Structured beam propagation through atomic media

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In this tutorial, we explore the propagation of Laguerre-Gaussian (LG) beam through atomic medium. Linear and non-linear atom-field interaction used to control polarization, phase and amplitude of beam, gives complete freedom to manipulate structured light. Therefore, the mechanism of efficient control of beam parameter can open up new avenues for high-resolution microscopy, high-density optical communication, realization of optical tweezers for controlled manipulation and trapping of particles, and in micromachining.

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I. INTRODUCTION

Interactions of electromagnetic(em) fields with atomic medium have been studied in great detail in the past few decades[1–4]. The ability to control the optical properties of an atomic medium using coherent em fields has attracted much attention of the researchers [5, 6]. The formation of atomic coherences holds the key of control over the optical properties of the medium. Generally atoms are characterized by different atomic states and coupling of em fields to these states leads to the creation of atomic coherence. This induced coherence not only changes the optical properties of the medium but also affects the interacting em fields. Moreover controlling atomic coherences using two coherent em fields can result in interesting coherence effects which have immense application in many areas of optics.

Interaction of two coherent em fields with an atomic medium is studied using a three level configuration, where both the fields act on different transitions but share a common atomic excited state. These fields create atomic coherences between different transition pathways of the three level configuration which can interfere with each other. The interference among the atomic coherences lead to exciting quantum optical phenomena such as coherent population trapping (CPT) [2, 7], electromagnetically induced transparency (EIT) [8–10], lasing without inversion (LWI) [11–13] and saturated absorption techniques (SAT) [14, 15]. Among the above quantum effects CPT was the first novel phenomena to be observed. CPT was first experimentally observed by Alzetta *et. al.,* in sodium vapor in a three level atomic configuration [16]. In this phenomena the atoms are prepared in coherent superposition of states which is immune to absorption of radiation *i.e.,* the atoms get trapped in the superposed state and cannot be excited to any other state [7]. In CPT both the fields are of comparable strength whereas one of the field becomes weak in case of EIT [9, 10, 17]. EIT is a technique for making an otherwise optically thick gaseous medium transparent to a resonant weak(probe) field by applying another strong(control) field. This induced transparency is a result of destructive interference between two excitation paths of a weak probe field interacting with the three level system. The reduced absorption of the probe field near the atomic resonance is also accompanied by a rapidly varying atomic dispersion. In 1990, Harris et al. [9] first proposed the EIT phenomena in a three level atomic system. The first experimental observation of EIT was reported by Boller *et. al.,*

[10] in 1991. Since then EIT has become a versatile tool for creating and manipulating various optical properties of the medium as well as that of em fields.

In the recent years EIT in three level atomic system has shown a very rich variety of applications in many areas starting from information processing to spectroscopy. Modification of three level systems to four level atomic systems or closed loop systems can result in new type of quantum interference effects such as Double dark resonance (DDR) [18, 19] and phase-dependent EIT (PDEIT) [20]. DDR has been demonstrated in a four-level system, where the probe absorption spectrum is characterized by two EIT windows, separated by a sharp absorption peak. The appearance of the central narrow structure is due to the coherent interaction between the two dark states, which greatly enhances the Kerr nonlinearity [21, 22]. Modification of the medium susceptibility by DDR also has many applications in imaging [23, 24], optical switching [25], and high precision atom localization [26]. Further, relative phase difference of the em fields can dramatically modify the EIT features in a closed-loop atomic system and create PDEIT. PDEIT is an effective tool in achieving high precision atom localization [27], frequency conversion [8], generation of structured beams [28, 29] etc. In general, open and closed-loop multi-level atomic systems allow us to manipulate both the medium and the em fields in a more sophisticated manner than in conventional EIT systems [21, 26, 27] and can have greater applications in various areas of optics.

The beams carrying singularities in phase has emerged as a topic of intensive study in the classical as well quantum domain [30, 31]. The azimuthal varying phase structure $e^{il\phi}$ of the beam corresponds to the origin of the orbital angular momentum [32]. Various methods such as cylindrical lens pairs [33], computer generated hologram [34], spatial light modulator [35] etc., have been utilized to produce phase singularity. Light beam possessing vortex singularity has broad applications in optical communication [36, 37], super-resolution imaging [38, 39], optical tweezers [40–42], nonlinear phenomena [43–45], etc. Several systems including multi-core supermode optical fiber [46], photonic crystal [46, 47] and atomic vapor media [48–50] are used for singularity based applications. Specially, nonlinear optical medium has been recognized as an excellent system for studying the generation, conversion and manipulation of vortex singularity because of its highly adaptable absorptive, dispersive and diffractive properties [51–53].

II. MAXWELL'S EQUATION

Light is an electromagnetic wave (em) consisting of oscillating electric and magnetic field vectors. The propagation of light (em) through an optical medium is governed by four fundamental Maxwell's equations, which in Gaussian units can be written as,

$$\vec{\nabla} \cdot \vec{D} = 4\pi\varrho, \qquad \text{(Gauss's Law)} \tag{1a}$$

$$\vec{\nabla} \cdot \vec{B} = 0, \tag{1b}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \qquad \text{(Faraday's Law)} \tag{1c}$$

$$\vec{\nabla} \times \vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{D}}{\partial t}, \quad \text{(Ampere's Law).} \tag{1d}$$

Here $\vec{E}$, $\vec{H}$ are the time-averaged values of the basic electric and magnetic field vectors at some space-time point $(\vec{r}, t)$, rather than the instantaneous values and c is the velocity of light in free space [54]. The electric displacement $\vec{D}$ and the magnetic induction $\vec{B}$ have "additive relations", coming from the interaction of matter and field,

$$\vec{D} = \vec{E} + 4\pi\vec{\mathcal{P}}, \tag{2a}$$

$$\vec{B} = \vec{H} + 4\pi\vec{\mathcal{M}}. \tag{2b}$$

Here $\vec{\mathcal{P}}$ and $\vec{\mathcal{M}}$ are the *electric* and *magnetic polarizations* respectively. In Eq. (2a) the contribution from the multipole moments (such as electric quadrapole moments) has been neglected, since at optical frequency the electric dipole moment is the most dominant quantity. In free space, both $\vec{\mathcal{P}}$ and $\vec{\mathcal{M}}$ vanish, hence, these quantities represent the influence of matter on the field. In this tutorial, the system under consideration is *non magnetic* ($\vec{\mathcal{M}} = 0$), and *non conducting* ($\vec{J} = 0$), together with no free charge ($\varrho = 0$). Applying the curl operator in Eq. (1c), taking appropriate time derivatives and using the constitutive relation for $\vec{B}$, we obtain

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(\vec{E} + 4\pi\vec{\mathcal{P}}) = 0. \tag{3}$$

Since $\vec{D} = \vec{E} + 4\pi\vec{P}$, for a charge free isotropic medium, $\vec{\nabla} \cdot \vec{D} = 0$, and thus $\vec{\nabla} \cdot \vec{E} = 0$. Therefore, the simplified wave equation can be written as

$$\vec{\nabla}^2\vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2 \vec{\mathcal{P}}}{\partial t^2}. \tag{4}$$

This equation has the form of an inhomogeneous wave equation. The source term which appears on the right-hand side of this equation represents the nonlinear response of the medium. In the absence of the source term, the above Eq.(4) reduces to

$$\vec{\nabla}^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0. \tag{5}$$

In the optical wavelength limit, the variations of the field along the transverse directions is small. Therefore the general solution of this equation can be written in the form,

$$\vec{E} = \vec{E}_1(z - ct) + \vec{E}_2(z + ct), \tag{6}$$

where $\vec{E}_1$ and $\vec{E}_2$ are arbitrary functions. We see that the argument of $\vec{E}_1$ is unchanged when (z, t) is replaced by $(z + ct, t + \tau)$, where τ is an arbitrary parameter. Hence $\vec{E}_1(\vec{E}_2)$ represents a field which is propagating with velocity c in the positive (negative) z-direction.

