n(X(t_n)).$$

Let us suppose that $M$ can be complexified to a nice complex symplectic manifold $Y$ with complex structure $I$ and complex symplectic form $\Omega$ – nice means that the $\sigma$-model of $Y$ exists as a UV-complete theory (for example, this is true if $Y$ is a complete hyper-Kahler manifold) with a holomorphic symplectic form $\Omega$ such that $\Omega|_M = \omega$. The space $\mathcal{W}_0$ of maps $X : S^1 \to M$ can be complexified to the space of $\mathcal{W}$ maps $X : S^1 \to Y$. Let us also assume that the functions $f_1, \cdots, f_n$ that we consider on $M$ are restrictions to $M$ of holomorphic functions on $Y$. Then deformation quantization of $M$ can be recast in the following form: the integrand of the path integral, namely

$$DX \, \exp\left(\frac{\mathrm{i}}{\hbar}\int_D (X^*(\Omega)\right) \, f_1(X(t_1)f_2(X(t_2))\cdots f_n(X(t_n))$$

can be viewed formally as a holomorphic form of top degree on the infinite-dimensional complex manifold $\mathcal{W}$.

A holomorphic form of top degree can be integrated on a middle-dimensional real cycle $\Gamma$. In the present case this would be a path integral

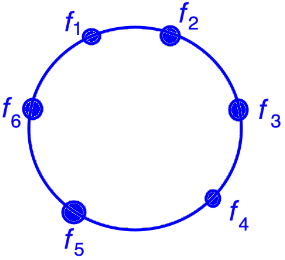
$$\int_\Gamma DX\,\exp\left(\frac{\mathrm{i}}{\hbar}\int_D (X^*(\Omega)\right)\, f_1(X(t_1)f_2(X(t_2))\cdots f_n(X(t_n)).$$

There is a standard choice of $\Gamma$, namely the original integration cycle $\mathcal{W}_0$ of maps $S^1 \to M$ (rather than to $Y$). However, any other $\Gamma$ will give something that satisfies the general conditions of deformation quantization. This statement generalizes the following fact: let $X$ be an algebraic variety of finite dimension, $\Lambda$ a holomorphic form of top degree, and $\Gamma$ a middle-dimensional cycle. Then if $X$ and $\Lambda$ depend on some parameters, the periods

$$\int_\Gamma \Lambda$$

will often satisfy differential equations that can be proved by integrating by parts, and if so those equations do not depend on the choice of $\Gamma$.

But what is a useful middle-dimensional integration cycle $\Gamma \subset \mathcal{W}$ other than the standard cycle $\mathcal{W}_0$ of maps to $M$? There is a simple answer: with $D$ a disc of boundary $S^1$, we take $\Gamma$ to consist of maps $X : S^1 \to Y$ that can be extended to pseudoholomorphic maps $X : D \to Y$ (pseudoholomoprhic in an almost complex structure $K$ in which $\omega = \mathrm{Im}\,\Omega$ is of type $(1,1)$ and positive; if $Y$ is actually hyper-Kahler, then $K$ can be chosen to be part of the hyper-Kahler structure of $Y$).

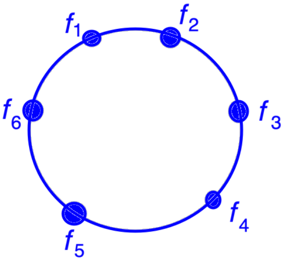


We consider the $A$-model on the disc $D$ and observe that (with our choice $B = \omega_J$) the action is

$$-\mathrm{i} \int_D X^*(\Omega) + \{Q, \cdots\}$$

Standard $A$-model arguments give localization on the space $\Gamma$ of pseudoholomorphic maps and the $A$-model path integral can indeed be interpreted as

$$\int_\Gamma DX \, \exp\left(\frac{\mathrm{i}}{\hbar}\int_D (X^*(\Omega)\right) \, f_1(X(t_1) f_2(X(t_2))\cdots f_n(X(t_n)).$$




I've shaded the disc because now we are actually doing a path integral on the disc, not on a circle.

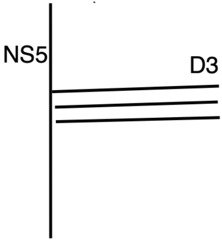
I've set $\hbar = 1$ in the final formulas, because although deformation quantization is usually formulated over a formal power series ring $\mathcal{C}[[\hbar]]$, we are interested in a situation in which it makes sense to take $\hbar$ to be a nonzero constant, and since I have not told you how to normalize the symplectic form, we lose no generality by setting $\hbar = 1$. It is because I want deformation quantization of $Y$ to be "exact," not just a formal procedure over a ring of formal power series, that I say $Y$ should be "nice."

