Categorifying a cluster algebra isomorphism of Hernandez-Leclerc

by Théo Pinet (13/11/2023)
Université Paris Cité / Université de Montréal

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Motivation

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$$\begin{array}{ccl}
\mathfrak{g} = \text{(fin-dim'l) simple Lie algebra} / \mathbb{C} & \xrightarrow{\text{affinization}} & \hat{\mathfrak{g}} = \text{untwisted affine Kac-Moody algebra} \\
 & & \Big\downarrow \scriptstyle{q\text{-group at level } 0} \quad \color{blue}{q \in \mathbb{C}^\times \setminus \{\text{roots of } 1\}} \\
 & & \mathcal{U}_q(\hat{\mathfrak{g}}) = \text{quantum loop algebra} \color{red}{= \text{Hopf algebra}}
\end{array}$$

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$q \in \mathbb{C}^\times \setminus \{\text{roots of } 1\}$ — q-group at level 0

$\mathcal{U}_q(\hat{\mathfrak{g}})$ = quantum loop algebra = Hopf algebra

Chevalley generating set $\{e_i, f_i, k_i^{\pm 1}\}_{i=0}^{n}$
with relations

$k_i k_j = k_j k_i$

$e_i f_j - f_j e_i = \delta_{i,j} \dfrac{k_i - k_i^{-1}}{q^{d_i} - q^{-d_i}}$

+ more...

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Rem. $\exists$ anti-isomorphism $\Xi : \mathcal{U}_q(\hat{\mathfrak{g}}) \xrightarrow{\sim} \mathcal{U}_{q^{-1}}(\hat{\mathfrak{g}})$
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(2) $\mathcal{O} \subseteq$ modules over Borel subalgebra (think BGG)

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monoidal categorifications of same cluster algebra

$\exists$ ring iso.
$D : K_0(\mathcal{O}^+) \xrightarrow{\sim} K_0(\mathcal{O}^-)$
(Groth. ring $[V][W] = [V \otimes W]$)
sending simple classes on simple classes

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Q. Precise link $\mathcal{O}^+ \rightsquigarrow \mathcal{O}^-$?

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for $1 \le i \le n$ $\begin{cases} (1) \ e_i v = 0 \text{ and} \\ (2) \ \Psi_i(z) v = \sum_{m\ge 0} (\phi_{i,m}^+ v) z^m \end{cases}$

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Context

$U_q(\hat{g})$

$\rightsquigarrow$ Chevalley generators $\{e_i, f_i, k_i^{\pm 1}\}_{i=0}^n \rightsquigarrow t^X = \langle k_i^{\pm 1} \mid 1 \le i \le n \rangle$

$\rightsquigarrow$ Drinfeld generators $\left\{ x_{i,r}^{\pm}, \phi_{i,\pm m}^{\pm} \;\middle|\; \begin{matrix} 1 \le i \le n \\ r \in \mathbb{Z} \\ m \in \mathbb{Z}_{\ge 0} \end{matrix} \right\} \rightsquigarrow U_q(h^+) = \langle \phi_{i,m}^+ \mid \begin{matrix} 1 \le i \le n \\ m \in \mathbb{Z}_{\ge 0} \end{matrix} \rangle$

2 commutative subalgebras of $U_q(\hat{b})$ $\rightsquigarrow$ 2 notions of weight

1st one: $\left. \begin{matrix} V = U_q(\hat{b})\text{-module} \\ \mu = (\mu_i)_{i=1}^n \in (\mathbb{C}^\times)^n \end{matrix} \right\} \rightsquigarrow V_\mu = \{ v \in V \mid k_i v = \mu_i v \ \ \forall i \}$ and $P(V) = \{ \mu \mid V_\mu \neq 0 \}$

Def (HJ, 2012): $V = U_q(\hat{b})$-module is in $\mathcal{O}$ iff $V = \bigoplus_\mu V_\mu$ with

(1) $\dim V_\mu < \infty$ $(\forall \mu)$

(2) $P(V)$ contained in finitely many cones of the form $D(\lambda) = \{ \mu \mid \mu \le \lambda \}$

canonical

Let $\mathfrak{r} = (\mathbb{C}(z)_0)^n$ where

$\mathbb{C}(z)_0$ = multiplicative group of rational functions that are regular & non-zero at $z=0$

Then, $\forall$simple V in $\mathcal{O}$, $\exists!\, \Psi = (\Psi_i(z))_{i=1}^n \in \mathfrak{r}$ and $\exists!$ (up to scalar) $v \in V \setminus \{0\}$ s.t.

for $1 \le i \le n$
(so $e_0 v$ can be $\neq 0$)

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This induces a bijection

$\{\text{simples in } \mathcal{O}\}/_{\cong} \xleftrightarrow{1:1} \mathfrak{r}$

$L(\Psi) \longleftarrow \Psi$

(HJ, 2012)

highest ℓ-weight

ex: Fix $1 \le i \le n$ and $a \in \mathbb{C}^{\times}$. Define $\Psi_{i,a} = (1, \dots, 1, \overset{i^{th}\text{-position}}{1-az}, 1, \dots, 1) \in \mathfrak{r}$.

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CAN HOWEVER CIRCUMVENT OBSTRUCTIONS BY CHANGING QUANTUM PARAMETER

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follow from def + prop. of antipode

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