Landau-Ginzburg Theory of Magnets

- Statistical mechanics of an equilibrium system is given by the partition function:

$$\mathcal{Z} = \sum_{\{\mu\}} e^{-\beta \mathcal{H}(\mu)} \quad ; \ \mu \equiv \text{Microstate}.$$

- Standard prescription:

① Write down the microscopic hamiltonian $\mathcal{H}$. For example, for a classical model of ferromagnets

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \bar{S}_i \cdot \bar{S}_j$$

- i, j are lattice indices.
- $\langle i, j \rangle$ means j is one of the nearest neighbors of i.

② Write down the partition f^n:

$$\mathcal{Z} = \sum_{\{\mu\}} e^{\beta J \sum_{\langle ij \rangle} \bar{S}_i \cdot \bar{S}_j}$$

- The microstate is given by the configurations $\{\bar{S}_1, \bar{S}_2, \dots, \bar{S}_N\}$ of the spins in the system.
- For example, for Ising model $S_i = \pm 1$
- For the XY-model $S_i = (\cos\theta_i, \sin\theta_i)$, $\theta_i \in [0, 2\pi)$. In this case, the sum can be replaced by integrals over θ_i.

③ The free energy is obtained by the relation:

$$\mathcal{Z} = e^{-\beta F}$$

$$\text{Or,} \quad F = -K_B T \ln \mathcal{Z}.$$

Field theory.

- In general, it is not simple to calculate the free energy for a large macroscopic system from the microscopic hamiltonian.

- Landau - Prescription.

  - Landau proposed that at the macroscopic level, a system can be described by simpler effective free energies that follows certain conditions.

① Order parameter : The free energy is a functional of the order parameter field:

$$F = \int d^d\bar{r}\ f[\phi(r)] \ ; \ f \text{ is the free energy density.}$$

② SYMMETRY : The macroscopic free energy must have the same symmetries as the microscopic hamiltonian.

- For Ising model, flipping the sign of spin i, keeps the Hamiltonian invariant, when $H=0$.

$$\mathcal{H} = -J\sum_{\langle ij\rangle} S_i S_j$$

$$S_i \rightarrow -S_i \ \ \& \ \ S_j \rightarrow -S_j$$

$$\mathcal{H}' = -J\sum_{\langle ij\rangle}(-S_i)(-S_j)$$
$$= -J\sum_{\langle ij\rangle} S_i S_j$$
$$= \mathcal{H}.$$

- For $O(n)$ model [$n=2$ is XY model]

$$\mathcal{H} = -J\sum_{\langle ij\rangle} \bar{S}_i \cdot \bar{S}_j$$

This Hamiltonian is symmetric w.r.t n-dimensional rotations R_n

$$\bar{S}_i' = R_n \bar{S}_i \qquad \bar{S}_j' = R_n \bar{S}_j$$

$$\bar{S}_i' \cdot \bar{S}_j' = (R_n \bar{S}_i)\cdot(R_n \bar{S}_j)$$

- Demonstration for $n=2$

$$R_2 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$R_2 \bar{s}_i = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} S_{i,x} \\ S_{i,y} \end{bmatrix} = \begin{bmatrix} \cos\theta\, S_{i,x} - \sin\theta\, S_{i,y} \\ \sin\theta\, S_{i,x} + \cos\theta\, S_{i,y} \end{bmatrix}$$

$$\therefore (R_2 \bar{s}_i)\cdot(R_2 \bar{s}_j) = (R_2 \bar{s}_i)^T (R_2 s_j)$$

$$= \{ (\cos\theta\, S_{i,x} - \sin\theta\, S_{i,y})(\cos\theta\, S_{j,x} - \sin\theta\, S_{j,y}) + (\sin\theta\, S_{i,x} + \cos\theta\, S_{i,y})(\sin\theta\, S_{j,x} + \cos\theta\, S_{j,y}) \}$$

$$= (\cos^2\theta\, S_{i,x} S_{j,x} + \sin^2\theta\, S_{i,y} S_{j,y} - \cos\theta \sin\theta\, \cancel{S_{i,y} S_{j,x}} - \cos\theta\, \cancel{\sin\theta}\, S_{i,x} S_{j,y}) + (\sin^2\theta\, S_{i,x} S_{j,x} + \cos^2\theta\, S_{i,y} S_{j,y} + \cos\theta\, \cancel{\sin\theta}\, S_{i,y} S_{j,x} + \cos\theta\, \cancel{\sin\theta}\, S_{i,x} S_{j,y})$$

$$= S_{i,x} S_{j,x} + S_{i,y} S_{j,y}$$

$$= \bar{s}_i \cdot \bar{s}_j$$

- Therefore, the Hamiltonian is invariant w.r.t to global rotation of the spins.

