Minimax optimization

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$\forall \tilde{x}, \tilde{y}$,

$$\min_x f(x, \tilde{y}) \le f(\tilde{x}, \tilde{y}) \le \max_y f(\tilde{x}, y)$$

$$\Rightarrow \max_{\tilde{y}} \min_x f(x, \tilde{y}) \le \min_{\tilde{x}} \max_y f(\tilde{x}, y)$$

$$\min_{x \in \mathcal{X}} \max_{y \in \mathcal{Y}} f(x, y).$$

where f is convex in x and concave in y.

<u>Example</u>: $f(x,y) = x^T A y$.] bilinear matrix games.

<u>Constrained opt</u>: $\min_x f(x)$ s.t. $g(x) \le 0$

$$\equiv \min_x \underbrace{\max_{\lambda \ge 0} f(x) + \lambda g(x)}.$$

$$h(x) = \max_{\lambda \ge 0} f(x) + \lambda g(x) = f(x) \text{ if } g(x) \le 0$$
$$= \infty \text{ if } g(x) \ge 0.$$

So, $\min_x h(x) = \min_x f(x)$ s.t. $g(x) \le 0$.

<u>Weak duality</u>: $\max_y \min_x f(x,y) \le \min_x \max_y f(x,y)$.

<u>Sion's minimax theorem</u>: If f is convex - concave then

$$\min_x \max_y f(x,y) = \max_y \min_x f(x,y).$$

(x^*, y^*) Nash equilibrium if $\max_y f(x^*, y) = \min_x f(x, y^*)$.

<u>Theorem</u>: If f is cvx - ccv and G- Lipschitz then GDA will find ϵ- approx Nash equilibrium $(\hat{x}, \hat{y})$ i.e.,

$$\max_y f(\hat{x}, y) \le \min_x f(x, \hat{y}) + \epsilon.$$ in $\frac{4GR}{\epsilon^2}$

iterations.

(*)

Proof: $x_{t+1} = x_t - \eta \nabla_x f(x_t, y_t)$

$y_{t+1} = y_t + \eta \nabla_y f(x_t, y_t).$

$$\| x_{t+1} - x \|^2 = \| x_t - x\|^2 - 2\eta \langle \nabla_x f(x_t, y_t), x_t - x \rangle + \eta^2 \| \nabla_x f(x_t, y_t)\|^2$$

$$\| y_{t+1} - y\|^2 = \| y_t - y\|^2 + 2\eta \langle \nabla_y f(x_t, y_t), y_t - y \rangle + \eta^2 \| \nabla_y f(x_t, y_t)\|^2.$$

$$f(x_t, y_t) - f(x, y_t) \leq \langle \nabla_x f(x_t, y_t), -x + x_t \rangle$$

$$f(x_t, y) - f(x_t, y_t) \leq \langle \nabla_y f(x_t, y_t), -y_t + y \rangle.$$

So, $$\| x_{t+1} - x\|^2 + \| y_{t+1} - y\|^2 \leq \| x_t - x\|^2 + \| y_t - y\|^2$$
$$- 2\eta \left[f(x_t, y) - f(x, y_t) \right]$$
$$+ 2\eta^2 G^2.$$

So, $$\frac{1}{T} \sum_{t=0}^{T-1} \left[f(x_t, y) - f(x, y_t) \right] \leq \frac{\| x_0 - x\|^2 + \| y_0 - y\|^2}{2\eta T} + \eta G^2$$

$$\Rightarrow f(\bar{x}_T, y) - f(x, \bar{y}_T) \leq \frac{\| x_0 - x\|^2 + \| y_0 - y\|^2}{2\eta T} + \eta G^2.$$

$$\max_y f(\bar{x}_T, y) - \min_x f(x, \bar{y}_T) \leq \frac{R^2}{\eta T} + \eta G^2$$

$$= \frac{2GR}{\sqrt{T}} \text{ for } \eta = \frac{R}{G}.$$

~~n experts. Each expert predicts 0~~ (*)

There are n experts and T days.

On each day each expert predicts 0 or 1.

We do not know the quality of any expert.

Can we be competitive wrt the best expert?

