Large population limit of interacting particle systems for population dynamics

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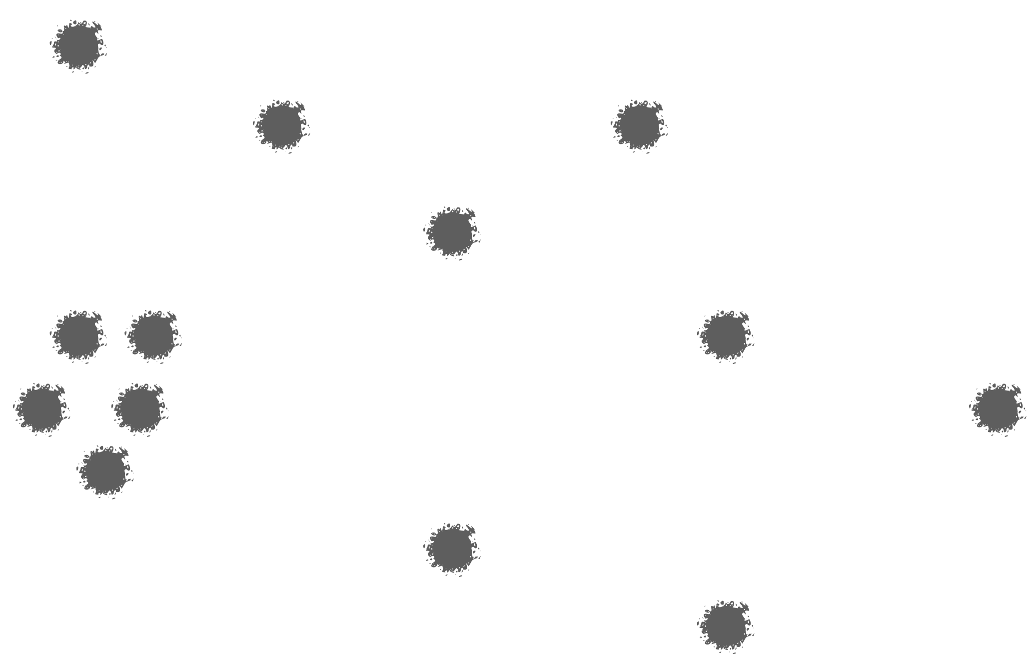
Population dynamics

Doubly nonlocal Fisher-KPP

$X \triangleq$ **compact Polish space**

(FKPP)
$$\partial_t u_t(x) = \mathrm{r_b}[u_t](x) - \mathrm{r_d}[u_t](x)\, u_t(x) + \int_X \big(u_t(y) - u_t(x)\big)\, \mathrm{r_h}[u_t](x,y)\, \pi(dy)$$

$\mathrm{r_b} \triangleq$ **birth,** $\mathrm{r_d} \triangleq$ **death/competition,** $\mathrm{r_h} \triangleq$ **hopping**



Population dynamics

Doubly nonlocal Fisher-KPP $\quad X \mathrel{\hat{=}}$ **compact Polish space**

(FKPP)

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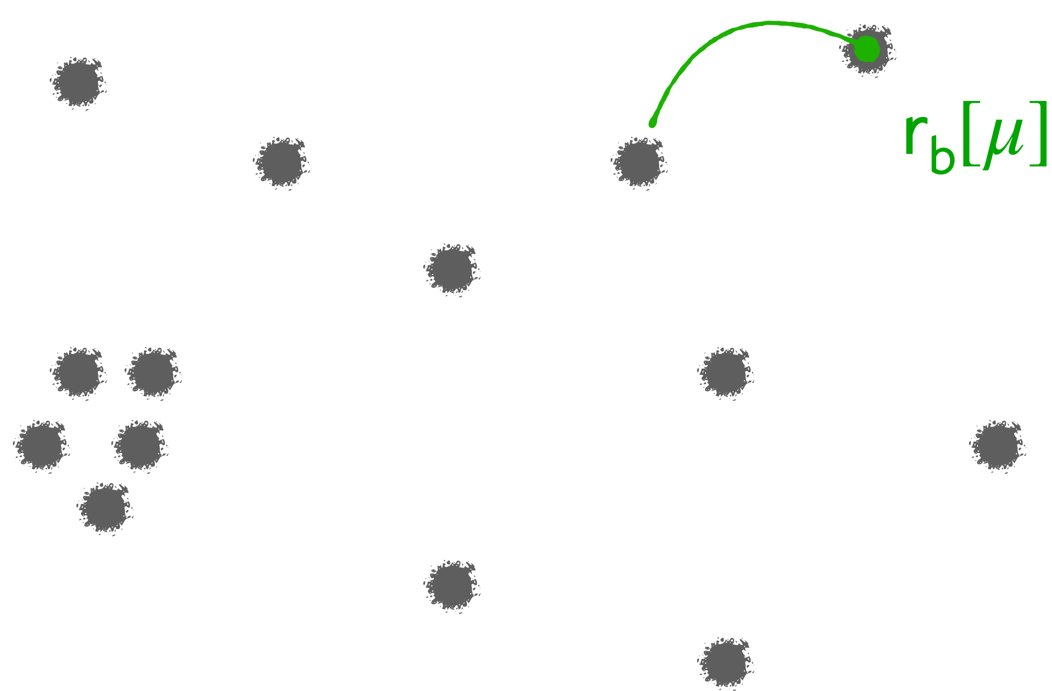