>> The macroscopic free energy has to obey this symmetry, which implies that it can be a f^n of scalars only.

$$f[\varphi] \equiv f[\{|\bar{\varphi}|, \varphi^2, \dots, |\nabla\varphi|, (\nabla\varphi)^2, \dots]$$

(3) <u>ANALYTICITY</u>: The free energy is an analytic f^n almost everywhere. Therefore, it can be expressed as a convergent power series of order parameter and its derivatives

$$f[\varphi(\bar{r})] = f_0 + a_1|\bar{\Phi}(\bar{r})| + a_2 \bar{\Phi}^2(\bar{r}) + \cdots + b_1|\nabla\bar{\Phi}(r)| + b_2(\nabla\Phi)^2 + \cdots$$

- The $|\Phi(\bar{r})|$ & $|\nabla\Phi(r)|$ terms are not analytic. Hence, we remove them from the power series.

$$f[\bar{\varphi}(\bar{r})] = f_0 + a_2\Phi^2(\bar{r}) + \cdots + b_2(\nabla\Phi)^2 + \cdots$$

④ <u>LOCALITY</u> : The interaction between microscopic constituents is sufficiently short-ranged such that f can be expressed in terms of the first few derivatives only.

$$f[\varphi(\bar{r})] = f_0 + a_1\varphi^2(\bar{r}) + a_2\varphi^4(\bar{r}) + \cdots + b_2(\nabla\varphi)^2$$

⑤ <u>SMALL ϕ</u> : Finally, we also make the assumption that the order parameter is small such that the power series in ϕ can be truncated after the first few terms; which gives the Ginzburg-Landau free energy. Conventionally, it is written as:

$$f[\varphi] = \frac{r}{2}\phi^2 + u\phi^4 + \frac{c}{2}(\nabla\phi)^2$$

⑩ Signs <u>of the terms</u>: The coefficients r, u, & c are f^ns of temp. T.

$$r = r_0(T-T_c) + r_1(T-T_c)^2 + \cdots$$

$$u = u_0 + u_1(T-T_c) + \cdots$$

$$c = c_0 + c_1(T-T_c) + \cdots$$

- r can change sign as a f^n of temperature.
- $u > 0$ to ensure that $f[\varphi]$ does not diverge to $-\infty$ for $\phi \to \infty$.
- If $u < 0$ then higher order terms are needed.
- $c > 0$ to ensure that <u>homogeneous</u> state is the equilibrium state
- c can be less than 0, but not for the simple magnets that we are studying here.

P.T.O.

Homogeneous equilibrium state : If $c > 0$, spatial variation in the order parameter field $\bar{\varphi}(\bar{r})$ will increase free energy.

- Hence, the equilibrium state is going to be homogeneous with the order parameter taking a constant value φ_0.

* The free energy functional transforms to a function of φ_0. & the minimum of f gives the value of φ_0.

$$f = \frac{r}{2}\phi_0^2 + u\phi_0^4$$

At minimum, $\frac{\partial f}{\partial \phi_0} = 0 \Rightarrow r\phi_0 + 4u\phi_0^3 = 0$

$$\Rightarrow \phi_0 = 0 \quad \text{or} \quad \pm\sqrt{\frac{-r}{4u}}$$

- $r \sim r_0(T-T_c)$

  » If $T > T_c$, $r > 0$; only solution is $\phi_0 = 0$

  » If $T \leq T_c$, $r \leq 0$; Free energy is minimized at $\pm\sqrt{\frac{-r}{4u}}$.

Correlation. & Susceptibility

* We would like to calculate the correlation f^n

$$G(\bar{r}, \bar{r}') = \langle \bar{\phi}(\bar{r}) \cdot \bar{\phi}(\bar{r}') \rangle$$

- We can calculate it directly or by using the fact that G is related to the susceptibility χ.