Algorithm: $W_0(i) = 1 \quad \forall i = 1, \dots n.$

On day t: Predict 1 w.p. $\dfrac{\sum_{i:\, Pred_t(i)=1} W_{t-1}(i)}{\sum_i W_{t-1}(i)}$

0 o.w.

$$W_t(i) = \begin{cases} W_{t-1}(i) & \text{if } Pred_t(i) \text{ was correct} \\ W_{t-1}(i)(1-\epsilon) & \text{if incorrect.} \end{cases}$$

Thm: Let M_T be the mispreds of our algorithm after t steps.

then, $M_T \le (1+\epsilon) \cdot M_T(i) + \dfrac{\log n}{\epsilon}$.

Proof: Let $\Phi_t = \sum_i W_t(i)$.

$$\Phi_{t+1} = \sum_i W_{t+1}(i)$$

$$= \sum_i W_t(i)\left(1 - \epsilon m_t(i)\right) = \Phi_t \left(1 - \epsilon \sum_i p_t(i)\, m_t(i)\right).$$

(*) $$\Phi_{t+1} = \Phi_t \left(1 - \epsilon\, \mathbb{E}[m_t]\right) \leq \Phi_t \cdot e^{-\epsilon \mathbb{E}[m_t]}.$$

On the other hand, $W_{t+1}(i) = W_0(i)(1-\epsilon)^{M_t(i)}$
$= (1-\epsilon)^{M_t(i)}$

So, ~~$W_0(i)$~~
$$(1-\epsilon)^{M_t(i)} \leq \Phi_0\, e^{-\epsilon \mathbb{E}[M_t]}$$
$$= n\, e^{-\epsilon \mathbb{E}[M_t]}.$$

So, $$\mathbb{E}[M_t] \leq \frac{\log n}{\epsilon} - M_t(i)\frac{\log(1-\epsilon)}{\epsilon}$$

For $0 \leq \epsilon \leq \frac{1}{2}$, we have $\log(1-\epsilon) \geq -\epsilon - \epsilon^2$

So, $$\mathbb{E}[M_t] \leq \frac{\log n}{\epsilon} + (1+\epsilon) M_t(i) \quad \forall i.$$
$$\leq M_t(i) + \underbrace{\epsilon \cdot T + \frac{\log n}{\epsilon}}$$
$$\leq 2\sqrt{T \log n}. \text{ if } \epsilon = \sqrt{\frac{\log n}{T}}.$$

Online grad. descent.	ST+

MAB with simple exploration :

EXP 3

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EXP 3

Algorithm

1. Set $p_1 = (\frac{1}{n}, \frac{1}{n}, \cdots)$; $C_0 = (0, \ldots 0)$

2. For $t = 1, 2, \ldots$

3. Choose expert i w.p. $(p_t)_i$; Observe $(c_t)_i$

   Let $\hat{c}_t = (0, \ldots 0, \frac{(c_t)_i}{(p_t)_i}, 0 \ldots 0)$

   set $(p_{t+1})_i \propto (p_t)_i \exp(-\eta \hat{c}_t)$.

Analysis: $KL(p \| p_{t+1}) = \sum_i p_i \log \frac{p_i}{(p_{t+1})_i}$

$$= \sum_i p_i \log \frac{p_i \, Z_t}{(p_t)_i \exp(-\eta \hat{c}_t)}$$

$$= \sum_i p_i \log \frac{p_i}{(p_t)_i} + \eta \sum_i p_i \hat{c}_t + \log Z$$

$$= KL(p \| p_t) + \eta \sum_i p_i \hat{c}_t + \log \sum_i (p_t)_i \exp(-\eta \hat{c}_t)$$

Fact: $\exp(-\alpha) \leq 1 - \alpha + \alpha^2$ for $0 \leq \alpha \leq \log_e 2$.