III. VECTOR AND SCALAR POTENTIALS

The Maxwell equations as described in Eq.(1) consist of a set of coupled first-order partial differential equations relating the various components of electric and magnetic fields. In absence of source term of Eq.(1a), first two Eqs.(1a) and Eqs.(1b) are known to be homogeneous equations and are decoupled. Last two equations Eqs.(1c) and Eqs.(1d) are coupled equations. They can be solved as they stand in simple situations. But it is often convenient to introduce scalar potential Φ and the vector potential $\vec{A}$ that can decouple the Faraday and Ampere equations. Since $\nabla \cdot \vec{B} = 0$, from vector identity we can define $\vec{B}$ in terms of a vector potential:

$$\vec{B} = \nabla \times \vec{A} \tag{7}$$

Then the other homogeneous equation in Eqs.(1c), Faraday's law, can be written

$$\nabla \times \left(\vec{E} + \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\right) = 0 \tag{8}$$

This means that the quantity with vanishing curl in Eq.(8) can be written as the gradient of some scalar function, namely, a scalar potential Φ:

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \tag{9}$$

The definition of $\vec{B}$ and $\vec{E}$ in terms of the potentials $\vec{A}$ and Φ according to Eq.(7) and Eq.(9) satisfies identically the two homogeneous Maxwell equations. The dynamic behavior of $\vec{A}$ and Φ can be used for determining the two inhomogeneous equations as stated in Eqs.(1c) and Eqs.(1d). We consider the propagation of electromagnetic radiation in vacuum, therefore the inhomogeneous equations in Eqs.(1c) and Eqs.(1d) can be caste into the following form

$$\left(\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}\right) = 0 \tag{10a}$$

$$\left(\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}\right) - \nabla\left(\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right) = 0 \tag{10b}$$

We have now reduced the set of four Maxwell equations to two equations. But they are still coupled equations. The uncoupling can be accomplished by exploiting the arbitrariness involved in the definition of the potentials. Since $\vec{B}$ is defined through Eq.(7) in terms of $\vec{A}$, the vector potential is arbitrary to the extent that the gradient of some scalar function Λ can be added. Thus $\vec{B}$ and $\vec{E}$ remain unchanged by the transformation,

$$\vec{A} \to \vec{A}' = \vec{A} + \nabla\Lambda, \quad \text{and} \quad \Phi \to \Phi' = \Phi - \frac{\partial\Lambda}{\partial t} \tag{11}$$

The freedom implied in the above Eq.(12) means that we can choose a set of scaler Φ and vector potentials $\vec{A}$ to satisfy the Lorenz condition,

$$\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t} = 0 \tag{12}$$

The above Lorentz gauge condition will uncouple the pair of equations Eqs.(10a) and (10b) and leave two inhomogeneous wave equations, one for scaler potential Φ and one for vector potentials $\vec{A}$.

IV. LINEAR AND ANGULAR MOMENTUM OF LIGHT

The wave equation for the vector potential $\vec{A}$ is expressed as

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0 \tag{13}$$

Let us assume that the solution for Eq.(13) can be written as

$$\vec{A} = \hat{x}\; u\,(x, y, z)\; e^{ikz - i\omega t} \tag{14}$$

With the help of Eq.(7) and taking into account the paraxial approximation, the magnetic field $\vec{B}$ can be read as

$$\vec{B} = ik\left(\hat{y}\,u + \frac{i}{k}\frac{\partial u}{\partial y}\hat{z}\right)e^{ikz-i\omega t} \tag{15}$$

From Ampere's law, the electric field expression can be obtained under the paraxial approximation as follow:

$$\vec{E} = ik\left(\hat{x}\,u + \frac{i}{k}\frac{\partial u}{\partial x}\hat{z}\right)e^{ikz-i\omega t} \tag{16}$$

We should note that there is an axial, z , dependent component to both the electric $\vec{E}$ and magnetic $\vec{B}$ fields, as there is for the TEM modes of a real laser. Let us evaluate the real part of the time averaged Poynting vector

$$\begin{aligned}\langle\vec{S}\rangle =& \tfrac{c}{4\pi}\langle\vec{E}\times\vec{B}\rangle = \tfrac{c}{8\pi}(\vec{E}^*\times\vec{B}+\vec{E}\times\vec{B}^*)\\ =& \quad \tfrac{ick}{8\pi}(u\nabla u^* - u^*\nabla u) + \tfrac{ck^2}{4\pi}|u|^2\hat{z}.\end{aligned} \tag{17}$$

In the case of cylindrical symmetry field of the form

$$u(r,\phi,z) = u_0(r,z)\,e^{-il\phi}\;e^{-\frac{ikr^2z}{2(z^2+z_R^2)}}\;e^{i(2p+l+1)tan^{-1}\left(\frac{z}{z_R}\right)} \tag{18}$$

which obeys the conditions of the paraxial approximation, it is easy to derive from eq. (17) that for a LG beam $\langle\vec{S}\rangle$ possesses nonzero component along the $\hat{r},\ \hat{\phi},\ \hat{z}$ direction

$$\langle\vec{S}\rangle = \frac{ck}{8\pi}\left(\frac{2krz}{z^2+z_R^2}|u|^2\hat{r} + \frac{2l}{r}|u|^2\hat{\phi}\right) + \frac{ck^2}{4\pi}|u|^2\hat{z}. \tag{19}$$

The component $\left\langle\vec{S}\,(r,\ \varphi,\ z)\right\rangle_r$ relates to the spread of the beam as it propagates, whereas $\left\langle\vec{S}\,(r,\ \varphi,\ z)\right\rangle_\phi$ gives rise to orbital angular momentum in the ϕ-direction. The linear momentum in the direction of propagation is denoted by $\left\langle\vec{S}\,(r,\ \varphi,\ z)\right\rangle_z$.

Problem 1. The cycle-average theorem: Show that if $\mathcal{A}$ and $\mathcal{B}$ are two complex quantities which both vary with time as $e^{\imath\omega t}$ then the average over a cycle of the oscillation of the product of the real parts of $\mathcal{A}$ and $\mathcal{B}$ is given by

$$\overline{(\mathrm{Re}\ \mathcal{A}) \times (\mathrm{Re}\ \mathcal{B})} = \frac{1}{2}(\mathrm{Re}\ \mathcal{A}\mathcal{B}^*) \tag{20}$$

where the star denotes complex conjugation.

Problem 2. In free space, an electric field is given by $\mathbf{E} = \hat{x}E_0\left[\cos\left(k\,z - \omega\,t\right) + \cos\left(k\,z + \omega\,t\right)\right]$.

(a) Find the magnetic field $\mathbf{B}$.

(b) Determine the Poynting vector and its time average over one cycle.

(c) Find the time averged electric and magnetic energy densities.

(d) Does this field represent a wave? If so, what kind of wave does it represent ?

Problem 3.The radiation field in an empty cubic cavity of side L satisfies the wave equation $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A} = 0$ together with the Coulomb gauge condition $\nabla \cdot \mathbf{A} = 0$. Show that the solution that satisfies the boundary conditions has components

$$\begin{aligned} A_x(\mathbf{r},t) &= A_x(t)\cos(k_x x)\sin(k_y y)\sin(k_z z), \\ A_y(\mathbf{r},t) &= A_y(t)\sin(k_x x)\cos(k_y y)\sin(k_z z), \\ A_z(\mathbf{r},t) &= A_z(t)\sin(k_x x)\sin(k_y y)\cos(k_z z), \end{aligned} \tag{21}$$

where $\vec{A}(t)$ is independent of position and the wave vector $\mathbf{k} = ik_x + jk_y + kk_z$ has components $k_{x_i} = n_{x_i}\pi x_i/L$. Hence show that the integers n_x, n_y, n_z are restricted in that only one of them can be zero at a time.