* To see this relationship, let's consider the Ising model

$$\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} s_i s_j - \sum_i H_i s_i$$

$$Z = \sum_{\{s_i\}} e^{-\beta\mathcal{H}}$$

$$= \sum_{\{s_i\}} \exp\left(\sum_{\langle ij \rangle} \beta J_{ij} s_i s_j + \sum_i \beta H_i s_i\right)$$

$$\frac{\partial Z}{\partial(\beta H_i)} = \sum_{\{S_i\}} S_i \, e^{-\beta \mathcal{H}}$$

$$\Rightarrow \frac{1}{Z}\frac{\partial Z}{\partial(\beta H_i)} = \frac{1}{Z}\sum_{\{S_i\}} S_i \, e^{-\beta\mathcal{H}} = \langle S_i \rangle$$

Similarly,

$$\frac{\partial^2 Z}{\partial(\beta \mathcal{H}_j)\,\partial(\beta\mathcal{H}_i)} = \sum_{\{S_i\}} S_i S_j \, e^{-\beta \mathcal{H}_j}$$

$$\therefore \frac{\partial}{\partial(\beta H_j)}\left(\frac{1}{Z}\frac{\partial Z}{\partial \beta \mathcal{H}_i}\right) = \frac{1}{Z}\frac{\partial^2 Z}{(\partial\beta H_j)(\partial\beta H_i)} - \frac{1}{Z^2}\frac{\partial Z}{\partial\beta H_j}\frac{\partial Z}{\partial \beta H_i}$$

$$\therefore \underbrace{\frac{\partial \langle S_i\rangle}{\partial(\beta H_j)}}_{\chi(i,j)} = \underbrace{\langle S_i S_j\rangle - \langle S_i\rangle\langle S_j\rangle}_{G(i,j)}$$

$$\Rightarrow \frac{1}{\beta}\chi(i,j) = G(i,j)$$

$$\therefore \boxed{\chi(i,j) = \beta \, G(i,j)}$$

We can similarly show that

$$\chi(\bar{r},\bar{r}') = \beta \, G(\bar{r},\bar{r}')$$

- When the system is translationally invariant:

$$\boxed{\chi(\bar{r}-\bar{r}') = \beta \, G(\bar{r}-\bar{r}')}$$

Therefore, finding correlation f^n boils down to finding the susceptibility.

Q: How do we find susceptibility from the free energy?

A: We first note that GL free energy is not Helmholtz free energy or Gibbs free energy. $\varphi = m$, which is a generalized displacement.

- To get GL free energy, we do a Legendre transform.

$$f(\phi(\bar{r}_i), T) = f_H(H(\bar{r}_i), T) + \sum_i H(\bar{r}_i)\,\phi(\bar{r}_i)$$

• Define $H_i \equiv H(\bar{r}_i)$ $\phi_i \equiv \phi(\bar{r}_i)$

$$\therefore f(\phi_i, T) = f_H(H_i, T) + \sum_i H_i \phi_i$$

$$\therefore df = df_H + \sum_i H_i d\phi_i + \sum_i \phi_i dH_i$$

$$\sum_j \frac{\partial f}{\partial \phi_j} d\phi_j = \sum_j \frac{\partial f_H}{\partial H_j} dH_j + \sum_j H_j d\phi_j + \sum_j \phi_j dH_j$$

$$\Rightarrow \frac{\partial f}{\partial \phi_j} = H_j \quad . \text{ Hence } \frac{\partial H_j}{\partial \phi_i} = \frac{\partial^2 f}{\partial \phi_i \partial \phi_j}$$

Since $\phi_i \equiv m_i$ $\frac{\partial H_j}{\partial \phi_i} = \chi_{ij}^{-1}$

$$\therefore \boxed{\chi_{ij}^{-1} = \frac{\partial^2 f}{\partial \phi_i \partial \phi_j}}$$

• Similarly, it can be shown that (for $n=1$)

$$\chi^{-1}(\bar{r}, \bar{r}') = \frac{\delta^2 f}{\delta\phi(\bar{r})\,\delta\phi(\bar{r}')},$$ where $\frac{\delta f}{\delta \phi(\bar{r})}$ is the function derivative.

$$f = \frac{r}{2}\phi^2(\bar{r}) + u\phi^4(\bar{r}) + \frac{c}{2}(\nabla\phi(\bar{r}))^2$$

$$\Rightarrow \frac{\delta f}{\delta\phi(\bar{r})} = r\phi(\bar{r}) + 4u\phi^3(\bar{r}) - c\nabla^2\phi(\bar{r})$$

$$\therefore \frac{\delta^2 f}{\delta\phi(\bar{r})\,\delta\phi(\bar{r})} = (r + 12u\phi^2 - c\nabla^2)\,\delta(\bar{r}-\bar{r}') = \chi^{-1}(\bar{r}-\bar{r}')$$

Hence, $$\beta G(\bar{r}-\bar{r}') = \frac{1}{(r+12u\phi^2 - c\nabla^2)\,\delta(\bar{r}-\bar{r}')}$$

• Because of translational invariance, we can fourier transform

$$\chi^{-1}(\bar{q}) = r + 12u\phi^2 + cq^2$$

$$\therefore \chi(\bar{q}) = \frac{1}{r + 12u\phi^2 + cq^2}$$

Susceptibility & Correlation in O(n) model.