Population dynamics

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(FKPP)

$$+ \int_X \big(u_t(y) - u_t(x)\big)\, \mathrm{r_h}[u_t](x,y)\, \pi(dy)$$

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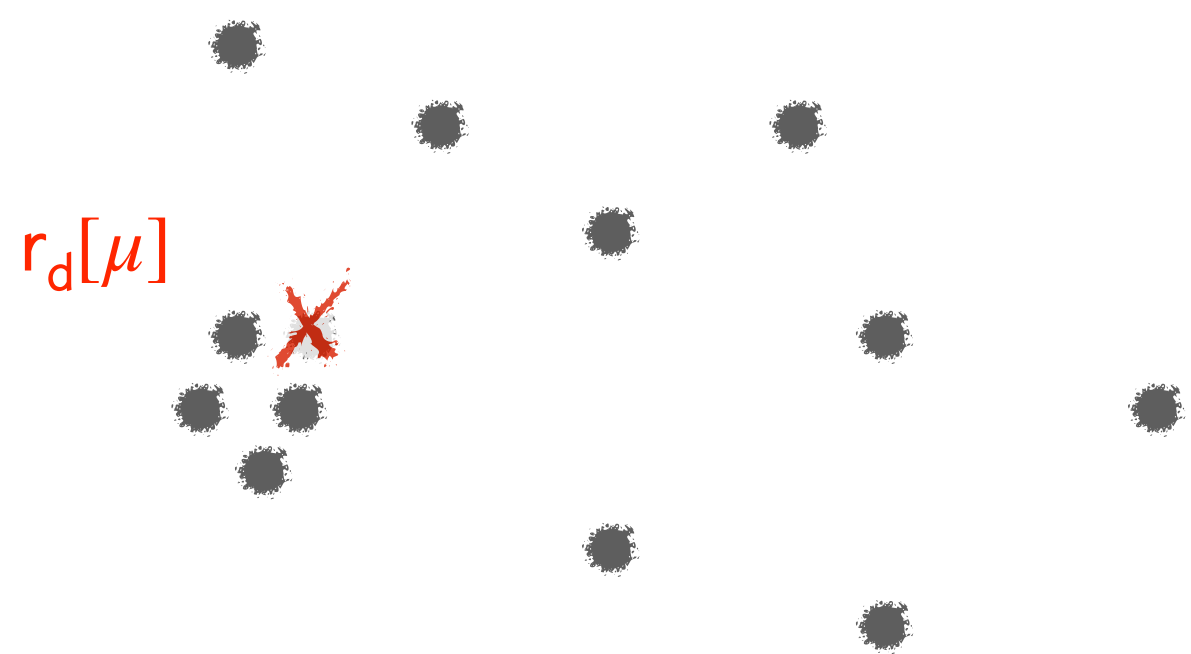

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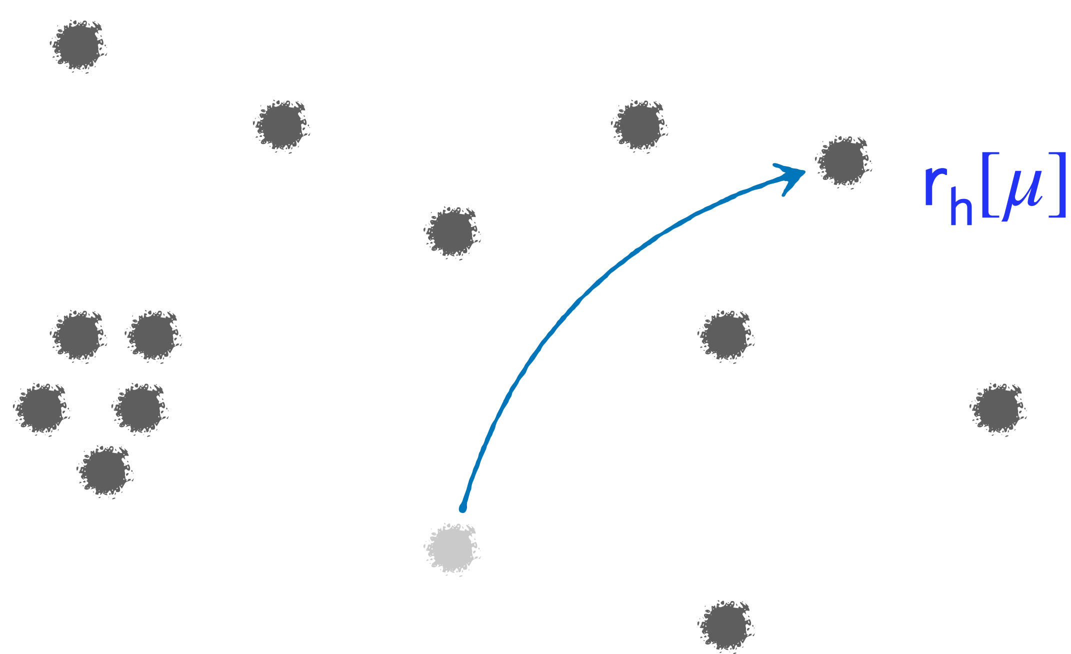
Reference measure

$\mathrm{r}_{\mathrm{b}} \mathrel{\hat{=}}$ **birth**, $\mathrm{r}_{\mathrm{d}} \mathrel{\hat{=}}$ **death/competition**, $\mathrm{r}_{\mathrm{h}} \mathrel{\hat{=}}$ **hopping**



Population dynamics

Doubly nonlocal Fisher-KPP

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Reference measure

$\mathrm{r_b} \triangleq$ **birth,** $\mathrm{r_d} \triangleq$ **death/competition,** $\mathrm{r_h} \triangleq$ **hopping**

Multicomponent

(SIS)

$$\partial_t u_t = -\,\phi \star v_t\, u_t + \beta\, v_t$$
$$\partial_t v_t = \phi \star v_t\, u_t - \beta\, v_t$$

$$\longrightarrow \quad \partial_t v_t = \phi \star v_t\,(\alpha - v_t) - \beta\, v_t$$

Bock-Fattler-Rodiah-T. 2017

Population dynamics

Doubly nonlocal Fisher-KPP $\qquad X \mathrel{\hat{=}}$ **compact Polish space**

(FKPP)

$$\partial_t \mu_t = \mathscr{V}[\mu_t]$$

$$\mathscr{V}[\mu_t] \mathrel{\hat{=}} \kappa_{\mathsf{bd}}[\mu_t] + \kappa_{\mathsf{h}}[\mu_t] : \mathscr{M}^+(X) \to \mathscr{M}^+(X)$$

where

$$\kappa_{\mathsf{bd}}[\mu](dx) = \mathsf{r}_{\mathsf{b}}[\mu](dx) - \mathsf{r}_{\mathsf{d}}[\mu](x)\,\mu(dx)$$

$$\kappa_{\mathsf{h}}[\mu](dx) = \int_X \mathsf{r}_{\mathsf{h}}[\mu](y,dx)\,\mu(dy) - \int_X \mathsf{r}_{\mathsf{h}}[\mu](x,dy)\,\mu(dx)$$

$\mathsf{r}_{\mathsf{b}} \mathrel{\hat{=}}$ **birth,** $\qquad \mathsf{r}_{\mathsf{d}} \mathrel{\hat{=}}$ **death/competition,** $\qquad \mathsf{r}_{\mathsf{h}} \mathrel{\hat{=}}$ **hopping**

Question: Is (FKPP) derived from microscopic dynamics?

