Calculations

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1 Berry's Operator

The operator, $\vec{\nabla}_B(\vec{A}.\vec{B}) = \vec{A}.\vec{\nabla}\vec{B}$ is defined by,

$$(A.\vec{\nabla}\vec{B})_{x_i} = \sum_{j=1}^{3} A_{x_j} \frac{\partial}{\partial x_i} B_{x_j}$$

where x_1, x_2, and $x_3 = x$, y, z.
(from eq. 1.4 of [1])

2 The Momentum Density

$$\begin{aligned} P &\propto \text{Re}\left[\vec{E}^* \times \vec{H}\right] \\ &\propto \text{Im}\left[\vec{E}^* \times \vec{(}\nabla \times \vec{E})\right] \qquad \left(-i\frac{\omega}{c}\mu\vec{H} = -\vec{\nabla} \times \vec{E}\right) \\ &= \text{Im}\ [\vec{E}^*.\nabla\vec{E} - (\vec{E}^*.\vec{\nabla})\vec{E}] \end{aligned} \tag{1}$$

(Wikipedia: Vector calculus identities, Section: Cross product rule)
Consider,

$$\begin{aligned} \vec{\nabla} \times (\vec{E}^* \times \vec{E}) &= \vec{E}^*(\vec{\nabla}.\vec{E}) - \vec{E}(\vec{\nabla}.E^*) + (\vec{E}.\vec{\nabla})\vec{E}^* - (\vec{E}^*.\vec{\nabla})\vec{E} \\ &= -2i\ \text{Im}\ [(\vec{E}^*.\vec{\nabla})\vec{E}] \text{ (Charge free Maxwell's eqns)} \\ \Rightarrow \text{Im}\ [(\vec{E}^*.\vec{\nabla})\vec{E}] &= \frac{i}{2}\vec{\nabla} \times (\vec{E}^* \times \vec{E}) \end{aligned}$$

Since $\frac{i}{2}\vec{\nabla} \times (\vec{E}^* \times \vec{E})$ is real,

$$\frac{i}{2}\vec{\nabla} \times (\vec{E}^* \times \vec{E}) = -\text{Im}\ [\frac{1}{2}\vec{\nabla} \times (\vec{E}^* \times \vec{E})]$$

From eq(1),

$$\begin{aligned} \Rightarrow \vec{P} &\propto \text{Im}\ [\vec{E}^*.\nabla\vec{E} + \frac{1}{2}\vec{\nabla} \times (\vec{E}^* \times \vec{E})] \\ &= \text{Im}\ [\vec{E}^*.\nabla\vec{E}] + \frac{1}{2}\vec{\nabla} \times \vec{S} \end{aligned}$$

where $\vec{S}$ is the spin angular momentum density.

The first term may be recognized as the canonical momentum, whereas the second term is the spin momentum.

Using the 'electric–magnetic democracy'[1] principle, we should also add the contribution from the magnetic part, thus modifying the equation to,

$$\begin{aligned}
\Rightarrow \vec{P} &\propto \frac{1}{2}\text{Im }[\vec{E}^*.\nabla\vec{E} + \vec{H}^*.\nabla\vec{H} + \frac{1}{2}\vec{\nabla}\times(\vec{E}^*\times\vec{E} + \vec{H}^*\times\vec{H})] \\
&= \frac{1}{2}\text{Im }[\vec{E}^*.\nabla\vec{E} + \vec{H}^*.\nabla\vec{H}] + \frac{1}{2}\vec{\nabla}\times S \\
&= \vec{P}^{can} + \vec{P}^{spin}
\end{aligned}$$

3 Force equation

From [2], the optical force terms of complex fields and dipole moments,

$$F \propto \text{Re }[\vec{d}_e^*.\vec{\nabla}\vec{E} + \vec{d}_m^*.\vec{\nabla}\vec{H}]$$

where $\vec{d}_e = \alpha_e\vec{E}, \vec{d}_m = \alpha_m\vec{H}$, are complex electric and magnetic dipole moments and α_e, α_m are complex electric and magnetic polarizabilities.