So far, we have considered $n=1$ model, e.g. Ising model. We now consider $n=2$ model which has a component parallel to the spontaneously chosen direction & a direction perpendicular to it.

$$\bar{\varphi} = \varphi_0 \hat{e}_1$$

- Let's consider fluctuations about the ordered state

$$\bar{\varphi} = \varphi_0 \hat{e}_1 + \varphi_0 (\varphi_{\parallel} \hat{e}_1 + \varphi_{\perp} \hat{e}_2) \qquad \varphi_{\parallel}, \varphi_{\perp} \ll 1$$

- Substitute these expression for $\bar{\varphi}$ in the expression for the free energy and retain terms upto quadratic order in $\varphi_{\parallel}$ & $\varphi_{\perp}$.

  - You should get:

$$F[\phi_{\parallel}, \phi_{\perp}] = F_0 + \int d^d\bar{r} \, [\tfrac{1}{2}(r\phi_0^2 + 12u\phi_0^4)\,\phi_{\parallel}^2(\bar{r}) + \tfrac{c}{2}(\nabla\phi_{\parallel}(\bar{r}))^2 + \tfrac{1}{2}(r\phi_0^2 + 4u\phi_0^4)\,\phi_{\perp}^2(\bar{r}) + \tfrac{c}{2}(\nabla\phi_{\perp}(\bar{r}))^2$$

$$F_0 = V\left(\tfrac{1}{2} r\phi_0^2 + u\phi_0^4\right)$$

- Also, note that $\chi^{-1}_{\alpha\beta}(\bar{r}, \bar{r}') = \dfrac{\delta^2 F}{\delta\phi_\alpha(\bar{r})\,\delta\phi_\beta(\bar{r}')}$

$$\Rightarrow F = \int d^d\bar{r}\, d^d\bar{r}'\, \chi^{-1}_{\alpha\beta}(\bar{r}, \bar{r}')\, \phi_\alpha(\bar{r})\, \phi_\beta(\bar{r}') + \cdots$$

- In our case,

$$F = \int d^d\bar{r}\, d^d\bar{r}' \, [\phi_0^2\, \chi^{-1}_{\parallel}(\bar{r}, \bar{r}')\, \phi_{\parallel}(\bar{r})\, \phi_{\parallel}(\bar{r}') + \chi^{-1}_{\perp}(\bar{r}, \bar{r}')\, \phi_0^2\, \phi_{\perp}\phi_{\perp} + F_0] + \cdots$$

- What are $\phi_{\parallel}$ & $\phi_{\perp}$ in terms of $\bar{\phi}$? To understand this question assume that the order parameter points along some arbitrary direction $\hat{e} = (e_1, e_2, \ldots)$

$$\phi_0 \phi_{\parallel} = \bar{\phi} \cdot \hat{e} \qquad \& \qquad \phi_0 \phi_{\perp} = \bar{\phi} - (\bar{\phi}\cdot\hat{e})\hat{e}$$

$$\therefore \quad \phi_0^2 \phi_{\parallel}\phi_{\parallel} = (\vec{\phi}\cdot\hat{e})(\bar{\phi}\cdot\hat{e})$$

$$= \sum_{\alpha\beta} \phi_\alpha \phi_\beta e_\alpha e_\beta$$

$$\phi_0^2\, \bar{\phi}_\perp \cdot \bar{\phi}_\perp = \left[\bar{\phi}(\bar{r}) - (\bar{\phi}(\bar{r}).\hat{e})\hat{e}\right]\cdot\left[\bar{\phi}(\bar{r}') - (\bar{\phi}(\bar{r}').\hat{e})\,\hat{e}\right]$$

$$= \varphi(\bar{r})\cdot\varphi(\bar{r}') - 2(\bar{\varphi}(\bar{r})\cdot\hat{e})^2 + (\varphi(\bar{r}).\hat{e})^2$$

$$= \bar{\varphi}(\bar{r})\cdot\bar{\varphi}(\bar{r}') - (\varphi_{\parallel})^2$$

$$= \sum_\alpha \varphi_\alpha \varphi_\alpha - \sum_{\alpha\beta} \varphi_\alpha \varphi_\beta e_\alpha e_\beta$$