Problem 4. The angular momentum of a distribution of electromagnetic fields in vacuum is given by

$$\vec{L} = \frac{1}{\mu_0 c^2}\int d^3x\ \ \vec{x} \times (\vec{E} \times \vec{B})$$

where the integration is over all space.

(a) For fields produced a finite time in the past (and so localized to a finite region of space) show that provided the magnetic field is eliminated in favor of the vector potential **A**, the angular momentum can be written in the form

$$\mathbf{L} = \frac{1}{\mu_0 c^2} \int d^3x \left[\mathbf{E} \times \mathbf{A} + \sum_{j=1}^{3} E_j (\mathbf{x} \times \nabla) A_j \right]$$

The first term is sometimes identified with the "spin" of the photon and the second with its "orbital" angular momentum because of the presence of the angular momentum operator $L_{op} = -i(x \times \nabla)$.

(b) Consider an expansion of the vector potential in the radiation gauge in terms of plane waves:

$$\vec{A}(\vec{x}, t) = \sum_{\lambda} \int \frac{d^3k}{(2\pi)^3} \left[\boldsymbol{\epsilon}_\lambda(\vec{k}) a_\lambda(\vec{k}) e^{i\vec{k}\cdot\vec{x} - \imath\omega t} + c.c. \right]$$

The polarization vectors $\boldsymbol{\epsilon}_\lambda(\vec{k})$ are conveniently chosen as the positive and negative helicity vectors $\boldsymbol{\epsilon}_\pm = (1/\sqrt{2})(\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2)$ where $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ are real orthogonal vectors in the $x-y$ plane whose positive normal is in the direction of $\vec{k}$.
Show that the time average of the first (spin) term of $\vec{L}$ can be written as

$$\vec{L}_{spin} = \frac{2}{\mu_0 c} \int \frac{d^3k}{(2\pi)^3} \vec{k} [|a_+(\vec{k})|^2 - |a_-(\vec{k})|^2]$$

Can the term "spin" angular momentum be justified from this expression? Calculate the energy of the field in terms of the plane wave expansion of **A** and compare.

Problem 5. A circularly polarized plane wave moving in the z direction has a finite extent in the x and y directions. Assuming that the amplitude modulation is slowly varying (the wave is many wavelengths broad), show that the electric and magnetic fields are given by

$$\vec{E}(x, y, z, t) \simeq \left[E_0(x, y)(\vec{e}_1 \pm i\vec{e}_2) + \frac{i}{k} \left(\frac{\partial E_0}{\partial x} \pm i \frac{\partial E_0}{\partial y} \right) \vec{e}_3 \right] e^{ikz - i\omega t}$$

$$\vec{B} \simeq \mp i\sqrt{\mu\epsilon}\, \vec{E}$$

where $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ are unit vectors in the x, y, z directions.

Problem 6. For the circularly polarized wave of Problem 4 with $E_0(x, y)$ a real function of x and y. calculate the time-averaged component of angular momentum parallel to the direction of propagation. Show that the ratio of this component of angular momentum to the energy of the wave is

$$\frac{L_3}{U} = \pm\omega^{-1}$$

Interpret this result in terms of quanta of radiation (photons). Show that for a cylindrically symmetric, finite plane wave, the transverse components of angular momentum vanishes.

V. BASIC BEAM PROPAGATION EQUATION

The light beam is confined to regions near the axis along which it propagates. Such as in the case of a propagating laser beam, the Helmholtz equation takes a much simpler form, called the paraxial wave equation. A solution to the paraxial wave equation is the fundamental (or TEM_{00}) Gaussian beam. Description of propagation of such a beam through simple optical systems, such as a transparent homogeneous medium, an interface of two transparent media, a thin lens, etc. can be obtained algebraically without having to solve a differential equation or to evaluate an integral at every step of the way. In this section we will first derive the paraxial wave equation, then obtain the fundamental solution to it and then describe the method used for paraxial propagation. The energy flow is primarily along one direction (our chosen z-direction), therefore the linearly-polarized electric field propagating in the z-direction can be described by

$$\vec{E}(\vec{r}, t) = \hat{e}\mathcal{E}_0(x, y, z)e^{-i(\omega t - kz)} + \text{c.}c. \tag{22}$$

where $\hat{e}$ is the direction of polarization, ω is the central angular frequency of the field, and the wave number, $k = \omega/c$. The complex conjugate (c.c.) term has been added to the electric field expression to make it real. The Laplacian ∇^2 can be written as

$$\nabla^2 = \nabla_\perp^2 + \partial^2/\partial z^2 \tag{23}$$

where $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ represents the spatial derivative in transverse direction. Now the spatial and temporal derivation of the electric field are given as

$$\nabla^2 \vec{E} = \hat{e}\left(\nabla_\perp^2 \mathcal{E}_0 + \frac{\partial^2 \mathcal{E}_0}{\partial z^2} + 2ik\frac{\partial \mathcal{E}_0}{\partial z} - k^2 \mathcal{E}_0\right) e^{i(kz-\omega t)} + \text{c.c.}, \tag{24a}$$

$$\frac{\partial^2 \vec{E}}{\partial t^2} = \hat{e}\left(-\omega^2 \mathcal{E}_0\right) e^{i(kz-\omega t)} + \text{c.c.}, \tag{24b}$$

Now substituting Eqs. (24a), and (24b) into Eq. (5), and canceling out the common phase term, $e^{i(kz-\omega t)}$, we get

$$\nabla_\perp^2 \mathcal{E} + 2ik\frac{\partial \mathcal{E}}{\partial \zeta} = 0. \tag{25}$$

In the above derivation, we assume that the longitudinal variations of the envelope function $\mathcal{E}_0$ are such that following inequality holds good

$$\left|k\frac{\partial \mathcal{E}_0}{\partial z}\right| \gg \left|\frac{\partial^2 \mathcal{E}_0}{\partial^2 z}\right| \tag{26}$$

Equation (25) is also commonly referred to as the paraxial wave equation. For a radially symmetric beam, the cylindrical coordinates are convenient because the dependence of $\mathcal{E}_0$ on the azimuthal angle ϕ drops off and $\mathcal{E}_0$ becomes a function of only two variables, r and z. In cylindrical coordinates, (r, ϕ, z), the Laplacian takes the form

$$\frac{\partial^2 \mathcal{E}_0}{\partial r^2} + \frac{1}{r}\frac{\partial \mathcal{E}_0}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \mathcal{E}_0}{\partial \phi^2} + 2ik\frac{\partial \mathcal{E}_0}{\partial z} = 0 \tag{27}$$

VI. LAGUERRE-GAUSSIAN (LG) BEAMS

In cylindrical coordinates (r, ϕ, z), the paraxial wave equation is as described in Eq. (27). General solutions that are functions of the cylindrical coordinates, expressed in terms of Laguerre polynomials, were obtained in the early work by Kogelnik and Li, [5]. Here the detailed derivation leading to the solution, which are not easy to find, are presented leading to the LG beam expression using the procedure outlined in Ref. [5]. A Laguerre Gaussian beam with angular mode number l has been shown to possess orbital angular momentum of amount $l\hbar$ per photon [6]. Following Ref [5] we assume that the solution for Eq. (27) can be written as

$$\mathcal{E}_0(r, \phi, z) = \xi^{\frac{l}{2}} L(\xi)\, e^{i\left[P(z) + \frac{kr^2}{2q(z)} + l\phi\right]} = \xi^{\frac{l}{2}} L(\xi)\, F \tag{28}$$

where $\xi = 2r^2/w^2(z)$ and $P(z)$, $q(z)$ and $w(z)$ are to be determined very shortly. After lengthy but straightforward calculations, we obtain transverse variation of the field $\mathcal{E}_0$

$$\nabla_\perp^2 \mathcal{E}_0 = \frac{8\xi^{\frac{l}{2}}}{w^2(z)} \left(\xi \frac{\partial^2 L}{\partial \xi^2} + (1 + l + 2im_1\xi) \frac{\partial L}{\partial \xi} + \{im_1(l+1) - m_1^2\xi\} L \right) F \tag{29}$$

where the parameter $m_1(z) = kw^2/4q$. The differentiation with respect to z can produce following expression

$$\frac{\partial \mathcal{E}_0}{\partial z} = \xi^{\frac{l}{2}} \left[i\frac{\partial P}{\partial z} L - i\frac{kw^2\xi}{4q^2} \frac{\partial q}{\partial z} L - \frac{l}{w} \frac{\partial w}{\partial z} L - \frac{2\xi w'}{w} \frac{\partial L}{\partial \xi} \right] F \tag{30}$$