$$F = \text{Re }[\alpha_e^*\vec{E}^*.\vec{\nabla}\vec{E} + \alpha_m^*\vec{H}^*.\vec{\nabla}\vec{H}]$$

$$= \text{Re }[\alpha_e]\text{Re }[\vec{E}^*.\vec{\nabla}\vec{E}] + \text{Re }[\alpha_m]\text{Re }[\vec{H}^*.\vec{\nabla}\vec{H}] - \text{Im }[\alpha_e^*]\text{Im }[\vec{E}^*.\vec{\nabla}E] - \text{Im }[\alpha_m^*]\text{Re }[\vec{H}^*.\vec{\nabla}H]$$

$$= \text{Re }[\alpha_e]\text{Re }[\vec{E}^*.\vec{\nabla}E] + \text{Re }[\alpha_m]\text{Re }[\vec{H}^*.\vec{\nabla}H] + 2\mu\omega\text{Im }[\alpha_e]P_e^{can} + 2\epsilon\omega\text{Im }[\alpha_m]P_m^{can}$$

From,

$$w = \frac{1}{2}(\epsilon|E|^2 + \mu|H|^2)$$

$$\vec{\nabla}w = \Big[\epsilon\text{Re }[\vec{E}^*.\vec{\nabla}\vec{E}] + \mu\text{Re }[\vec{H}^*.\vec{\nabla}\vec{H}]\Big] = \vec{\nabla}w_e + \vec{\nabla}w_m$$

$$\Rightarrow F \propto \frac{c^2}{2\omega}\Big[\mu\text{Re }[\alpha_e]\vec{\nabla}w_e + \epsilon\text{Re }[\alpha_m]\vec{\nabla}w_m\Big] + \mu\text{Im }[\alpha_e]P_e^{can} + \epsilon\text{Im }[\alpha_m]P_m^{can}$$

4 Spin angular momentum density

Evanescent fields are given by,

$$E \propto \frac{1}{\sqrt{1+|m|^2}} \left[\hat{x} + m\frac{k}{k_z}\hat{y} - i\frac{\kappa}{k_z}\hat{z} \right] e^{ik_z z - \kappa x}$$

where, $k = k_z\,\hat{z} + i\kappa\,\hat{x}$, and $k^2 = k_z^2 - \kappa^2$.
The corresponding magnetic field,

$$\begin{aligned} H &= \frac{1}{k\sqrt{1+|m|^2}}[\vec{k} \times \vec{E}] \\ &= \frac{1}{k\sqrt{1+|m|^2}} \left[-mk\hat{x} + \left[k_z - \frac{\kappa^2}{k_z} \right] \hat{y} + \frac{i\kappa k m}{k_z}\hat{z} \right] e^{ik_z z - \kappa x} \\ &= \frac{1}{\sqrt{1+|m|^2}} \left[-m\hat{x} + \frac{k}{k_z}\hat{y} + \frac{i\kappa m}{k_z}\hat{z} \right] e^{ik_z z - \kappa x} \end{aligned}$$

Consider,

$$\vec{E}^* \times \vec{E} \propto \frac{1}{(1+|m|^2)} \left[-\frac{2i\mathrm{Re}(m)k\kappa}{k_z^2}\hat{x} + \frac{2i\kappa}{k_z}\hat{y} + \frac{2i\mathrm{Im}(m)k}{k_z}\hat{z} \right] e^{-2\kappa x}$$

Now consider,

$$\vec{H}^* \times \vec{H} \propto \frac{1}{(1+|m|^2)} \left[\frac{2i\mathrm{Re}(m)k\kappa}{k_z^2}\hat{x} + |m|^2\frac{2i\kappa}{k_z}\hat{y} + \frac{2i\mathrm{Im}(m)k}{k_z}\hat{z} \right] e^{-2\kappa x}$$

Therefore,

$$\begin{aligned} \vec{S} &\propto \frac{1}{2}\mathrm{Im}\,(E^* \times E + H^* \times H) \\ &= \left[\frac{\kappa}{k_z}\hat{y} + \frac{2\mathrm{Im}(m)}{1+|m|^2}\frac{k}{k_z}\hat{z} \right] e^{-2\kappa x} \end{aligned} \tag{2}$$

From eq. (5) from the paper [3],

$$\vec{S}_\perp \propto \frac{\mathrm{Re}\,\vec{k} \times \mathrm{Im}\,\vec{k}}{(\mathrm{Re}\,\vec{k})^2} = \frac{\kappa}{k_z}\hat{y}$$

This agrees with our derivation.