$$= \sum_{\alpha\beta} (\varphi_\alpha \varphi_\beta \delta_{\alpha\beta} - \varphi_\alpha \varphi_\beta e_\alpha e_\beta)$$

$$= \sum_{\alpha\beta} \varphi_\alpha \varphi_\beta (\delta_{\alpha\beta} - e_\alpha e_\beta)$$

$$\therefore\ F = \int d^d\bar{r}\, d^d\bar{r}' \left[\chi_\parallel^{-1}(\bar{r},\bar{r}')\, e_\alpha e_\beta + \chi_\perp^{-1}(\bar{r},\bar{r}')\,(\delta_{\alpha\beta} - e_\alpha e_\beta)\right]\phi_\alpha \phi_\beta$$

$$\therefore\ \chi_{\alpha\beta}^{-1}(\bar{r},\bar{r}') = \chi_\parallel^{-1}(\bar{r},\bar{r}')\, e_\alpha e_\beta + \chi_\perp^{-1}(\bar{r},\bar{r}')\,(\delta_{\alpha\beta} - e_\alpha e_\beta)$$

Also.
$$\chi_{\alpha\beta}^{-1} = \frac{\delta F}{\delta\phi_\alpha(\bar{r})\,\delta\phi_\beta(\bar{r}')}$$

- Now, let's take $n=2$

$$F = F_0 + \int d^d\bar{r} \cdot \left[\tfrac{1}{2}(r + 12u\,\phi_0^2)\,\phi_0^2\phi_\parallel^2 + \tfrac{1}{2}(r + 4u\,\phi_0^2)\,\phi_0^2\phi_\perp^2 + \tfrac{c}{2}(\nabla\phi_0\phi_\parallel)^2 + \tfrac{c}{2}(\nabla\phi_0\phi_\perp)^2\right]$$

$$\chi_\parallel^{-1}(\bar{r},\bar{r}') = \frac{1}{\phi_0^2}\frac{\delta^2 F}{\delta\phi_\parallel(\bar{r})\,\delta\varphi_\parallel(\bar{r}')}$$

$$= (r + 12u\,\phi_0^2 - c\nabla^2)\,\delta(\bar{r}-\bar{r}')$$

$$\Rightarrow\ \chi_\parallel^{-1}(\bar{q}) = (r + 12u\,\phi_0^2 + cq^2)$$

$$\chi_\perp(\bar{r},\bar{r}') = \frac{\delta^2 F}{\delta\phi_\perp(\bar{r})\,\delta\phi_\perp(\bar{r}')}\frac{1}{\phi_0^2} = (r + 4u\,\phi_0^2 - c\nabla^2)\,\delta(\bar{r}-\bar{r}')$$

$$\therefore\ \chi_\perp(\hat{q}) = r + 4u\phi_0^2 + cq^2$$

We also have.

$$G_{\parallel}(\bar{q}) = K_B T\, \chi_{\parallel}(\bar{q}) \;\&\; G_{\perp}(\bar{q}) = K_B T\, \chi_{\perp}(\bar{q})$$

$$\begin{aligned}\chi_{\parallel}^{-1}(\bar{q}) &= r + cq^2 \quad T > T_c \\ &= -2r + cq^2 \quad T \leq T_c\end{aligned}$$

$$\therefore \lim_{q\to 0} \chi_{\parallel}^{-1}(\bar{q}) = r \,,\; T > T_c$$
$$= -2r \,.\; T < T_c$$

$$\begin{aligned}\chi_{\perp}^{-1}(\bar{q}) &= r + cq^2 \quad T > T_c \\ &= cq^2 \quad T < T_c\end{aligned}$$

$$\Rightarrow \lim_{q\to 0} \chi_{\perp}^{-1}(\bar{q}) = r \,,\; T > T_c$$
$$= 0 \,,\; T < T_c$$

- Recall that $F_{\perp} = \int d^d\bar{r}'\, d^d\bar{r}\; \chi_{\perp}^{-1}(\bar{r}-\bar{r}')\, \phi_0^2\, \phi_{\perp}(\bar{r})\, \phi_{\perp}(\bar{r}')$

Fourier transforming the R.H.S gives.

$$F_{\perp} = \chi_{\perp}^{-1}(\bar{q})\, \phi_0^2 |\phi_{\perp}(q)|^2$$

- Therefore long length scale ($q \to 0$) fluctuations in directions $\perp$ to the order param directions cost no free energy.
  - They are "gapless" or "massless".
- The fluctuations along the $\parallel$ directions are "gapped" or "massive".