$\Leftarrow -\exp(-\alpha) \leq -1 + 2\alpha$

$\Leftarrow \exp(\alpha) \leq 2$

$\eta \hat{c}_t \leq \log 2$

(*)

So, $KL(p \| p_{t+1}) \leq KL(p \| p_t) + \eta \cancel{\mathbb{E}_p[c_t]} \sum_i p_i (\hat{c}_t)_i$

$$+ \log \sum_i (p_t)_i \left[1 - \eta \hat{c}_t + \eta^2 \hat{c}_t^2\right]$$

$$= KL(p \| p_t) + \eta \, \mathbb{E}_p[c_t] + \log\left[1 - \eta \, \mathbb{E}_{p_t}[\hat{c}_t] + \eta^2 \, \mathbb{E}_{p_t}[\hat{c}_t^2]\right]$$

$$\leq KL(p \| p_t) + \eta \, \mathbb{E}_p[c_t] - \eta \, \mathbb{E}_p[\hat{c}_t] + \eta^2 \cancel{\mathbb{E}_{p_t}[\hat{c}_t^2]} \sum_i (p_t)_i (\hat{c}_t)_i^2$$

$\boxed{\begin{array}{l} \eta \hat{c}_t < 1 \\ \log(1-x) \overset{?}{\leq} -x \cdot \text{ for } 0 \leq x \leq 1. \\ 1 - x \leq e^{-x} \end{array}}$

$$\cancel{\mathbb{E}_p[\hat{c}_t] - \mathbb{E}_p[c_t] \leq \frac{KL(p\|p_t) - KL(p\|p_{t+1})}{\eta} + \eta \, \mathbb{E}_{p_t}[\hat{c}_t^2]}$$

$$KL(p \| p_{t+1}) \leq KL(p \| p_t) + \eta \sum_i p_i (\hat{c}_t)_i - \eta \sum_i (p_t)_i (\hat{c}_t)_i + \eta^2 \sum_i (p_t)_i (\hat{c}_t)_i^2$$

Taking expectations,

$$\mathbb{E}\left[KL(p \| p_{t+1})\right] \leq \mathbb{E}\left[KL(p \| p_t)\right] + \eta \sum_i p_i \, \mathbb{E}\left[(\hat{c}_t)_i\right] - \eta \sum_i (p_t)_i \, \mathbb{E}\left[(\hat{c}_t)_i\right] + \eta^2 \sum_i (p_t)_i \, \mathbb{E}\left[(\hat{c}_t)_i^2\right]$$

$$= \mathbb{E}\left[KL(p \| p_t)\right] + \eta \, \mathbb{E}_p[c_t] - \eta \, \mathbb{E}_{p_t}[c_t] + \eta^2 \sum_i (p_t)_i \frac{(c_t)_i^2}{(p_t)_i}$$

So, $\mathbb{E}[KL(p \| p_{t+1})] \leq \mathbb{E}[KL(p \| p_t)] - \eta\left(\mathbb{E}_{p_t}[c_t] - \mathbb{E}_p[c_t]\right) + \eta^2 \cdot n$ (*)

$$\mathbb{E}_{p_t}[c_t] - \mathbb{E}_p[c_t] \leq \frac{\mathbb{E}[KL(p \| p_t)] - \mathbb{E}[KL(p \| p_{t+1})]}{\eta} + \eta n.$$

~~Choose~~ $\cancel{\eta}$

Average $\Rightarrow$ Average Subopt $\leq \dfrac{KL(p \| p_0)}{\eta T} + \eta n.$

$$\leq \frac{\log n}{\eta T} + \eta n.$$

Choose $\eta = \sqrt{\dfrac{\log n}{nT}} \longrightarrow 2\sqrt{\dfrac{n \log n}{T}}.$

Simple exploration

For $t = 1, \dots T$

let b_t be a Bernoulli random variable w.p. $\delta = 1$
w.p. $1-\delta = 0$.

If $b_t = 1$ [do uniform exploration]

Pick $i_t \in \{1, \dots n\}$ unif. at random.

let $\hat{\ell}_t(i) = \begin{cases} \dfrac{n}{\delta} \cdot \ell_t(i_t) & \text{if } i = i_t \\ 0 & \text{o.w.} \end{cases}$

let $\hat{f}_t(x) = \hat{\ell}_t^T x$ and $x_{t+1} = \mathcal{A}(\hat{f}_1, \dots \hat{f}_t)$