A simple nice case is that $Y = T^*W$ is a cotangent bundle of some other complex manifold $W$ with the standard complex symplectic structure. Then one can show that the deformation-quantized ring $\mathcal{A}$ is a ring of holomorphic differential operators acting on the line bundle $K^{1/2} \to W$, where $K$ is the canonical bundle of $W$. If one "sheafifies" the situation by working locally on $W$, one gets the sheaf of holomorphic differential operators on $W$, acting on $K^{1/2}$.

In the case of geometric Langlands, we take $Y$ to be $\mathcal{M}_H(G)$, the moduli space of $G$-Higgs bundles on $C$. This is birational to $T^*\mathcal{M}(G)$ where $\mathcal{M}(G)$ is the moduli space of holomorphic $G$-bundles on $C$. So $\mathcal{A} = \mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$ is the sheaf of holomorphic differential operators on $\mathcal{M}(G)$. If $\mathcal{B}$ is any other brane, then $\mathrm{Hom}(\mathcal{B}, \mathcal{B}_{\mathrm{cc}})$ is going to be a module for $\mathcal{A} = \mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$. In other words, the category on the $A$-model side of the geometric Langlands duality is a category of, roughly speaking, $\mathcal{D}$-modules on $\mathcal{M}(G)$.

Mathematically, it is important to work with $\mathcal{D}$-modules on the stack $\mathrm{Bun}_G$, not on a finite-dimensional moduli space $\mathcal{M}(G)$. Physically, what that means is that it is important that a version of $\mathcal{B}_{\mathrm{cc}}$ can be defined directly in four dimensions, not only after reducing to a two-dimensional description. Kapustin and I had a version of that in our original paper, but D. Gaiotto and I found an improved version a few years later ("Knot Invariants From Four-Dimensional Gauge Theory," 2010).

The brane $\mathcal{B}_{\rm cc}$ descends from a "deformed" version of Neumann boundary conditions (Neumann for gauge fields would be $n^k F_{kl} = 0$, where $n$ is the normal vector to the boundary, extended to other fields in a supersymmetric fashion, and by deformed Neumann I mean that $n^k F_{ik}$ is expressed in terms of other fields, with no normal derivatives). This deformed Neumann boundary condition can be realized in string theory by a D3-NS5 system with a Chan-Paton curvature on the NS5-brane:



This description shows that the dual of $\mathcal{B}_{\rm cc}$ is a boundary condition that comes from the D3-D5 system of string theory (also deformed). That made it possible to answer a question that earlier had been obscure from a physical point of view, though the answer was known mathematically: what is the $B$-model dual of $\mathcal{B}_{\rm cc}$? $\mathcal{B}_{\rm cc}$ is an $A$-brane of $\mathcal{M}_H(G)$ so its dual will be a $B$-brane of $\mathcal{M}_H(G^\vee)$ – in other words, a coherent sheaf, or a complex of coherent sheaves, on $\mathcal{M}_H(G^\vee)$. The $\mathcal{D}$-module corresponding to $\mathcal{B}_{\rm cc}$ itself is the sheaf of differential operators on $\mathcal{M}(G)$, and its dual in view of the results of Beilinson and Drinfeld is the structure sheaf of the variety of opers, a very special Lagrangian submanifold $L_{\rm op} \subset \mathcal{M}_H(G^\vee)$. A gauge theory explanation of that fact comes from the duality between D3-NS5 and D3-D5.

In summary the main ideas in the gauge theory/geometric Langlands correspondence are

- electric-magnetic duality $G \leftrightarrow G^\vee$ of supersymmetric gauge theory in four dimensions
- "twisting" to make a dual pair of topological field theories
- compactification to two dimensions on a Riemann surface $C$
- the dual theories have dual sets of line operators, with 't Hooft operators that are related to geometric Hecke transformations, and their dual Wilson operators
- on one side of the duality there is a distinguished brane $\mathcal{B}_{\mathrm{cc}}$, establishing a map from $A$-branes to $\mathcal{D}$-modules on $\mathrm{Bun}_G$
- on the other side, one is studying $B$-branes on $\mathcal{M}_H(G^\vee)$ in the complex structure in which it parametrizes flat bundles.