- Let's look at the correlation f^n for the $\perp$ fluctuations. ($T \leq T_c$)

$$G_{\perp}(\bar{q}) = K_B T\, \chi_{\perp}(\bar{q}) = \frac{K_B T}{cq^2}$$

$$\therefore G_{\perp}(r) \sim \int \frac{d^d q}{cq^2} e^{i\bar{q}\cdot\bar{r}} = \int \frac{q^{d-1}\,dq}{cq^2}\, d\Omega\, e^{iqr\cos\theta}$$
$$\sim r^{2-d}\,(\cdot) \quad \text{if } d \neq 2$$
$$\sim \ln r\,(\cdot) \quad \text{if } d = 2$$

Functional Derivative

1. Functional Derivative

Definition:

Let $F\{\eta(\bar{r})\}$ be a functional of $\eta(\bar{r})$. The functional derivative of F w.r.t. η is defined as:

$$\frac{\delta F}{\delta \eta(\bar{r}')} = \lim_{\epsilon\to 0} \frac{F\{\eta(\bar{r})+\epsilon\,\delta(\bar{r}-\bar{r}')\} - F\{\eta(\bar{r})\}}{\epsilon} \qquad (1)$$

\# Show that:

(1) $\delta/\delta\eta(r) \int d^d r'\, \eta(\bar{r}') = 1$ Using the def^n (1), we have for $F\{\eta(\bar{r})\} \equiv \int d^d\bar{r}\, \eta(\bar{r})$:

$$\delta F/\delta\eta(\bar{r}) = \int dr' [\eta(\bar{r}') + \epsilon\,\delta(\bar{r}'-\bar{r}) - \eta(r')]/\epsilon$$
$$= \int d\bar{r}'\, \delta(\bar{r}'-\bar{r}) = 1$$

(2) $\delta\eta(\bar{r}')/\delta\eta(\bar{r}) = \delta(\bar{r}-\bar{r}')$ $\Rightarrow F\{\eta(\bar{r})\} = \eta(\bar{r})$

$$\lim_{\epsilon\to 0} [\eta(r') + \epsilon\,\delta(\bar{r}'-\bar{r}) - \eta(\bar{r}')]/\epsilon = \delta(\bar{r}'-\bar{r})$$

(3) $\frac{\delta}{\delta\eta(\bar{r})} \int d^d\bar{r}' \frac{1}{2}(\nabla\eta(\bar{r}'))^2 = -\nabla^2\eta(\bar{r})$ $\Rightarrow F\{\eta(\bar{r})+\epsilon\,\delta(\bar{r}'-\bar{r})\} = \int d^d\bar{r}'\, \frac{1}{2}[\nabla(\eta(\bar{r}')+\epsilon\,\delta(\bar{r}'-\bar{r}))]^2$

$$= \int d^d\bar{r}' \frac{1}{2}[(\nabla\eta)^2 + 2\epsilon\,\nabla\eta(\bar{r}')\nabla\delta(\bar{r}'-\bar{r}) + O(\epsilon^2)]$$

$$\therefore \frac{\delta F}{\delta\eta(r)} = \lim_{\epsilon\to 0}[F\{\eta(r')+\epsilon\,\delta(\bar{r}'-\bar{r})\} - F\{\eta(\bar{r}')\}]/\epsilon$$
$$= \int d^d\bar{r}'\, \nabla\eta(\bar{r}')\nabla\delta(\bar{r}'-\bar{r})$$
$$= \nabla\eta(r')\,\delta(\bar{r}'-\bar{r})\Big|_{\text{Surface}} - \int \delta(\bar{r}'-r)\nabla^2\eta(\bar{r}')\, d^d r'$$
$$= -\nabla^2\eta(\bar{r})$$

(4) $\frac{\delta}{\delta\eta(r)} \int d^d\bar{r}'\, |\bar{\eta}(\bar{r}')|^n = n\,\bar{\eta}(\bar{r})\, |\bar{\eta}(\bar{r})|^{n-2}$

$$F\{\bar{\eta}(\bar{r})\} = \int d^d\bar{r}' |\bar{\eta}(\bar{r}')|^n = \int d^d\bar{r}' \, (\bar{\eta}(\bar{r})\cdot\bar{\eta}(\bar{r}))^{n/2}$$