Is (FKPP) a (generalized) gradient flow?

Agenda

Configuration space

Jump processes on configuration spaces

Towards the Liouville equation

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Configuration space

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Towards the Liouville equation

Configuration Spaces

For each $N \in \mathbb{N}_0$, define

$$\mathbb{X}^N := \left\{ (x^1, \ldots, x^N) \in X^N \ : \ x^i \neq x^j, \ j \neq i \right\} / \sim$$

where $(x^1, \ldots, x^N) \sim (x^{\sigma_1}, \ldots, x^{\sigma_N})$ for every permutation σ

Reference measure $\quad \pi \in \mathscr{M}^+(X) \longrightarrow \dfrac{\pi^{\otimes N}}{N!} \in \mathscr{M}^+(\mathbb{X}^N)$

Configuration space $\quad \mathbb{X} := \bigsqcup_{N \in \mathbb{N}_0} \mathbb{X}^N \qquad \Lambda := e^{-\pi(X)} \sum_{N \in \mathbb{N}_0} \dfrac{\pi^{\otimes N}}{N!} \in \mathscr{P}(\mathbb{X})$

Normalised Lebesgue-Poisson

Eg. **Guichardet** 1972, **Obata** 1987, **Albeverio-Kondratiev-Röckner** 1998

Configuration Spaces

Configuration space

$$\mathbb{X} := \bigsqcup_{N\in\mathbb{N}_0} \mathbb{X}^N \qquad \Lambda := e^{-\pi(X)} \sum_{N\in\mathbb{N}_0} \frac{\pi^{\otimes N}}{N!} \in \mathscr{P}(\mathbb{X})$$

Normalised Lebesgue-Poisson

Observations

1) $\mathbf{x} \in \mathbb{X}^N \longleftrightarrow \gamma = \{x^1, \dots, x^N\} \overset{\mathbb{L}}{\longleftrightarrow} \nu = \sum_{x\in\gamma} \delta_x \in \mathscr{M}^+(X)$

2) $\Pi := \mathbb{L}_\# \Lambda \in \mathscr{P}(\mathscr{M}^+(X))$

3) $\mathbb{E}\left[\mathrm{N}(\mathbf{x})\right] = \int_{\mathbb{X}} \mathrm{N}(\mathbf{x})\, \Lambda(d\mathbf{x}) = \pi(X) = O(1)$

Eg. **Guichardet** 1972, **Obata** 1987, **Albeverio-Kondratiev-Röckner** 1998

Configuration Spaces

Configuration space

$$\mathbb{X} := \bigsqcup_{N \in \mathbb{N}_0} \mathbb{X}^N \qquad \Lambda^n := e^{-n\pi(X)} \sum_{N \in \mathbb{N}_0} \frac{(n\pi)^{\otimes N}}{N!} \in \mathscr{P}(\mathbb{X})$$

Scaled Normalised Lebesgue-Poisson

Observations

1) $\mathbf{x} \in \mathbb{X}^N \longleftrightarrow \gamma = \{x^1, \dots, x^N\} \overset{\mathbb{L}^n}{\longleftrightarrow} \nu = \frac{1}{n} \sum_{x \in \gamma} \delta_x \in \mathscr{M}^+(X)$

2) $\Pi^n := \mathbb{L}^n_\# \Lambda^n \in \mathscr{P}(\mathscr{M}^+(X))$

3) $\mathbb{E}^n \left[\mathsf{N}(\mathbf{x}) \right] = \int_{\mathbb{X}} \mathsf{N}(\mathbf{x}) \, \Lambda^n(d\mathbf{x}) = n\pi(X) = O(n)$

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Configuration Spaces

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2) $\Pi^n := \mathbb{L}^n_\# \Lambda^n \in \mathscr{P}(\mathscr{M}^+(X))$

3) $\mathbb{E}^n\big[|\nu|\big] = \int_{\mathscr{M}^+(X)} |\nu|\, \Pi^n(d\nu) = \pi(X)$

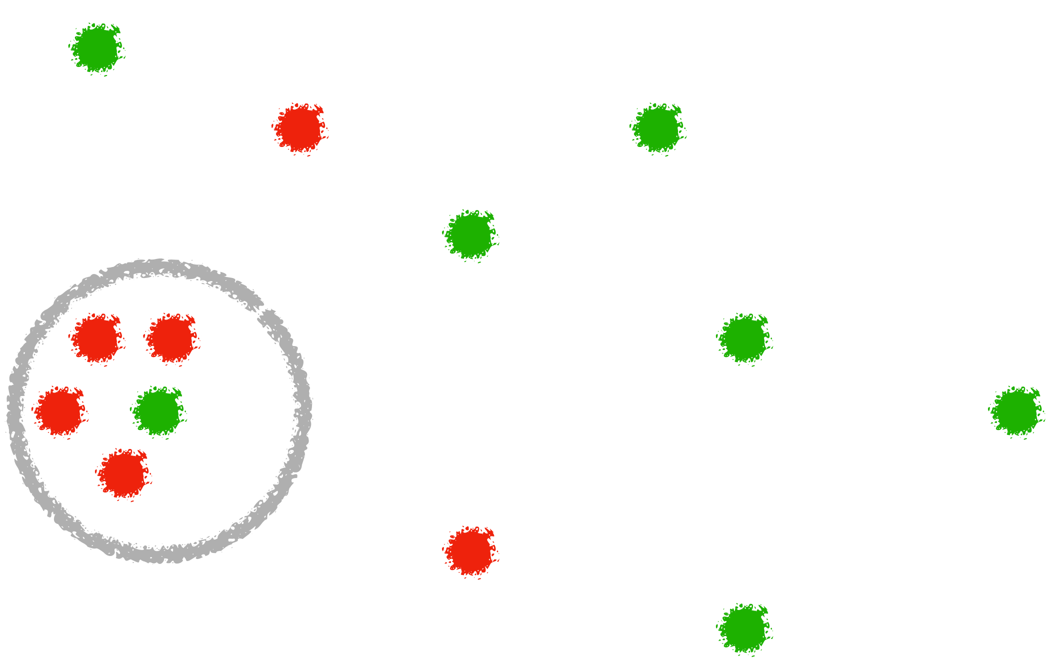
Eg. **Guichardet** 1972, **Obata** 1987, **Albeverio-Kondratiev-Röckner** 1998