The paraxial wave equation, *i.e.,* Eq. (25), then can be expressed as

$$\begin{aligned} &\left[\xi \tfrac{\partial^2 L}{\partial \xi^2} + (1 + l + 2im_1\xi) \tfrac{\partial L}{\partial \xi} + \{im_1(l+1) - m_1^2\xi\} L \right] = \\ &\quad - \tfrac{2ikw^2}{8} \left[i\tfrac{\partial P}{\partial z} L - i\tfrac{kw^2\xi}{4q^2} \tfrac{\partial q}{\partial z} L - \tfrac{lL}{w} \tfrac{\partial w}{\partial \xi} - \tfrac{2\xi}{w} \tfrac{\partial w}{\partial \xi} \tfrac{\partial G}{\partial \xi} \right] \end{aligned} \tag{31}$$

In case of TEM_{00}, near the axis of propagation, we can ignore $\partial^2 L/\partial \xi^2$ and $\partial L/\partial \xi$ and equating the terms with same power of ξ, we obtain

$$-m_1^2 \xi L = -\left(\frac{kw^2}{4q} \right)^2 \frac{dq}{dz} \xi L \tag{32}$$

The above equation is equivalent provided first-order differential equations of $dq/dz = 1$ because paraxial Helmholtz equation should not diverge at $r \to \infty$. Analogous to the Gaussian beam solution, the form of $1/q(z)$ can be taken as

$$\frac{1}{q(z)} = \frac{1}{R(z)} + \frac{i\lambda}{\pi w^2} \tag{33a}$$

$$\left(\frac{1}{q(z)} \right)^2 = \left[\left(\frac{1}{R} \right)^2 - \left(\frac{\lambda}{\pi w^2} \right)^2 \right] + \frac{2i\lambda}{w^2 R} \tag{33b}$$

$$\frac{d}{dz} \left(\frac{1}{q(z)} \right) = \frac{1}{R(z)} \frac{dR}{dz} + \frac{2i\lambda}{\pi w^3} \frac{dw}{dz} \tag{33c}$$

$$-\frac{d}{dz} \left(\frac{1}{q(z)} \right) = \frac{dq/dz}{q^2} = \frac{1}{q^2} = \left[\left(\frac{1}{R} \right)^2 - \left(\frac{\lambda}{\pi w^2} \right)^2 \right] + \frac{2i\lambda}{w^2 R} \tag{33d}$$

From the above equation, the real and imaginary parts

$$R(z) = 1 - \left(\frac{\lambda R}{\pi w^2} \right)^2, \qquad \frac{dw}{dz} = \frac{w}{R} \tag{34}$$

The solutions for $P(z)$ and $L(\xi)$ are still to be derived. Let us consider the expression $m_1(z) = kw^2/4q$ and insert it in Eq. (31) and rearrange the terms so that

$$\left[\xi \frac{\partial^2 L}{\partial \xi^2} + (1 + l + 2im_1\xi) \frac{\partial L}{\partial \xi} + im_1(l+1) L \right] = -\frac{2ikw^2}{8} \left[i\frac{\partial P}{\partial z} L - \frac{lL}{w} \frac{\partial w}{\partial \xi} - \frac{2\xi}{w} \frac{\partial w}{\partial \xi} \frac{\partial L}{\partial \xi} \right], \tag{35}$$

where we have used the relation Eq. (34). Using separation of variables, we show that the expression on the left side of Eq. (35) is proportional to L and the proportionality constant is $-p$

$$\left[\xi\frac{\partial^2 L}{\partial\xi^2}+(1+l-\xi)\frac{\partial L}{\partial\xi}\right]=-pL \tag{36a}$$

$$\frac{kw^2}{4}\left[\frac{\partial P}{\partial z}L-i\frac{\xi}{R}\frac{\partial L}{\partial\xi}\right]-i\frac{kw^2}{4}L+\frac{l+1}{2}L=-pL \tag{36b}$$

We ignore the $\partial L/\partial\xi$ term as being small near the axis of propagation, Eq.(36b) can be solved with the explicit expressions for $w(z)$ and $R(z)$ obtained from Eq.(34) and it can be expressed as

$$e^{iP(z)}=\frac{w(0)}{w(z)}e^{i(1+l+2p)tan^{-1}[\frac{z}{z_R}]} \tag{37}$$

The parameters $|l|$ and p are integers 0, 1, 2,$\cdots$. The Eq. (36a) is identical with the differential equation satisfied by the Laguerre [55] polynomials L_p^l under the assumption that l and p are nonnegative integer values. Thus the full solution to the paraxial wave equation in cylindrical coordinates can be written as

$$\mathcal{E}_{0p}^{l}(r,\phi,z)=\mathcal{E}_0\frac{w(0)}{w(z)}\left(\frac{\sqrt{2}r}{w(z)}\right)^{l}L_p^l(\xi)\,e^{-\frac{r^2}{w^2(z)}-i\frac{kr^2}{2R(z)}-il\phi+(2p+l+1)tan^{-1}\frac{z}{z_R}} \tag{38}$$

where $|l|$ and p are integers 0, 1, 2, etc. This matches the expression found in [56] for a Laguerre-Gaussian beam. l and p are referred to as the angular and radial mode numbers of the beam.

VII. INTERACTION OF RADIATION WITH MATTER

The light field (em) is treated classically by Maxwell's equations while the atom is considered to have quantized energy levels and is treated by the Schrödinger equation. For simplicity, the atom is assumed to have a single electron of charge e and mass m interacting with an external electromagnetic field. The interaction between atom and field is described by the following Hamiltonian

$$\mathcal{H}=\frac{[\vec{P}-e\vec{A}(\vec{r},t)]^2}{2m}+e\Phi(\vec{r},t)+V(r), \tag{39}$$

where $\vec{P}$ is the momentum of the electron, $\vec{A}(\vec{r},t)$ and $\Phi(\vec{r},t)$ are the vector and scalar potentials of the external field respectively. Here $V(r)$ is a central potential experienced by

the bound electron due to the presence of motionless nucleus. Quantization of the electron motion can be done by replacing the classical variable with operators, *e.g.*,

$$\vec{P} \longrightarrow -i\hbar\vec{\nabla}, \mathcal{H} \longrightarrow i\hbar\partial/\partial t. \tag{40}$$

Here $\hbar = h/2\pi$, where h is the Planck's constant. Therefore, the motion of electron is described by the Schrödinger equation

$$\begin{aligned} i\hbar\frac{\partial|\Psi(\vec{r},t)\rangle}{\partial t} &= \left\{\frac{[-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}(\vec{r},t)]^2}{2m} + V(r) + e\Phi(\vec{r},t)\right\}|\Psi(\vec{r},t)\rangle \\ &= (\mathcal{H}_{\mathcal{O}} + \mathcal{H}_{\mathcal{I}})|\Psi(\vec{r},t)\rangle, \end{aligned} \tag{41}$$

where the unperturbed Hamiltonian is given by

$$\mathcal{H}_{\mathcal{O}} = -\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(r) + e\Phi \tag{42}$$

and the interaction Hamiltonian involves only the vector potential $\vec{A}$:

$$\mathcal{H}_{\mathcal{I}} = \frac{e}{2mc}\left[2i\hbar\vec{A}(\vec{r},t)\cdot\vec{\nabla} + i\hbar\vec{\nabla}\cdot\vec{A}(\vec{r},t)\right] + \frac{e^2}{2mc^2}\vec{A}(\vec{r},t)\cdot\vec{A}(\vec{r},t) \tag{43}$$