$$F\{\bar{\eta}(\bar{r}) + \bar{\epsilon}\,\delta(\bar{r}'-\bar{r})\} = \int d^d\bar{r}' \left[(\bar{\eta}(\bar{r}') + \bar{\epsilon}\,\delta(\bar{r}-\bar{r}'))\cdot(\bar{\eta}(\bar{r}') + \bar{\epsilon}\,\delta(\bar{r}-\bar{r}'))\right]^{n/2}$$
$$= \int d^d\bar{r}' \left[|\bar{\eta}(\bar{r}')|^2 + 2\bar{\epsilon}\cdot\bar{\eta}(\bar{r}')\delta(\bar{r}'-\bar{r}) + \dots\right]^{n/2}$$
$$\simeq \int d^d\bar{r}'\, |\bar{\eta}(\bar{r}')|^n \left(1 + n\bar{\epsilon}\cdot\frac{\bar{\eta}(\bar{r}')\delta(\bar{r}'-\bar{r})}{|\eta(\bar{r}')|^2}\right)$$
$$= \int d^d\bar{r}' \left[|\eta(\bar{r}')|^n + n\,\bar{\epsilon}\cdot\bar{\eta}(\bar{r}')\,\delta(\bar{r}'-\bar{r})\, |\eta(\bar{r}')|^{n-2}\right]$$

To calculate the functional derivative take $\bar{\epsilon}$ along a given direction, say x direction $\bar{\epsilon} = \epsilon\hat{x}$, then

$$\frac{\delta F}{\delta\bar{\eta}(r)} = n\,\eta_x(\bar{r})\, |\eta(r)|^{n-2}$$

Repeating this analysis for other directions, we find that, in general.

$$\boxed{\frac{\delta F}{\delta\bar{\eta}(\bar{r})} = n\,\bar{\eta}(\bar{r})\,|\eta(r)|^{n-2}}$$

$\Rightarrow \frac{\delta\int |\bar{P}|^2 d\bar{r}'}{\delta\bar{P}} = 2\bar{P}$ $\quad \frac{\delta\int |P|^4 dr'}{\delta\bar{P}} = 4\bar{P}\,|P|^2 \quad \dots$

⑤ $F\{\bar\eta(\bar r)\} = \int |\bar\eta|^2 \nabla\cdot\bar\eta \; d^d\bar r \quad \Rightarrow \quad \frac{\delta F}{\delta\bar\eta(\bar r)} = 2\bar\eta\,\nabla\cdot\bar\eta - \nabla|\bar\eta|^2$

$$F\{\bar\eta(\bar r)+\bar\epsilon\,\delta(\bar r'-\bar r)\} = \int \Big[\big(\eta(\bar r')+\bar\epsilon\,\delta(\bar r'-\bar r)\big)\cdot\big(\eta(\bar r')+\bar\epsilon\,\delta(\bar r'-\bar r)\big)\Big]\Big[\nabla\cdot\bar\eta + \nabla\cdot\big(\bar\epsilon\,\delta(\bar r'-\bar r)\big)\Big]\, d^d\bar r'$$

$$\simeq \int \big[|\eta(\bar r')|^2 + 2\bar\epsilon\,\delta(\bar r'-\bar r)\cdot\bar\eta(\bar r')\big]\big[\nabla\cdot\eta + \bar\epsilon\cdot\nabla\delta(\bar r'-\bar r)\big]\, d^d\bar r'$$

$$\simeq \int \big[|\bar\eta|^2\nabla\cdot\eta + 2(\bar\epsilon\cdot\bar\eta)\,\nabla\cdot\eta\;\delta(\bar r'-\bar r) + \bar\epsilon\cdot\nabla\delta(\bar r'-\bar r)\,|\eta|^2\big]\, d^d r'$$

$$\therefore\; F\{\eta(r')+\bar\epsilon\,\delta(\bar r'-\bar r)\} - F\{\eta(r')\} = 2\int \bar\epsilon\cdot\bar\eta\;\nabla\cdot\eta\;\delta(\bar r'-\bar r)\, d^d\bar r' + \int \bar\epsilon\cdot\nabla\delta(\bar r'-\bar r)\,|\eta|^2$$

Take $\bar\epsilon = \epsilon\hat x$ $\quad \therefore\; \frac{\delta F}{\delta\eta} = 2\int \eta_x\,\nabla\cdot\eta\;\delta(\bar r'-\bar r) + \int \partial_x\delta(\bar r'-\bar r)\,|\eta|^2$

$$= [2\bar\eta\,\nabla\cdot\eta - \nabla|\eta|^2]_x .$$

$$\therefore\; \frac{\delta F}{\delta\bar\eta} = 2\bar\eta\,\nabla\cdot\bar\eta - \nabla|\bar\eta|^2$$