Configuration Spaces

Multicomponent Configurations

$$\mathbb{Y} := \bigsqcup_{\mathbf{N} \in \mathbb{N}_0^2} \mathbb{Y}^{\mathbf{N}} \qquad \mathbb{Y}^{\mathbf{N}} := \mathbb{X}^{N_1} \times \mathbb{X}^{N_2}$$

$$\mathbb{Y}_{\neq} := \left\{ \boldsymbol{\gamma} \in \mathbb{Y} \;:\; \boldsymbol{\gamma} = (\gamma_1, \gamma_2),\;\; \gamma_1 \cap \gamma_2 = \varnothing \right\}$$



Configuration Spaces

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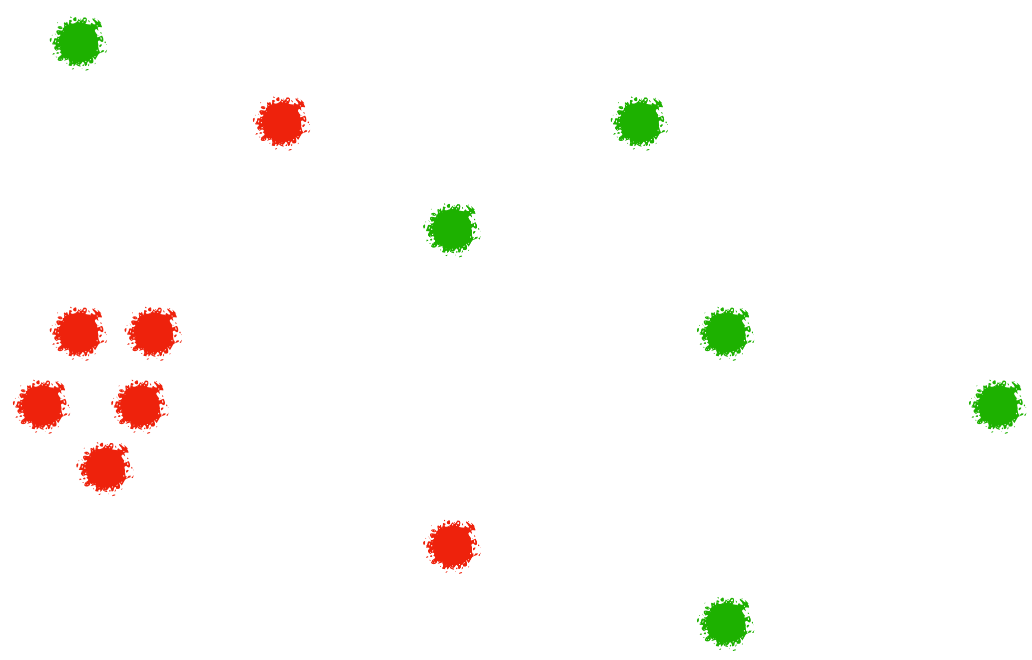