In passing we note that, the transformations $\vec{A} \longrightarrow \vec{A'} = \vec{A} + \frac{\hbar}{e}\vec{\nabla}\chi$ and $\Phi \longrightarrow \Phi' = \Phi - \frac{\hbar}{e}\frac{\partial\chi}{\partial t}$, leave the $\vec{E}$ and $\vec{B}$ as invariant quantities which are thus gauge independent. Here, χ is any arbitrary scalar function. This allows one to choose a suitable gauge to simplify a given problem. Here we are working in the radiation gauge in which $\Phi(\vec{r},t) = 0$ and $\vec{\nabla}\cdot\vec{A} = 0$. Under the radiation gauge condition, the interaction Hamiltonian becomes

$$\mathcal{H}_{\mathcal{I}} = \frac{ie\hbar}{mc}\vec{A}(\vec{r},t)\cdot\vec{\nabla} + \frac{e^2}{2mc^2}\vec{A}(\vec{r},t)\cdot\vec{A}(\vec{r},t) \tag{44}$$

The dipole moment approximation is often used in quantum optics, which simplifies the interaction Hamiltonian term [57]. This approximation assumes that the whole atom is submerged in a plane em wave described by a vector potential, $\vec{A}(\vec{r}+\vec{r}_0,t)$, which is assumed to have no spatial variation in the vicinity of the atom whose nucleus is located at $\vec{r}_0$. For such a case,

$$\begin{aligned} \vec{A}(\vec{r}+\vec{r}_0,t) &= \vec{A}(t)exp\left[i\vec{k}\cdot(\vec{r}+\vec{r_0})\right] \\ &= \vec{A}(t)exp(i\vec{k}\cdot\vec{r}_0)(1 + i\vec{k}\cdot\vec{r} + \cdots) \end{aligned} \tag{45}$$

Taking $\vec{k}\cdot\vec{r} \ll 1$, we obtain

$$\vec{A}(\vec{r}+\vec{r}_0,t) \approx \vec{A}(t)exp(i\vec{k}\cdot\vec{r}_0). \tag{46}$$

Using the unitary transformation $|\Psi(\vec{r},t)\rangle = e^{\frac{ie}{\hbar}\vec{r}\cdot\vec{A}_0}|\psi(\vec{r},t)\rangle$ in Eq. (41), we get

$$\begin{aligned} i\hbar\frac{\partial|\psi(\vec{r},t)\rangle}{\partial t} &= \left\{\frac{\hbar^2}{2m}\vec{\nabla}^2 + V(r) - e\vec{r}\cdot\vec{E}(t)\right\}|\psi(\vec{r},t)\rangle \\ &= (\mathcal{H}_{\mathcal{O}} + \mathcal{H}_{\mathcal{I}})|\psi(\vec{r},t)\rangle \end{aligned} \tag{47}$$

The atom-field interaction Hamiltonian in the semiclassical picture is given by

$$\mathcal{H}_{\mathcal{I}} = -e\vec{r}\cdot\vec{E} = -\vec{d}\cdot\vec{E} \tag{48}$$

where the dipole moment operator $\vec{d}$ is $e\vec{r}$.

VIII. INDUCED ATOMIC COHERENCES IN TWO LEVEL ATOMIC SYSTEMS

The coherent interaction of the external laser fields and atoms induces atomic coherence among the atomic states. In recent years, atomic coherence has played a central role in the precise control over the optical property of the medium. Next, we will show how this atomic coherence is created and manipulated using the density matrix formalism.

The simplest nontrivial problem involving atom-field interaction is the coupling of a two level atom with a quasi-monochromatic radiation field. In nature, real two level atoms do not exist. A two level description is found to be useful in explaining phenomena where two levels involved are in resonance (or near resonance) with the external radiation field, while all other levels are highly detuned. In this two level atomic system, we will introduce certain realistic approximations which bring the problem to a tractable form.

Let $|1\rangle$ and $|2\rangle$ represent the excited and ground states of the atom as shown in Fig (1). The respective eigenvaules of the states $|1\rangle$ and $|2\rangle$ are $\hbar\omega_1$ and $\hbar\omega_2$ for the unperturbed Hamiltonian $\mathcal{H}_0$. By using the completeness relation $|1\rangle\langle 1| + |2\rangle\langle 2| = 1$, we write the Hamiltonian $\mathcal{H}_0$

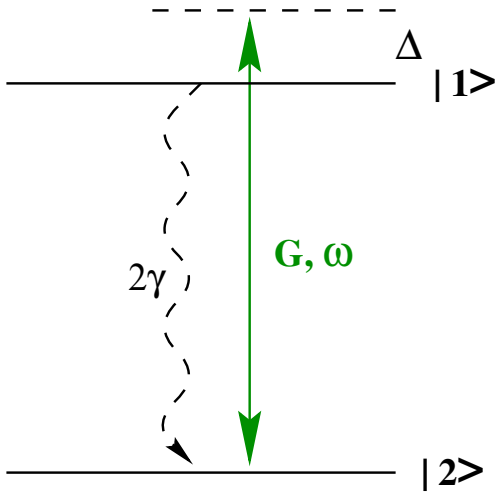

FIG. 1. Two level atomic system coupled to a control field with Rabi frequency $2G$.

$$\mathcal{H}_0 = \hbar\omega_1|1\rangle\langle 1| + \hbar\omega_2|2\rangle\langle 2| \tag{49}$$

and the wave function in the Schrödinger picture is as fellow:

$$|\psi\rangle = C_1|1\rangle + C_2|2\rangle, \tag{50}$$

where C_i $(i = 1, 2)$ is the probability amplitude of being in a state $|i\rangle$. Then the density matrix operator is defined as the projector $\rho = |\psi\rangle\langle\psi|$, which is given by in matrix form as

$$\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \tag{51}$$

where the matrix elements are given by

$$\rho_{11} = C_1C_1^*, \text{ probability of being in upper level} \tag{52a}$$

$$\rho_{22} = C_2C_2^*, \text{ probability of being in lower level} \tag{52b}$$

$$\rho_{12} = \rho_{21}^* = C_1C_2^*, \qquad \text{atomic coherence} \tag{52c}$$

An atomic coherence depends on the phase difference between C_1 and C_2. These can be related to the macroscopic property of the atomic medium. The dipole moment operator can be written as

$$\vec{d} = \vec{d}_{12}|1\rangle\langle 2| + \vec{d}_{21}|2\rangle\langle 1| \tag{53}$$

where the diagonal element $\vec{d}_{11}$ and $\vec{d}_{22}$ are zero as the dipole operator $\vec{d}$ has odd parity. Therefore, the elements of the dipole operator $\vec{d}$ will be non zero if and only if the states $|1\rangle$

and $|2\rangle$ have different parity. The two level atomic system is driven by a plane monochromatic laser field

$$\vec{E} = \hat{e}\mathcal{E}_0 e^{-i(\omega t - \vec{k}.\vec{r})} + \text{c.c.}, \tag{54}$$

where $\hat{e}$ and $\mathcal{E}_0$ are respectively the direction of polarization and constant amplitude for the continuous wave (cw) respectively. The carrier frequency ω of the cw is very close to the atomic transition frequency, $\omega_{_{12}}$, $(= \omega_{_1} - \omega_{_2})$ of the two level atomic system and is highly detuned with all other levels. The interaction Hamiltonian of the two level atom in the dipole moment approximation can be written as :

$$\begin{aligned} \mathcal{H}_I &= -\vec{d}\cdot\vec{E} \\ &= -(\vec{d}_{12}|1\rangle\langle 2| + \vec{d}_{21}|2\rangle\langle 1|)\cdot\vec{E}. \end{aligned} \tag{55}$$