5 Canonical Momentum

From eq. (6) from [3],

$$\vec{P}^{can} = \frac{1}{2}\text{Im}\,[\vec{E}^*.\nabla\vec{E} + \vec{H}^*.\nabla\vec{H}]$$

Therefore,

$$\begin{aligned}
(\vec{E}^*.\nabla\vec{E})_x &= E_x^*\frac{\partial}{\partial x}E_x + E_y^*\frac{\partial}{\partial x}E_y + E_z^*\frac{\partial}{\partial x}E_z \\
&= \left[1 + \left(\frac{|m|k}{k_z}\right)^2 + \frac{\kappa^2}{k_z^2}\right]\frac{-\kappa e^{-2\kappa x}}{1+|m|^2} \\
(\vec{E}^*.\nabla\vec{E})_y &= 0 \\
(\vec{E}^*.\nabla\vec{E})_z &= E_x^*\frac{\partial}{\partial z}E_x + E_y^*\frac{\partial}{\partial z}E_y + E_z^*\frac{\partial}{\partial z}E_z \\
&= \left[1 + \left(\frac{|m|k}{k_z}\right)^2 + \frac{\kappa^2}{k_z^2}\right]\frac{ik_z e^{-2\kappa x}}{1+|m|^2}
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\vec{H}^*.\nabla\vec{H})_x &= \left[|m|^2 + \frac{k^2}{k_z^2} + \left(\frac{m\kappa}{k_z}\right)^2\right]\frac{-\kappa e^{-2\kappa x}}{1+|m|^2} \\
(\vec{H}^*.\nabla\vec{H})_y &= 0 \\
(\vec{H}^*.\nabla\vec{H})_z &= \left[|m|^2 + \frac{k^2}{k_z^2} + \left(\frac{m\kappa}{k_z}\right)^2\right]\frac{ik_z e^{-2\kappa x}}{1+|m|^2}
\end{aligned}$$

Finally,

$$\vec{P}^{can} = k_z e^{-2\kappa x}\hat{z} = e^{-2\kappa x}\,\text{Re}\,\vec{k}$$

In agreement with eq.(9) from [3].

6 Spin Momentum

From eq. (7) from [3],

$$\vec{P}^{spin} \propto \frac{1}{2}\vec{\nabla}\times\vec{S}$$

where $\vec{S}$ is given by eq. 2,

$$\begin{aligned}
\vec{S} &= \left[\frac{\kappa}{k_z}\hat{y} + \sigma\frac{k}{k_z}\hat{z}\right]e^{-2\kappa x} \\
\Rightarrow \vec{P}^{spin} &\propto \left[\sigma\frac{k}{k_z}\hat{y} - \frac{\kappa}{k_z}\hat{z}\right]\kappa e^{-2\kappa x}
\end{aligned} \tag{3}$$

From (10) from [3],

$$\begin{aligned}
\vec{P}^{spin} &\propto -\frac{(\mathrm{Im}\ \vec{k})^2}{(\mathrm{Re}\ \vec{k})^2}\mathrm{Re}\ \vec{k} + \sigma k \frac{\mathrm{Re}\ \vec{k} \times \mathrm{Im}\ \vec{k}}{(\mathrm{Re}\ \vec{k})^2} \\
&= \left[-\frac{\kappa}{k_z}\hat{z} + \sigma\frac{k}{k_z}\hat{y}\right]\kappa
\end{aligned}$$

in agreement with eq. 3

References

[1] M. V. Berry, Optical currents, Journal of Optics A: Pure and Applied Optics 11 (9) (2009) 094001. doi:10.1088/1464-4258/11/9/094001.
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[2] K. Y. Bliokh, A. Y. Bekshaev, F. Nori, Extraordinary momentum and spin in evanescent waves, Nature Communications 5 (1) (2014) 3300. doi:10.1038/ncomms4300.
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[3] K. Y. Bliokh, Transverse spin and momentum in structured light: Quantum spin hall effect and transverse optical force, in: 2016 URSI International Symposium on Electromagnetic Theory (EMTS), 2016, pp. 345–348. doi:10.1109/URSI-EMTS.2016.7571393.