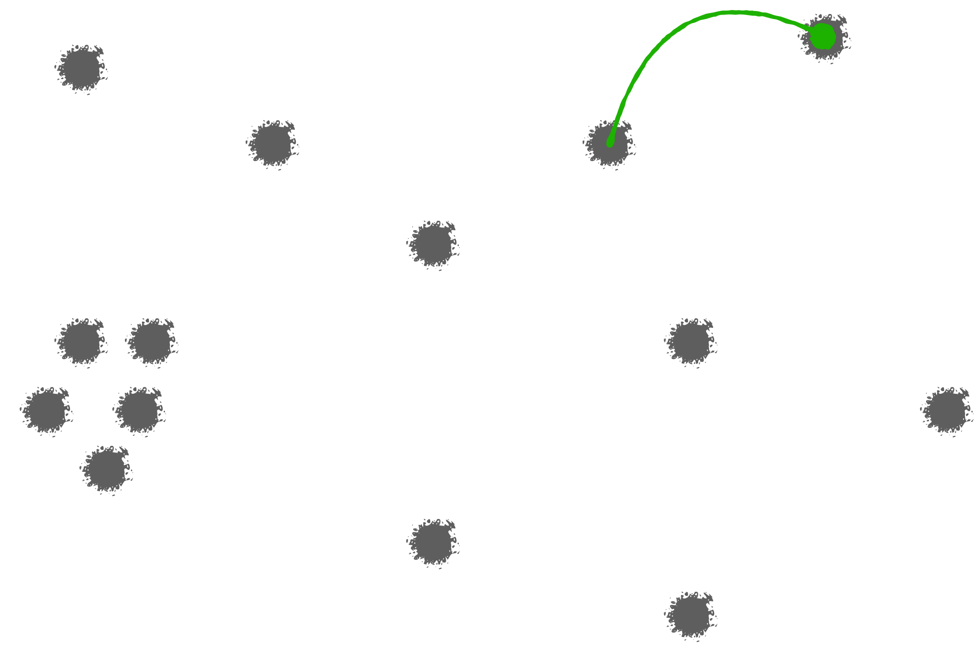
Agenda

Configuration space

Jump processes on configuration spaces

Towards the Liouville equation

Jump process



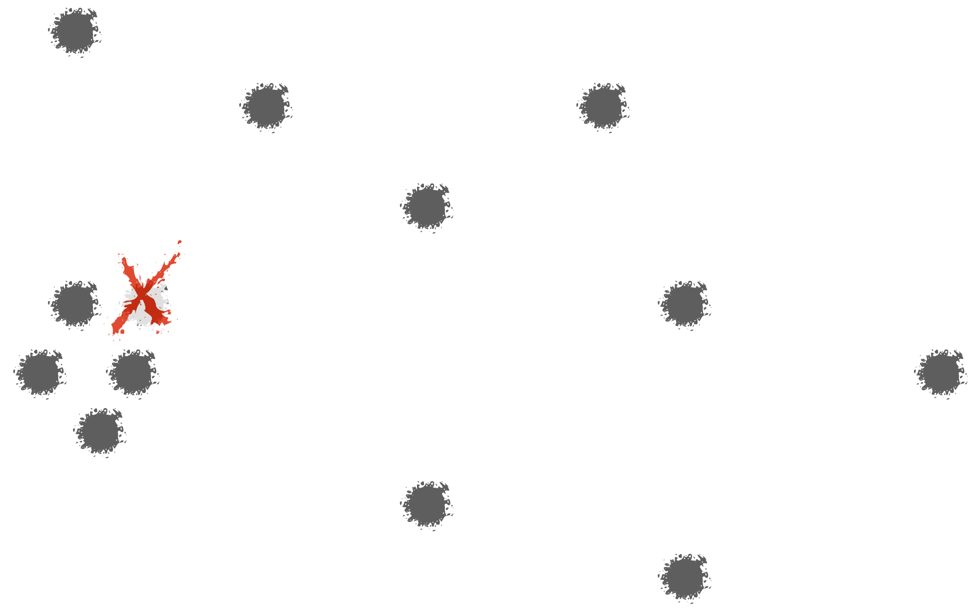
Birth $\nu \mapsto \nu + \frac{1}{n}\delta_x$

with rate $n\,\mathsf{r}_{\mathsf{b}}[\nu](dx)$

Generator

$$L^n_{\mathsf{b}}F(\nu) := \int_X n\left[F\left(\nu + \frac{1}{n}\delta_x\right) - F(\nu)\right]\mathsf{r}_{\mathsf{b}}[\nu](dx)$$

Jump process



Birth $\nu \mapsto \nu + \frac{1}{n}\delta_x$

with rate $n\, \mathrm{r_b}[\nu](dx)$

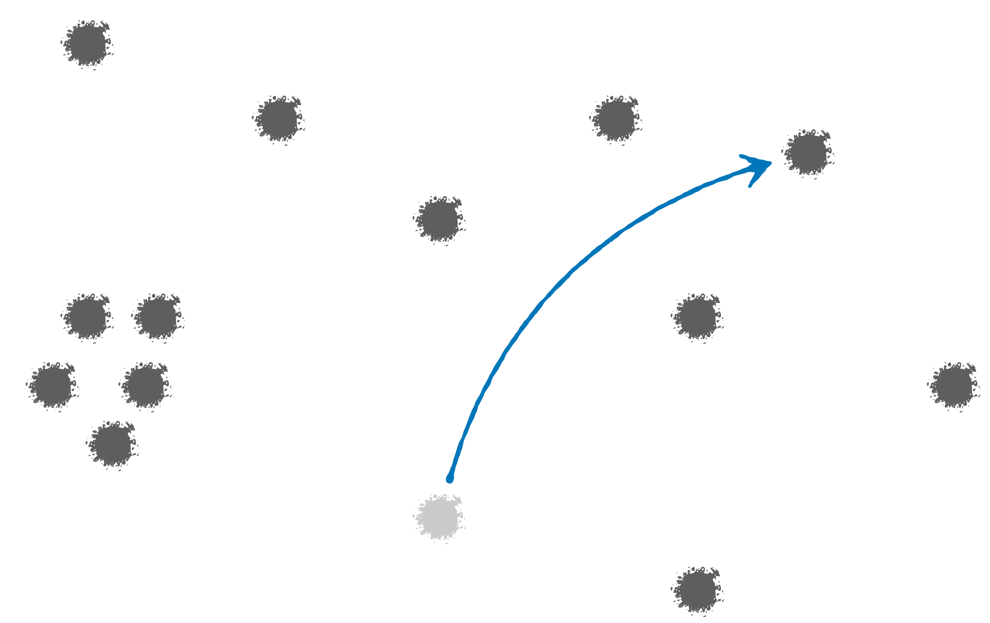
Death $\nu \mapsto \nu - \frac{1}{n}\delta_x$

with rate $n\, \mathrm{r_d}[\nu](x)\, \nu(dx)$

Generator

$$L_{\mathrm{d}}^{n} F(\nu) := \int_X n\Big[F\Big(\nu - \frac{1}{n}\delta_x\Big) - F(\nu)\Big]\, \mathrm{r_d}[\nu](x)\, \nu(dx)$$

Jump process



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with rate $n\, \mathrm{r_b}[\nu](dx)$