Therefore the total Hamiltonian of the atom is given by

$$\mathcal{H} = \hbar\omega_{12}|1\rangle\langle 1| - (\vec{d}_{12}|1\rangle\langle 2| + \vec{d}_{21}|2\rangle\langle 1|)\cdot\vec{E}, \tag{56}$$

where the energy of the ground state level $|2\rangle$ is taken to be zero. We now make use of the unitary transformation :

$$|\psi(t)\rangle = e^{-i\omega|1\rangle\langle 1|t}|\phi(t)\rangle; \tag{57}$$

to write the Schrödinger equation written as

$$i\hbar\frac{\partial|\phi(\vec{r},t)\rangle}{\partial t} = \mathcal{H}_{eff}|\phi(t)\rangle. \tag{58}$$

Here the effective Hamiltonian is

$$\mathcal{H}_{eff}/\hbar = -\Delta|1\rangle\langle 1| - (G|1\rangle\langle 2| + \text{h.c.}) - (\text{G}^{'}\text{e}^{-2\text{i}\omega\text{t}}|1\rangle\langle 2| + \text{h.c.}), \tag{59}$$

where $\Delta = \omega - \omega_{12}$, is the detuning of the control field from the atomic transition frequency. The coupling strengths G and $G^{'}$ are given by

$$G = \frac{\vec{d}_{12}\cdot\vec{\mathcal{E}}_0}{\hbar}e^{i\vec{\text{k}}\cdot\vec{r}}, \qquad G^{'} = \frac{\vec{d}_{12}\cdot\vec{\mathcal{E}}_0}{\hbar}e^{-i\vec{\text{k}}\cdot\vec{r}} \tag{60}$$

It should be noted that the effective Hamiltonian contains d.c terms; $-\Delta|1\rangle\langle 1| - (G|1\rangle\langle 2| + G^{*}|2\rangle\langle 1|)$ and a highly oscillating term (oscillating at frequency $2\omega_c$) related to $G^{'}$. The value of $G^{'}$ becomes important only when $G^{'} \approx \omega$. Therefore, the term $G^{'}$ can be neglected at optical frequency domain where $G^{'} \ll 2\omega$. This approximation is known as the *rotating wave*

approximation (RWA) which is used frequently in this work [58]. Therefore, the effective Hamiltonian becomes

$$\mathcal{H}_{eff}/\hbar = -\Delta|1\rangle\langle 1| - (G|1\rangle\langle 2| + G^*|2\rangle\langle 1|). \tag{61}$$

In writing Eq. (61) the RWA has been made to remove the explicit time dependence of $\mathcal{H}_{eff}$. The coupling strength $2G$ is called the Rabi frequency [59]. To obtain the dynamics of the density matrix equation using the Liouville equation

$$\dot{\rho} = -\frac{i}{\hbar}[H_{eff}, \rho]. \tag{62}$$

The dynamics of population and polarization of the atoms in the two-level configuration is given by

$$\dot{\rho}_{11} = -\dot{\rho}_{22} = iG\rho_{21} - iG^*\rho_{12}, \tag{63a}$$

$$\dot{\rho}_{12} = \dot{\rho}^*_{21} = i\Delta\rho_{12} + iG(\rho_{22} - \rho_{11}), \tag{63b}$$

where dot denotes $\partial/\partial t$. The density-matrix elements in the original frame are given by $\rho_{12}e^{-i\omega t}, \rho_{11}$, and ρ_{22}. These equations are known as the optical Bloch equations (OBE), in analogy to the Bloch equations in nuclear magnetic resonance. Note that $\dot{\rho}_{11} = -\dot{\rho}_{22}$, in accordance with the requirement of a closed two-level system [60], where the total population is conserved ($\rho_{11} + \rho_{22} = 1$). Let us assume that initially all the atoms are in the ground state $|2\rangle$ and thus $\rho_{22}(0) = 1$ with all other density-matrix elements beings zero. The solutions for the atomic population and atomic polarization can be obtained by solving the set of density matrix equations (63) and can be written as

$$\rho_{22} = \cos^2\left(\frac{\Omega t}{2}\right) + \frac{\Delta^2}{\Omega^2}\sin^2\left(\frac{\Omega t}{2}\right) \tag{64}$$

$$\rho_{12} = \frac{2G}{\Omega^2}\sin\left(\frac{\Omega t}{2}\right)\left\{\Delta\sin\left(\frac{\Omega t}{2}\right) + i\Omega\cos\left(\frac{\Omega t}{2}\right)\right\} \tag{65}$$

where $\Omega = \sqrt{(\Delta^2 + 4|G|^2)}$ is called the generalized Rabi frequency. For the zero detuning case, with $\omega = \omega_{12}$, Eq. (64) reduces to

$$\rho_{22} = \cos^2\left(\frac{\Omega t}{2}\right); \tag{66}$$

the atom oscillates symmetrically between its ground and excited states with an angular frequency Ω. The increase in the detuning of the field results in the increase of the Rabi

oscillation with a reduced amplitude. The two level atom interacting with a classical electromagnetic field has been beautifully discussed in the book by Allen and Eberly [61].

The above calculation does not include spontaneous emission. To include the same it is necessary to generalize these Bloch equations by inclusion of the effects of the spontaneous emission [62]. In the presence of spontaneous emission, the Bloch equations are modified to

$$\begin{aligned}
\dot{\rho}_{11} &= -2\gamma\rho_{11} + iG\rho_{21} - iG^*\rho_{12} \\
\dot{\rho}_{22} &= 2\gamma\rho_{11} - iG\rho_{21} + iG^*\rho_{12} \\
\dot{\rho}_{12} &= -[\gamma - i\Delta]\rho_{12} + iG(\rho_{22} - \rho_{11}) \\
\dot{\rho}_{21} &= -[\gamma + i\Delta]\rho_{12} + iG^*(\rho_{11} - \rho_{22})
\end{aligned} \tag{67}$$

where 2γ $(1/T_1)$ describes the decay rate of the atomic excited state $|1\rangle$ and γ $(1/T_2)$ is the decay rate of the atomic coherence. However, in cases where the collision between atoms play a significant role, the decay of the coherences and the populations are described by different decay parameters, and in those cases the parameters T_2 and T_1 are introduced to account for this difference.

The solution of the above equations are no longer purely oscillatory, as in the cases of Eqs. (64) and (65). The system now settles down into a steady state after a sufficiently long time ($t \gg 1/\gamma$). Then all time derivatives in Eqs. (67) are set equal to zero; it is then reduced to linear algebraic equations. The simultaneous equations for steady state density matrix elements are readily solved to give

$$\begin{aligned}
\rho_{11} &= \frac{|G|^2}{(\gamma^2 + \Delta^2) + 2|G|^2} \\
\rho_{12} &= \frac{iG(\gamma + i\Delta)}{(\gamma^2 + \Delta^2) + 2|G|^2}.
\end{aligned} \tag{68}$$

The induced polarization, say at frequency ω, is expressed in terms of the non-diagonal elements of the density matrix ρ_{12} and ρ_{21}:

$$\vec{\mathcal{P}} \equiv \mathcal{N}\langle\vec{d}\rangle \equiv \mathcal{N}\mathrm{T}r(\vec{d}\rho) \equiv \mathcal{N}(\vec{d}_{21}\rho_{12} + \mathrm{h.}c.). \tag{69}$$

Here $\mathcal{N}$ is the atomic density of the medium. Using the steady state value of ρ_{12}, one can easily calculate the susceptibility of the medium

$$\chi = \frac{\mathcal{N}|d_{12}|^2}{\hbar} \frac{i(\gamma + i\Delta)}{(\gamma^2 + \Delta^2) + 2|G|^2} \tag{70}$$

This is no longer a linear susceptibility because the strength of the field $\mathcal{E}_0$ is contained in the quantity G that appears in the denominator. These contributions are related to the nonlinear susceptibility [63], which controls varieties of higher-order processes that occur in nonlinear optics. Note that the imaginary part of the above susceptibility, which gives the absorption profile of the medium, is of the Lorentzian type. The full width at half maximum (FWHM) of the Lorentzian profile is $\gamma_c = \sqrt{\gamma^2 + 2|G|^2}$. Therefore, the width depends on the intensity of the applied field as shown in Fig. (2)(red-long dashed). The additional contribution to the line width is known as *power or saturation broadening* . The linear susceptibility expression can be obtained from Eq. (70) after dropping the $2|G|^2$ term in the denominator:

$$\chi = \frac{\mathcal{N}|d_{12}|^2}{\hbar} \frac{i}{(\gamma - i\Delta)}. \tag{71}$$

In the presence of a weak field, Re $[\chi]$ has a standard dispersive line shape i.e., the dispersion is anomalous, Im $[\chi]$ has also a Lorentzian shaped with natural line-width 2γ as shown in the Fig. (2) (black-dashed) . However, as can be seen from Fig. (2) for the case of a two-level system, in usual the medium the light pulses experience very large absorption in the vicinity of the sharp atomic resonance that prevents a clear observation of high anomalous dispersion. We next show that this susceptibility behavior can be drastically modified by applying an additional control field which will extend two level system to three level atomic configuration.