This story so far involves deformation quantization, not quantization. Let me pause to underline the difference, with the concrete example of a two-sphere

$$x^2 + y^2 + z^2 = j^2$$

viewed as a symplectic manifold with its usual rotation-invariant symplectic form ($\omega = \mathrm{d}x\mathrm{d}y/z$). In deformation quantization, we start with the commutative algebra of functions $\mathbb{C}[x, y, z]$ and we want to deform it to a noncommutative algebra. In general, one specifies that the leading noncommutative deformation should agree with the Poisson bracket, and asks if a family of associative algebras over $\mathbb{C}[\hbar]$, or possibly $\mathbb{C}[[\hbar]]$, exists with that property. In the present case, we can just write down the answer, which is given by the $\mathfrak{su}(2)$ Lie algebra

$$[x, y] = -\mathrm{i}\hbar z, \quad [y, z] = -\mathrm{i}\hbar x, \quad [z, x] = -\mathrm{i}\hbar y.$$

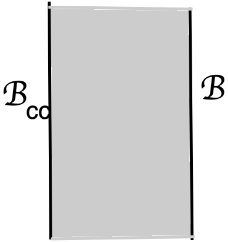
This makes sense for any value of $j^2$ and is *not* quantization. Quantization means finding a Hilbert space $\mathcal{H}$ ("of the appropriate size") that the algebra acts on. This does *not* exist for arbitrary $j$. To construct $\mathcal{H}$, one has to "quantize" the parameter $j$ – set it to preferred values – at which the Hilbert space exists. These special values correspond to the angular momenta in the real world of electrons, atoms, molecules, etc. It is because of this last step that the subject is called "quantization."

S. Gukov and I ("Branes and Quantization," 2008) asked whether the *A*-model in this general setup can describe quantization, and not just deformation quantization. The answer is yes, under certain conditions. Recall that in general, we are discussing a complex symplectic manifold $Y$ (in geometric Langlands, $Y$ is the moduli space of Higgs bundles) viewed as a real symplectic manifold with symplectic form $\omega = \mathrm{Im}\,\Omega$. Let us discuss Lagrangian *A*-branes, that is branes supported on a submanifold $L \subset Y$ that is Lagrangian for $\mathrm{Im}\,\Omega$. In many of the examples that are important in geometric Langlands, $L$ is actually a complex Lagrangian submanifold, that is, it is Lagrangian for $\Omega$, not just for $\mathrm{Im}\,\Omega$. (For example, the dual of a skyscraper sheaf on the $G^\vee$ side – supported on a point in $\mathcal{M}_H(G^\vee)$ – is a Lagrangian brane supported on a fiber of the Hitchin fibration of $Y = \mathcal{M}_H(G)$. This is a complex Lagrangian submanifold. That happened because a point in $\mathcal{M}_H(G^\vee)$ is "hyperholomorphic.") But in general $L$ need not be Lagrangian for $\Omega$ (just as a coherent sheaf on $\mathcal{M}_H(G^\vee)$ need not be hyperholomorphic).

Gukov and I considered the opposite case of an $A$-brane $\mathcal{B}$ whose support $M$ is Lagrangian for $\mathrm{Im}\,\Omega$ – as it must be – but is *symplectic* for $\mathrm{Re}\,\Omega$. (Our work was stimulated by a number of prior contributions, including computations by Aldi and Zaslow (2005) in illustrative special cases.) We argued that in this case $\mathcal{H} = \mathrm{Hom}(\mathcal{B}, \mathcal{B}_{\mathrm{cc}})$ represents a quantization of $M$ with symplectic structure $\mathrm{Re}\,\Omega$. For example, if $Y$ is an affine variety, then the holomorphic functions on $Y$, which are deformation quantized to get $\mathcal{A} = \mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$ can be regarded as analytic continuations of certain real analytic functions on $M$. There are enough of them to be a reasonable quantum-deformed algebra of observables. $\mathcal{H} = \mathrm{Hom}(\mathcal{B}, \mathcal{B}_{\mathrm{cc}})$ is an $\mathcal{A}$-module which one can show has appropriate properties to be a quantization of $M$. A formal argument reduces the description of $\mathcal{H}$ to the problem of quantizing $M$.

Suppose that $\mathcal{B}$ is a rank 1 $A$-brane whose support $M$ is Lagrangian for $\mathrm{Im}\,\Omega$ and symplectic for $\mathrm{Re}\,\Omega$. The Chan-Paton bundle of $\mathcal{B}$ will have to be a unitary line bundle with curvature $F$, such that $F + B|_M = 0$. Recall that we chose $B = \omega_J$. So we need $F = -\omega_J$. A prequantum line bundle $\mathcal{L}$, in the sense of geometric quantization, for quantization of $M$ with symplectic structure $\omega_J$ is supposed to be a unitary line bundle of curvature $+\omega_J$. So we can take $\mathcal{L}^{-1}$ for the Chan-Paton bundle of $\mathcal{B}$. Let me write $T$ for the unitary connection on $\mathcal{L}$.