Death $\nu \mapsto \nu - \frac{1}{n}\delta_x$

with rate $n\, \mathrm{r_d}[\nu](x)\, \nu(dx)$

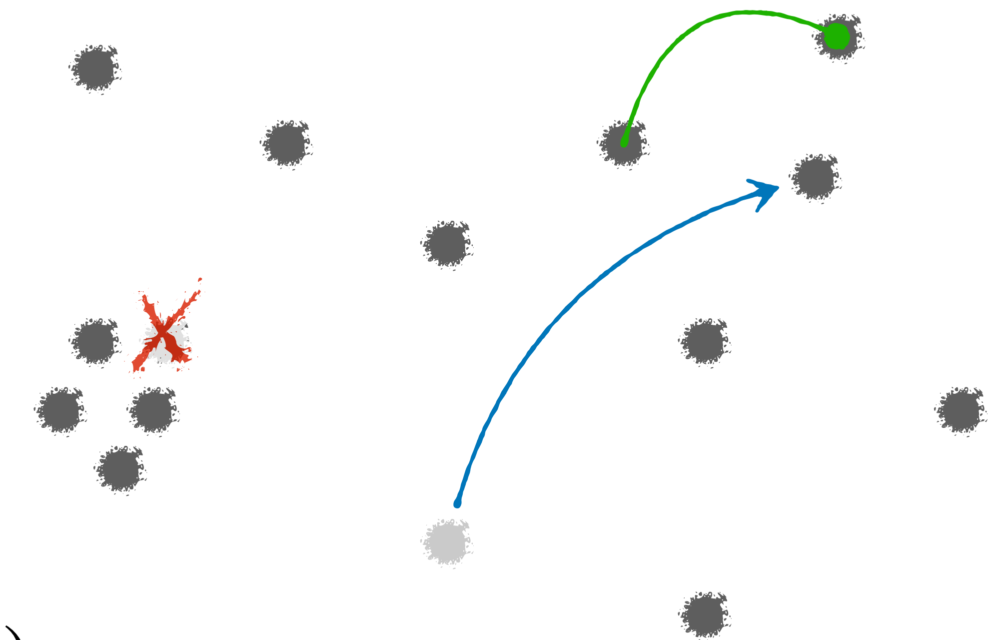
Hopping $\nu \mapsto \nu - \frac{1}{n}\delta_x + \frac{1}{n}\delta_y$

with rate $n\, \mathrm{r_h}[\nu](x, dy)\, \nu(dx)$

Generator

$$L^n_{\mathrm{h}} F(\nu) := \iint_{X\times X} n\Big[F\Big(\nu - \tfrac{1}{n}\delta_x + \tfrac{1}{n}\delta_y\Big) - F(\nu)\Big]\, \mathrm{r_h}[\nu](x, dy)\, \nu(dx)$$

Jump process



Generator $L^n := L^n_{\mathsf{b}} + L^n_{\mathsf{d}} + L^n_{\mathsf{h}}$

$$L^n F(\nu) = \int_{\mathscr{M}^+(X)} \big[F(\eta) - F(\nu)\big]\kappa^n(\nu, d\eta)$$

generates the process $t \mapsto \nu_t \in D([0,T]; \mathscr{M}^+(X))$

where $\mathsf{P}_t :=$ **Law** ν_t **satisfies**

Gradient structure $\Longleftrightarrow$ Detailed balance

Mielke-Peletier-Renger 2014

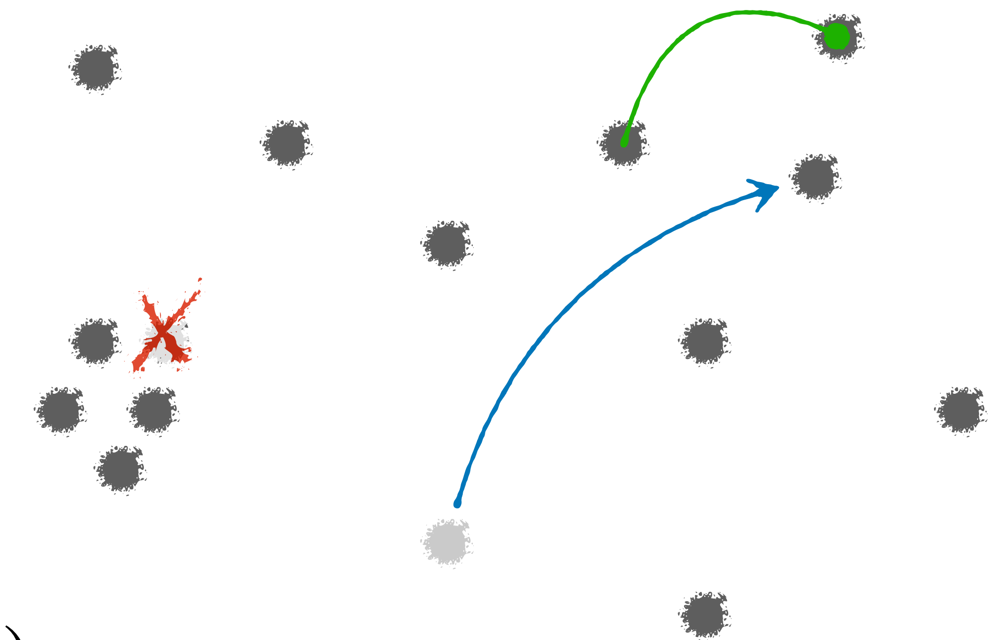
(FKE$_n$) $\partial_t \mathsf{P}_t = (L^n)^* \mathsf{P}_t \;\; = -\,\overline{\mathbf{div}}_{\mathscr{M}}\, \mathsf{J}_t$

$\kappa^n(\nu, d\eta)\,\Pi^n(d\nu)$ **is symmetric**

$$\mathsf{J}_t = \kappa^n(\nu, d\eta)\,\mathsf{P}_t(d\nu) = \mathsf{P}_t \odot \kappa^n$$

Forward Kolmogorov

Jump process



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$$\mathsf{J}_t = \partial_2 \mathscr{R}^*_n\big(\mathsf{P}_t, -\overline{\nabla}_{\mathscr{M}} \mathscr{E}_n(\mathsf{P}_t)\big)$$

$$\mathscr{E}_n(\mathsf{P}) = n^{-1}\, \mathrm{Ent}(\mathsf{P} \,|\, \Pi^n)$$

Forward Kolmogorov

Peletier-Rossi-Savaré-T. 2022

Jump process

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$$\mathscr{E}_n(\mathsf{P}) = n^{-1}\, \mathrm{Ent}(\mathsf{P} \,|\, \Pi^n)$$