⑥ $F\{\bar\eta\} = \int_{\bar r'} \nabla\cdot\bar\eta\;\delta\rho \quad \Rightarrow \quad \frac{\delta F}{\delta\bar\eta} = -\nabla\delta\rho$

$F\{\bar\eta+\bar\epsilon\,\delta(\bar r'-\bar r)\} = \int_{r'} \nabla\cdot\big(\bar\eta+\bar\epsilon\,\delta(\bar r'-\bar r)\big)\,\delta\rho . \qquad \int_{r'} \equiv \int d^d\bar r'$

$$\therefore\; F\{\eta+\bar\epsilon\,\delta(\bar r-\bar r')\} - F\{\eta\} = \int_{r'} \nabla\cdot\big(\bar\epsilon\,\delta(\bar r'-\bar r)\big)\,\delta\rho = \int_{r'} \bar\epsilon\cdot\nabla\delta(\bar r'-\bar r)\,\delta\rho .$$

$\bar\epsilon = \epsilon\hat x$

$$\therefore\; \left(\frac{\delta F}{\delta\bar\eta}\right)_x = \int_{r'} \nabla\delta(\bar r'-\bar r)\,\delta\rho = -\nabla\delta\rho = -\nabla\rho \qquad \left[\because\; \delta\rho = \rho-\rho_0 \;\&\; \rho_0 = \text{Const}\right]$$

APPLICATION

Free energy functional

$$F_p = \int_{\bar r}\left[\frac{\tilde\alpha(\rho)}{2}|\bar p|^2 + \frac{\tilde\beta}{4}|\bar p|^4 + \frac{\tilde K}{2}(\nabla\bar p)^2 + \frac{w}{2}|\bar p|^2\nabla\cdot\bar p - w_1\nabla\cdot\bar p\,\frac{\delta\rho}{\rho_0} + \frac{A}{2}\left(\frac{\delta\rho}{\rho_0}\right)^2\right]$$

$$\frac{\delta F}{\delta\bar p} = \tilde\alpha(\rho)\bar p + \tilde\beta|\bar p|^2\bar p - \tilde K\nabla^2\bar p + w\,\bar p(\nabla\cdot\bar p) - \frac{w}{2}\nabla|\bar p|^2 + \frac{w_1}{\rho_0}\nabla\delta\rho$$

$$= [\tilde\alpha + \tilde\beta|p|^2]\bar p - \tilde K\nabla^2\bar p + w\,\bar p(\nabla\cdot\bar p) - \frac{w}{2}\nabla|\bar p|^2 + w_1\nabla(\rho/\rho_0)$$

• $\partial_t\bar p + \lambda(\bar p\cdot\nabla)\bar p = -\frac{1}{\gamma}\,\delta F/\delta\bar p + \bar f = -\frac{1}{\gamma}(\tilde\alpha+\tilde\beta|p|^2)\bar p + \frac{\tilde K}{\gamma}\nabla^2\bar p - \frac{w_1}{\gamma}\nabla(\rho/\rho_0) + \frac{w}{\gamma}\left(\frac12\nabla|p|^2 - \bar p(\nabla\cdot\bar p)\right) + \bar f$

$$\therefore\; \partial_t\bar p + \lambda(\bar p\cdot\nabla)\bar p = -(\alpha+\beta|\bar p|^2)\bar p + K\nabla^2\bar p - v_1\nabla(\rho/\rho_0) + \frac{\lambda}{2}\nabla|p|^2 - \lambda\bar p(\nabla\cdot\bar p) + \bar f$$

• $\frac{\delta F}{\delta\rho} = \frac{\delta}{\delta\rho}\int_{r'}\left[-w_1\nabla\cdot\bar p\,\frac{\rho}{\rho_0} + \frac{A}{2\rho_0^2}[\rho^2 - 2\rho\rho_0]\right] = -w_1\frac{\nabla\cdot\bar p}{\rho_0} - \frac{A}{\rho_0} + \frac{A\rho}{\rho_0^2} = -\frac{w_1\nabla\cdot\bar p}{\rho_0} + \frac{A}{\rho_0}\left(1-\frac{\rho}{\rho_0}\right)$

$$= -\frac{w_1\nabla\cdot\bar p}{\rho_0} + \frac{A}{\rho_0^2}\delta\rho$$

$$\therefore\; \frac{\delta F}{\delta\rho} = -\frac{w_1\,\nabla\cdot\bar p}{\rho_0} + \frac{A}{\rho_0^2}\delta\rho$$