Gukov and I showed that for the $(\mathcal{B}_{\mathrm{cc}}, \mathcal{B})$ strings




are governed by an effective action

$$I = -\mathrm{i} \int x^*(T) + \{Q, \cdots\}.$$

So, as I tried to explain yesterday, we can define $\mathcal{H} = \mathrm{Hom}(\mathcal{B}, \mathcal{B}_{\mathrm{cc}})$ as a quantization of $M$ with the prequantum line bundle $\mathcal{L}$, and it will automatically admit an action of the quantum-deformed algebra of functions $\mathcal{A} = \mathrm{Hom}(\mathcal{B}_{\mathrm{cc}}, \mathcal{B}_{\mathrm{cc}})$.

I have actually omitted so far a key point: what structure is needed so that $\mathcal{H} = \mathrm{Hom}(\mathcal{B}, \mathcal{B}_{\mathrm{cc}})$ is a Hilbert space, not just a vector space? In general, if $\mathcal{B}_1, \mathcal{B}_2$ are any two branes, there is a nondegenerate bilinear (not hermitian) pairing $\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2) \otimes \mathrm{Hom}(\mathcal{B}_2, \mathcal{B}_1) \to \mathbb{C}$. To get a hermitian inner product on $\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)$ is therefore the same as finding an antilinear map from $\mathrm{Hom}(B_2, \mathcal{B}_1)$ to $\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)$. Gukov and I showed that there is such an antilinear map, which I will call $\Theta_\tau$, if there is an antiholomorphic map $\tau : Y \to Y$ with $M$ as a component of its fixed point set. I won't explain the definition of $\Theta_\tau$ but I will explain why one should expect the existence of such a $\tau$ to be the right criterion. If holomorphic functions on $Y$ act on $\mathcal{H}$ and $\mathcal{H}$ is a Hilbert space, one will have a notion of which holomorphic functions on $Y$ are hermitian. Existence of $\tau$ gives a natural notion: a holomorphic function on $Y$ is "real" if it is real when restricted to $M$.

Given $\tau$, the hermitian product on $\mathrm{Hom}(\mathcal{B}_1, \mathcal{B}_2)$ for any pair of $\tau$-invariant branes is defined by

$$\langle \psi, \chi \rangle = (\Theta_\tau \psi, \chi)$$

where $\Theta_\tau$ is the antillinear mapping that is defined using $\tau$, and $(\ ,\ )$ is the bilinear pairing of the *A*-model. This pairing is always nondegenerate but in this generality it is not always positive definite. For the more specific case relevant to quantization, the best we can say is that the pairing on $\mathrm{Hom}(\mathcal{B}, \mathcal{B}_{\mathrm{cc}})$ is positive definite if one is sufficiently close to a classical limit.

In this approach to quantization, if one is given a real symplectic manifold $M$ that one wants to quantize, one has to find a complexification of $M$ to a complex symplectic manifold $Y$ with some appropriate properties, and then one can use the $A$-model of $Y$ to quantize $M$. In particular, $Y$ should have (1) an antiholomorphic involution $\tau$ with $M$ as a component of its fixed point set, and (2) a well-defined $A$-model, not just in the sense of formal power series. The known sufficient condition for (2) is that $Y$ should have a complete hyper-Kahler metric.

Quantization by branes might be compared loosely to geometric quantization, which is the closest there has been to a systematic approach to quantization. In geometric quantization, to quantize $M$, one has to find a "polarization" of $M$ (roughly a maximal set of Poisson-commuting variables). Given a polarization, geometric quantization gives a recipe of quantization. Geometric quantization – or quantization by branes – can give good results if there is a natural polarization – or a natural choice of complexification – for the problem at hand.

A limitation of my paper with Gukov is that we did not understand very much about how to compare brane quantization to geometric quantization. Gaiotto and I were motivated in the last few months to look at this more closely as background to understanding the work of Etinghof, Frenkel, and Kazhdan that I mentioned at the beginning. We were able to find criteria under which geometric quantization and quantization by branes will agree. The basic idea is that they agree if the polarization used in geometric quantization can be analytically continued to a holomorphic polarization of the complexification that is used in brane quantization. This paper will be on the arXiv in a few days.

What I have said today will hopefully serve as useful background when we discuss the work of Etinghof, Frenkel, and Kazhdan (EFK) on Tuesday.