Variational Principle

$$(\mathsf{P}, \mathsf{J}) \in \textbf{argmin} \left\{ \mathscr{L}_n(\mathsf{Q}, \mathsf{G}) \;:\; \partial_t \mathsf{Q}_t + \overline{\mathbf{div}}_{\mathscr{M}} \, \mathsf{G}_t = 0, \;\; \mathsf{Q}_0 = \mathsf{P}_0 \right\}$$

where

$$0 \le \mathscr{L}_n(\mathsf{P}, \mathsf{J}) := \int_0^T \mathscr{R}_n(\mathsf{P}_t, \mathsf{J}_t) + \mathscr{D}_n(\mathsf{P}_t)\, dt + \mathscr{E}_n(\mathsf{P}_T) - \mathscr{E}_n(\mathsf{P}_0) \le 0$$

Chain rule Inequality

Generalized Minimizing-movement

Peletier-Rossi-Savaré-T. 2022

Jump process

Gradient structure ⇒ detailed balance

Mielke-Peletier-Renger 2014

$\kappa^n(\nu, d\eta)\,\Pi^n(d\nu)$ **symmetric**

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Peletier-Rossi-Savaré-T. 2022

Jump process

~~Gradient structur[illegible] ⇒ [illegible]iled balance~~

~~Mielke-Pelet[illegible]nger 2014~~

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where

$$\mathscr{L}_n(\mathsf{P}, \mathsf{J}) := \int_0^T \mathrm{Ent}\Big(\mathsf{J}_t \,|\, \mathsf{P}_t \odot \kappa^n\Big)\, dt \;\; \leq 0$$

Large deviations

Approximations +
Lower semicontinuity

Agenda

Configuration space

Jump processes on configuration spaces

Large population limit

Towards Louville

$$\partial_t \mathsf{P}_t^n + \overline{\mathbf{div}}_{\mathscr{M}}\, \mathsf{J}_t^n = 0$$

$$\mathscr{E}_n(\mathsf{P}) = n^{-1}\,\mathsf{Ent}(\mathsf{P}\,|\,\Pi^n)$$

Recall

$$L_{\mathsf{b}}^n F(\nu) = \int_X \overbrace{n\Big[F\Big(\nu + \frac{1}{n}\delta_x\Big) - F(\nu)\Big]}^{=\,\overline{\nabla}_{\mathsf{b}}^n F(\nu,x)}\, \mathsf{r}_{\mathsf{b}}[\nu](dx)$$

Cylindrical functions $\quad \mathbf{Cyl}_c(\mathscr{M}^+(X))$

$$F(\nu) = g\big(\langle f_1,\nu\rangle, \ldots, \langle f_m,\nu\rangle\big)$$

$$g \in C_c^\infty(\mathbb{R}^m)$$

$$f_l \in C(X), \;\; l = 1,\ldots,m$$

Large population limit

$$\overline{\nabla}_{\mathsf{b}}^n F(\nu,x) \quad \longrightarrow \quad \sum_{i=1}^m \partial_i g\big(\langle f_1,\nu\rangle, \ldots, \langle f_m,\nu\rangle\big)\, f_i(x) =: \nabla_{\mathscr{M}} F(\nu,x)$$

$$L_{\mathsf{b}}^n F(\nu) \quad \longrightarrow \quad \int_X \nabla_{\mathscr{M}} F(\nu,x)\, \mathsf{r}_{\mathsf{b}}[\nu](dx)$$

Towards Louville

$$\partial_t \mathsf{P}_t^n + \overline{\mathbf{div}}_{\mathscr{M}}\, \mathsf{J}_t^n = 0$$

$$\mathscr{E}_n(\mathsf{P}) = n^{-1}\,\mathsf{Ent}(\mathsf{P}\,|\,\Pi^n)$$

Recall

$$L_{\mathsf{d}}^n F(\nu) = \int_X \overbrace{n\Big[F\big(\nu - \tfrac{1}{n}\delta_x\big) - F(\nu)\Big]}^{= \overline{\nabla}_{\mathsf{d}}^n F(\nu, x)}\, \mathsf{r}_{\mathsf{d}}[\nu](x)\,\nu(dx)$$

Cylindrical functions $\mathbf{Cyl}_c(\mathscr{M}^+(X))$

$$F(\nu) = g\big(\langle f_1, \nu\rangle, \ldots, \langle f_m, \nu\rangle\big)$$

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Large population limit

$$\overline{\nabla}_{\mathsf{d}}^n F(\nu, x) \;\longrightarrow\; -\nabla_{\mathscr{M}} F(\nu, x)$$

$$L_{\mathsf{d}}^n F(\nu) \;\longrightarrow\; -\int_X \nabla_{\mathscr{M}} F(\nu, x)\, \mathsf{r}_{\mathsf{d}}[\nu](x)\,\nu(dx)$$

Towards Louville

$$\partial_t \mathsf{P}^n_t + \overline{\textbf{div}}_{\mathscr{M}}\, \mathsf{J}^n_t = 0$$

$$\mathscr{E}_n(\mathsf{P}) = n^{-1}\,\mathsf{Ent}(\mathsf{P}\,|\,\Pi^n)$$

Recall

$$L^n_{\mathsf{h}} F(\nu) = \iint_{X\times X} \overbrace{n\Big[F\big(\nu - \tfrac{1}{n}\delta_x + \delta_y\big) - F(\nu)\Big]}^{=\,\overline{\nabla}^n_{\mathsf{h}} F(\nu,x,y)}\, \mathsf{r}_{\mathsf{h}}[\nu](x,dy)\,\nu(dx)$$

Cylindrical functions

$\textbf{Cyl}_c(\mathscr{M}^+(X))$

$$F(\nu) = g\big(\langle f_1,\nu\rangle,\ldots,\langle f_m,\nu\rangle\big)$$

$$g \in C_c^\infty(\mathbb{R}^m)$$

$$f_l \in C(X),\;\; l = 1,\ldots,m$$

Large population limit

$$\overline{\nabla}^n_{\mathsf{h}} F(\nu,x,y) \quad\longrightarrow\quad \nabla_{\mathscr{M}} F(\nu,y) - \nabla_{\mathscr{M}} F(\nu,x)$$

$$L^n_{\mathsf{h}} F(\nu) \quad\longrightarrow\quad \iint_{X\times X} \big[\nabla_{\mathscr{M}} F(\nu,y) - \nabla_{\mathscr{M}} F(\nu,x)\big]\,\mathsf{r}_{\mathsf{h}}[\nu](x,dy)\,\nu(dx)$$