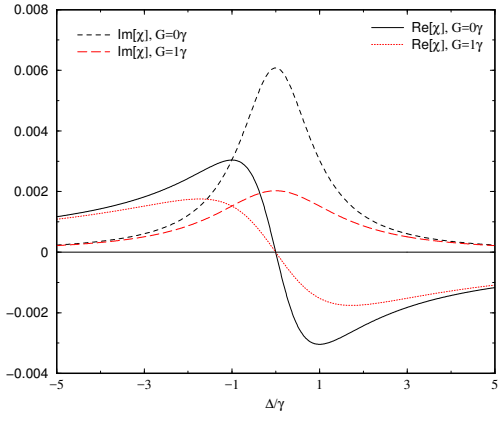

FIG. 2. Real $[\chi]$ and imaginary $[\chi]$ parts of the susceptibility of a two-level atom as a function of the atom-field detuning $\Delta = \omega - \omega_{12}$ in units of γ for two different field G=0, 1γ. The parameters of the above graph for ^{87}Rb vapor are chosen as density $\mathcal{N}= 2 \times 10^{12}$ atoms/cc, $\gamma = 3\pi \times 10^6$ rad/sec .

Problem 7. The density matrix operator for atoms in particular states may be represented by simply giving the entries for the matrix explicitly. For example, the ground state density matrix for a two-level atom is $\rho_g = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and the excited state matrix is $\rho_e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(a) Verify that these two representations are correct by calculating the occupation probabilities $\rho_{11} = \langle 1|\rho_g|1\rangle$ and $\rho_{22} = \langle 2|\rho_e|2\rangle$ for states $|1\rangle$ and $|2\rangle$. Thus, diagonal entries in ρ give occupation probabilities or populations of specific levels.

(b) Off-diagonal entries in ρ do not give populations. Show that $\rho_{12} = \langle 1|\psi\rangle\langle\psi|2\rangle = 0$, if $|\psi\rangle$ is an eigenstate and $\rho_{12} = \langle 1|\psi\rangle\langle\psi|2\rangle \neq 0$, if $|\psi\rangle$ is a superposition state like $|\psi\rangle = c_1|1\rangle + c_2|2\rangle$.

(c) To illustrate the difference between pure and mixed case density matrices, now consider both a single two-level atom in a superposition state described by $\rho_{pure} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ and an ensemble of such atoms at infinite temperature. Half the ensemble atoms are in the ground state and half are in the excited state, giving the matrix $\rho_{mix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$

Show that ρ pure can be transformed using the rotation matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ to give ρ_g or ρ_e, but that it cannot be rotated to give ρ_{mix} above. (This illustrates the fact that a single interaction with light which causes rotations in the Hilbert space of the atom cannot transform an entire system of independent atoms into the same state at the same time. Only operations on coherently prepared ensembles provide total system control.)

Problem 8. For a system with Hamiltonian $H = H_0 + V$, where V is a perturbation:

(a) Show the equations of motion of the density matrix in the Schrodinger, Interaction, and Heisenberg pictures are

$$i\hbar\dot{\rho}_s = [H, \rho_s], \qquad i\hbar\dot{\rho}_I = [V_I, \rho_I], \text{and } i\hbar\dot{\rho}_H{=}0$$

(b) What can be said in general about eigenvalues and eigenfunctions of V if it is Her-

mitian?

(c) Not all interactions and perturbations are Hermitian. Write down the Schrodinger equation in matrix form for a four-level system subject to a Hermitian Hamiltonian H_0 and a perturbative, non-Hermitian interaction V which couples only levels $|3\rangle$ and $|4\rangle$. Find exact energy eigenvalues of the system in terms of matrix elements of H_0 and V using a diagonalization procedure, and compare their properties to eigenvalues of Hermitian operators.

Problem 9. To determine the spread expected in repeated measurements of the dipole moment $e\boldsymbol{r} = d\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, one can calculate the root-mean-squared fluctuation given by $\sigma = \sqrt{\langle (e\boldsymbol{r})^2 \rangle - \langle e\boldsymbol{r} \rangle^2}$. Assuming d is real, determine σ in terms of d and elements of the density matrix for a two-level atom. (Hint: The angled brackets in the expression for σ indicate expectation values or means, calculated as traces of the operator with the density matrix.)

Problem 10. Consider a three level ladder atomic system is subjected to two electromagnetic radiations with carrier frequencies ω_p and ω_c respectively, as shown in the Fig.3. The excited states $\mid 1\rangle$ and $\mid 2\rangle$ are under goes spontaneous decay $2\gamma_{31}$ and $2\gamma_{32}$.

(a) Write down the unperturbed H_0 and pertubed Hamiltonian H_I of the system.

(b) Use suitable unitary transformation U_o to express the above Hamiltonian in following effective Hamiltonian form

$$H_{eff} = -\hbar\Delta_p \mid 2\rangle\langle 2 \mid -\hbar\left(\Delta_p + \Delta_c\right) \mid 1\rangle\langle 1 \mid -hG \mid 1\rangle\langle 2 \mid -\hbar g \mid 2\rangle\langle|3 \mid + h.c. \tag{72}$$

(c) Write down the equations of motion for all elements of the density matrix in the given system.

(d) Use steady-state perturbation theory to solve the equations of part (c) by expressing $\rho_{ij} = \rho_{ij}^{(0)} + g\rho_{ij}^{(1)} + g^*\rho_{ij}^{(2)}$. Find out the linear susceptibility expression for the probe field which oscillates with ω_p by assuming all atoms are in ground state $\rho_{33}(0) = 1$ at time $t = 0$ and all other population and coherences are zero.

(e) Find out the eigen vectors and eigen states of the effective Hamiltonian H_{eff} under

two photon resonance condition i.e., $\Delta_p + \Delta_c = 0$.

(f) Plot the real and imaginary part of susceptibility $\chi(\Delta_p)$ expression as a function of Δ_p. The values of different parameters can be taken as $\gamma_{21} = \gamma_{32} = \gamma, g = 0.01\gamma, G = 2\gamma$, and $\Delta_c = 0\gamma$.

(g) Explain the nature of quantum interference in terms of dressed state analysis.

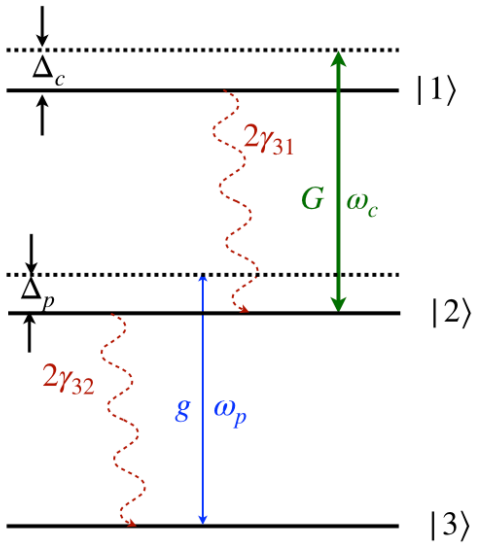

FIG. 3. Three level ladder atomic system coupled to a control and probe fields with Rabi frequencies $2G$ and $2g$, respectively.