Large population limit

$$\partial_t \mathsf{P}_t^n + \overline{\mathbf{div}}_{\mathscr{M}}\, \mathsf{J}_t^n = 0$$

$$\mathscr{E}_n(\mathsf{P}) = n^{-1}\,\mathrm{Ent}(\mathsf{P}\,|\,\Pi^n)$$

Large population limit $\quad \mathscr{V}[\nu] \triangleq \kappa_{\mathsf{bd}}[\nu] + \kappa_{\mathsf{h}}[\nu]$

$$L^n F(\nu) \longrightarrow \int_X \nabla_{\mathscr{M}} F(\nu, x)\,\mathscr{V}[\nu](dx) \qquad \forall\, F \in \mathbf{Cyl}_c(\mathscr{M}^+(X))$$

Result: The sequence of solutions $(\mathsf{P}^n)_{n\in\mathbb{N}}$ to **(FKE$_n$)** with initial data $\mathsf{P}_0 \in \mathscr{P}(\mathscr{M}^+(X))$ such that

$$\mathbf{sup}_{n\in\mathbb{N}}\mathscr{E}_n(\mathsf{P}_0) < +\infty$$

has a (narrow) accumulation point P satisfying

$$\textbf{(Li)} \qquad \partial_t \mathsf{P}_t + \mathbf{div}_{\mathscr{M}}(\mathscr{V}\mathsf{P}_t) = 0 \qquad \text{Liouville}$$

where

$$\mathbf{sup}_{t\in[0,T]}\mathscr{E}_\infty(\mathsf{P}_t) < +\infty, \qquad \mathscr{E}_\infty(\mathsf{P}) = \int_{\mathscr{M}^+(X)} \mathbf{Ent}(\nu\,|\,\pi)\,\mathsf{P}(d\nu)$$

Large population limit

$$\partial_t \mathsf{P}_t^n + \overline{\mathbf{div}}_{\mathscr{M}}\, \mathsf{J}_t^n = 0$$

$$\mathscr{E}_n(\mathsf{P}) = n^{-1}\,\mathrm{Ent}(\mathsf{P}\,|\,\Pi^n)$$

Result: The sequence of solutions $(\mathsf{P}^n)_{n\in\mathbb{N}}$ to **(FKE$_n$)** with initial data $\mathsf{P}_0 \in \mathscr{P}(\mathscr{M}^+(X))$ such that

$$\mathbf{sup}_{n\in\mathbb{N}}\mathscr{E}_n(\mathsf{P}_0) < +\infty$$

has a (narrow) accumulation point P satisfying

$$\textbf{(Li)} \qquad \partial_t \mathsf{P}_t + \mathbf{div}_{\mathscr{M}}(\mathscr{V}\mathsf{P}_t) = 0$$

Liouville

where

$$\mathbf{sup}_{t\in[0,T]}\mathscr{E}_\infty(\mathsf{P}_t) < +\infty, \qquad \mathscr{E}_\infty(\mathsf{P}) = \int_{\mathscr{M}^+(X)} \mathbf{Ent}(\nu\,|\,\pi)\,\mathsf{P}(d\nu)$$

If $\mathcal{S}_t : \mathscr{M}^+(X) \to \mathscr{M}^+(X)$ is the flow map for **(FKPP)** $\partial_t\mu_t = \mathscr{V}[\mu_t]$, then

$$\mathsf{P}_t = (\mathcal{S}_t)_\#\mathsf{P}_0 \qquad \textbf{for all } \; t\in[0,1]$$

Superposition Principle

In particular, if $\mathsf{P}_0 = \delta_\nu$ with $\mathrm{Ent}(\nu_0\,|\,\pi) < +\infty$, then $\mathsf{P}_t = \delta_{\mathcal{S}_t(\nu)}$

Large population limit

$$\partial_t \mathsf{P}_t^n + \overline{\mathbf{div}}_{\mathscr{M}}\, \mathsf{J}_t^n = 0$$

$$\mathscr{E}_n(\mathsf{P}) = n^{-1}\, \mathrm{Ent}(\mathsf{P}\,|\,\Pi^n)$$

Result: The sequence of solutions $(\mathsf{P}^n)_{n\in\mathbb{N}}$ to **(FKE$_n$)** with initial data $\mathsf{P}_0 \in \mathscr{P}(\mathscr{M}^+(X))$ such that

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where

$$\mathbf{sup}_{t\in[0,T]} \mathscr{E}_\infty(\mathsf{P}_t) < +\infty, \qquad \mathscr{E}_\infty(\mathsf{P}) = \int_{\mathscr{M}^+(X)} \mathbf{Ent}(\nu\,|\,\pi)\,\mathsf{P}(d\nu)$$

If **Detailed Balance** is satisfied then the GGF structure on **(FKE$_n$)** induces a GGF structure on **(FKPP)** via *Evolutionary Γ-convergence*

Sandier-Serfaty 2004

Hoeksema-T. 2023

Wrap Up

(FKPP) $\partial_t \mu_t = \mathscr{V}[\mu_t]$

Flow map

$\mathcal{S}_t : \mathscr{M}^+(X) \to \mathscr{M}^+(X)$

Entropic Propagation of Chaos

$\lim_{n\to\infty} \mathbf{Ent}(\mathsf{P}_t^n \,|\, \Pi_{\mu_t}^n) = 0$

(Li) $\partial_t \mathsf{P}_t + \mathbf{div}_{\mathscr{M}}\left(\mathscr{V}\mathsf{P}_t\right) = 0$

(FKE$_n$) $\partial_t \mathsf{P}_t^n + \overline{\mathbf{div}}_{\mathscr{M}}\, \mathsf{J}_t^n = 0$

$$\nu_t = \frac{1}{n}\sum_{x\in\gamma}\delta_x$$

$n \to \infty$

Work in progress

Marked configurations

Singular (detailed balance)

Infinite configurations

'Central limit'

Thank you!