Problem 11. Consider a three level Λ atomic system is subjected to two electromagnetic radiations with carrier frequencies ω_p and ω_c respectively, as shown in the Fig.4. The excited state $\mid 1\rangle$ under goes spontaneous decay $2\gamma_{31}$ and $2\gamma_{32}$.

(a) Write down the unperturbed H_0 and pertubed Hamiltonian H_I of the system.

(b) Use suitable unitary transformation U_o to express the above Hamiltonian in following effective Hamiltonian form $H_{eff} = -\hbar\Delta_p \mid 1\rangle\langle 1 \mid -\hbar\left(\Delta_c - \Delta_p\right) \mid 2\rangle\langle 2 \mid -hG \mid 1\rangle\langle 2 \mid -\hbar g \mid 1\rangle\langle|3 \mid + h.c.$

(c) Write down the equations of motion for all elements of the density matrix in the given system.

(d) Use steady-state perturbation theory to solve the equations of part (c) by expressing $\rho_{ij} = \rho_{ij}^{(0)} + g\rho_{ij}^{(1)} + g^*\rho_{ij}^{(2)}$. Find out the linear susceptibility expression for the probe field

which oscillates with ω_p by assuming all atoms are in ground state $\rho_{33}(0) = 1$ at time t=0 and all other population and coherences are zero.

(e) Find out the eigen vectors and eigen states of the effective Hamiltonian H_{eff} under two photon resonance condition i.e., $\Delta_p = \Delta_c = 0$.

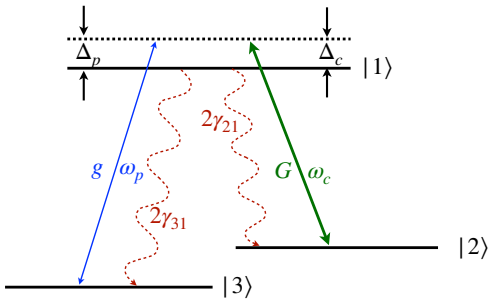

FIG. 4. Three level ladder atomic system coupled to a control and probe fields with Rabi frequencies $2G$ and $2g$, respectively.

IX. BEAM PROPAGATION EQUATION THROUGH ATOMIC MEDIA

Let us now consider the system under consideration is *non magnetic* ($\vec{\mathcal{M}} = 0$), and *non conducting* ($\vec{J} = 0$), together with no free charge ($\varrho = 0$). Therefore, the simplified wave equation in presence of induced polarization can be written as

$$\vec{\nabla}^2\vec{E} - \frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \frac{4\pi}{c^2}\frac{\partial^2\vec{\mathcal{P}}}{\partial t^2}. \tag{73}$$

This equation has the form of an inhomogeneous wave equation. The source term which appears on the right-hand side of this equation represents the nonlinear response of the medium. The exact solution of nonlinear wave equation requires strenuous effort. So, we employ few reasonable approximations to simplify the wave equation. We consider a quasi-monochromatic wave propagating in the z-direction through a dielectric medium. The electric field and induced medium polarization due to this field can be described as

$$\vec{E}(x,y,z,t) = \hat{e}\mathcal{E}(x,y,z)e^{-i(\omega t - kz)} + \text{c.c.}, \tag{74a}$$

$$\vec{P}(x,y,z,t) = \hat{e}\mathcal{P}(x,y,z)e^{-i(\omega t - kz)} + \text{c.c.}, \tag{74b}$$

The required space and time derivatives of Eq.(74a), and (74b) are given by

$$\nabla^2\vec{E} = \hat{e}\left(\nabla_\perp^2\mathcal{E} + \frac{\partial^2\mathcal{E}}{\partial z^2} + 2ik\frac{\partial\mathcal{E}}{\partial z} - k^2\mathcal{E}\right)e^{i(kz-\omega t)} + \text{c.c.}, \tag{75a}$$

$$\frac{\partial^2\vec{E}}{\partial t^2} = \hat{e}\left(-\omega^2\mathcal{E}\right)e^{i(kz-\omega t)} + \text{c.c.}, \tag{75b}$$

$$\frac{\partial^2\vec{P}}{\partial t^2} = \hat{e}\left(-\omega^2\mathcal{P}\right)e^{i(kz-\omega t)} + \text{c.c.}, \tag{75c}$$

where $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ represents the spatial derivative in transverse direction. We then assume that the variation of the envelope functions $\mathcal{E}$ and $\mathcal{P}$ in space and time within the time period and wavelength are very slow. This assumption is called the "*slowly varying envelope approximation*" (SVEA) and is mathematically written as:

$$|k\mathcal{E}| \gg \left|k^2\frac{\partial\mathcal{E}}{\partial z}\right| \gg \left|\frac{\partial\mathcal{E}^2}{\partial^2 z}\right|, \quad |k\mathcal{P}| \gg \left|k^2\frac{\partial\mathcal{P}}{\partial z}\right| \gg \left|\frac{\partial\mathcal{P}^2}{\partial^2 z}\right| \tag{76}$$

Substituting the above equations in Eq. (73) and neglecting the higher order time and space derivative we get:

$$\frac{\partial\mathcal{E}}{\partial z} = \frac{i}{2k}\nabla_\perp^2\mathcal{E} + 2\pi ik\mathcal{P}. \tag{77}$$

The above form of wave equation is known as paraxial wave equation. We can further expressed Eq. (77) in terms of Rabi frequency g as shown below

$$\frac{\partial g}{\partial z} = \frac{i}{2k}\nabla_\perp^2 g + 2\pi ik\chi g. \tag{78}$$

where χ is the susceptibility of the medium. The first term on right hand side is a second order partial derivative in the xy plane i.e. $\nabla_\perp^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ which describes inherent optical diffraction of the light beam. The second term on right hand side incorporates the dispersion and absorption profile of the medium.

One can explore a detailed theoretical explanation of the generation of structured beams by studying full density matrix equations with Maxwell's wave equation. To facilitate these structured beam generation, one may use a homogeneously as well as inhomogeneously broadened multi levels atomic system driven by two orthogonal polarization components of probe beam. In order to create phase-dependent atomic coherences, one may use a weak or strong magnetic field to couple the ground states. One can derive an analytical expression for the probe susceptibility in the weak as well as strong field regime. However, numerical solutions of density matrix equations at steady state limits is inevitable to obtain the

response of the medium at strong probe field intensities. The ground-state coherence created in the presence of strong TMF can be attributed to nonlinear magneto-optical rotation (NMOR). This can enhance the rotation of the structured beam. These studies open up new possibilities for generating a high-contrast structured beam in other closed-loop systems that display narrow EIT resonances. The steep variation of refractive index around the narrow transparency window is the main reason behind the formation of high-contrast beams. Thus an atomic medium with inhomogeneously broadened atomic system may be a suitable candidate for creating a diffraction-controlled high-contrast structured beam.

X. CONCLUSION

In conclusion, the absorptive properties of a resonant systems can be well controlled by the application of a suitable spatially dependent control beam. The effective manipulation of polarization structure of the probe beam can allow us to create a spot size as small as $0.1612\lambda^2$ [64]. This realization is possible due to the existence of a strong longitudinal polarization component [65]. The focusing property of CV beams has applications in single-molecule spectroscopy, STED [66], confocal microscopy [67–69], efficient laser cutting [70] and optical trapping of particles [71]. The inhomogeneous polarization distribution of VBs has application in polarization dependent measurements [72]. In quantum information, a high-dimensional Hilbert space [73] can be facilitated by the VBs that employ their polarization inhomogeneity and OAM. This high-dimensional Hilbert space can be leveraged to encode single-photon qubits [74].

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