Opinion Dynamics for Agents with Resource Limitations

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Decisions, Games, and Evolution
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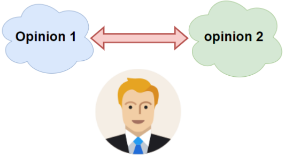
Opinion Dynamics

Consensus

Clustering

Polarization



Oscillations

Social Power

Opinion manipulation

Opinion and Influence Modeling

Opinion of an agent on a single topic is a scalar $x \in \mathbb{R}$

$x < 0$	$x = 0$	$x > 0$
Opposing Opinion	**Neutral Opinion**	**Favoring Opinion**

Magnitude denotes the extent to which the agent favors or opposes the topic

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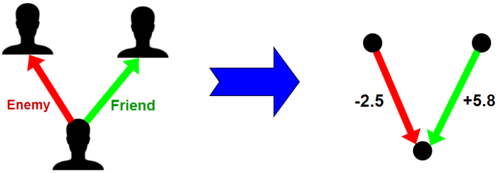
- *Modeling of social connections among the agents using a graph.*



Social ties ***Directed signed graph***

Literature Review

- Consensus: French-DeGroot Model (1974), Abelson's Model (1964).

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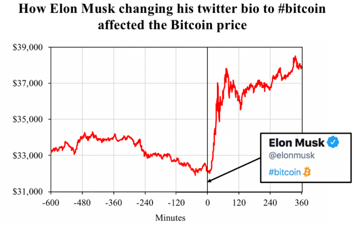
- Utility Maximization and Game theoretic approach: Ghaderi and Srikant (2014), Etesami et.al (2019), Etesami and Basar (2015), Etesami (2022), Park et.al (2022).

Motivation for Our Work

- Resources influence the opinions and social power.



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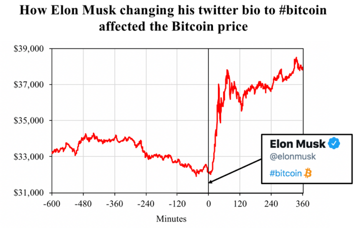
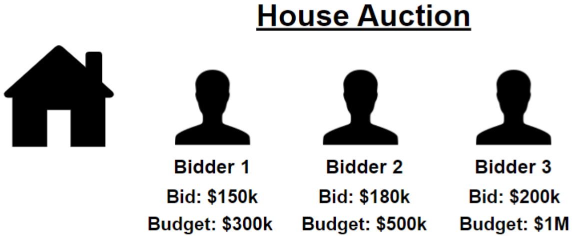


- Resources restrict extreme opinions.



Problem Formulation

- $\mathcal{A} := \{1, \cdots, n\}$: Set of agents discussing on a single topic.
- $z_i(t) \in \mathbb{R}$: Opinion of $i \in \mathcal{A}$ at time t.
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- $w_i \in \mathbb{R}_{>0}$: Importance that $i \in \mathcal{A}$ attaches to its preference *(stubbornness)*.
- $a_{ik} \in \mathbb{R}$: Influence weight of k on i.
  - $a_{ik} > 0$: k is a *friend* of i.
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$$U_i(\mathbf{z}) = \frac{-w_i}{2}\left(z_i - p_i\right)^2 - \sum_{k \in \mathcal{A}} \frac{a_{ik}}{2}\left(z_k - z_i\right)^2 - \frac{1}{4r_i} z_i^4$$

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  - penalizes extreme opinions.

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$$\dot{\mathbf{z}}_i := S_i(z_i) + C_i(\mathbf{z}), \forall i \in \mathcal{A}, \qquad (OD)$$

where

$$S_i(z_i) := -w_i \left[z_i - p_i \right] - \frac{z_i^3}{r_i} \quad (\mathit{Self}\ ),$$

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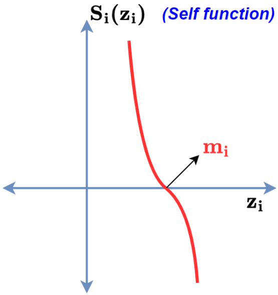

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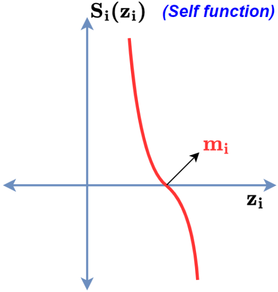
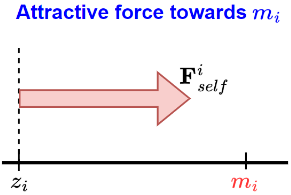


$\mathbf{S_i}(.)$ is strictly ↓ and $\mathbf{S_i(m_i) = 0}$

- $\mathcal{E}$: Set of equilibria of OD.
- $m_i \in \mathbb{R}$ is the opinion agent i would hold, if i was "isolated" from social influence.

Force analogy

- Force due to $S_i(z_i)$:



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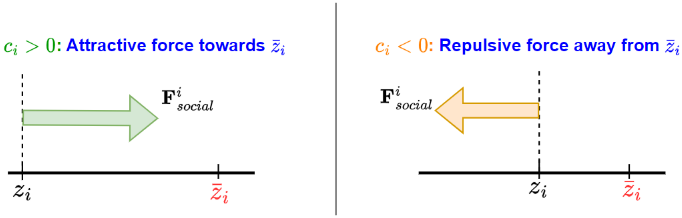

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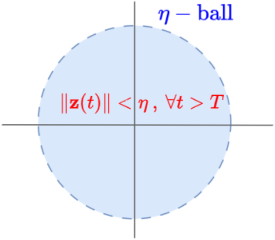
Theorem (Ultimate Boundedness)

Under OD, the opinions are ultimately bounded.



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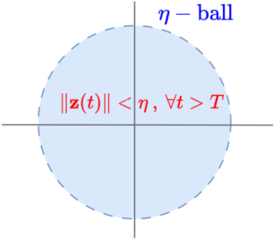

Theorem (Existence and Uniqueness of Equilibrium)

Let $\mathcal{N}_i^e$ denote the set of enemies of i. Suppose

$$w_i > \sum_{k \in \mathcal{N}_i^e} 2|a_{ik}|\ ;\ \forall i \in \mathcal{A} \quad \textit{(WAR: Weak antagonistic relations)}$$

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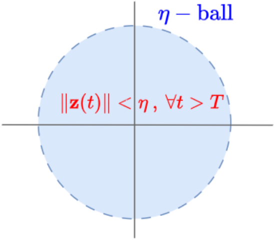

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Then, there exists an unique GES equilibrium point $\mathbf{z}^* \in \mathcal{E}$.

Ultimate Bound (No Antagonistic Relations)

- Let $\mathcal{M} := [m_{min}, m_{max}]$. Also define $\mathcal{V}_{max} := \{i \in \mathcal{A} \mid m_i = m_{\max}\}$ and $\mathcal{V}_{min} := \{i \in \mathcal{A} \mid m_i = m_{\min}\}$.

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- $\mathbf{z}(t)$ converges to positive invariant set $\mathcal{M}^n$.
- $\mathbf{z}(t)$ converges to interior of $\mathcal{M}^n$ in finite time iff every agent in $\mathcal{V}_{max}$ and $\mathcal{V}_{min}$ is directly/indirectly influenced by at least one agent in $\mathcal{A} \setminus \mathcal{V}_{max}$ and $\mathcal{A} \setminus \mathcal{V}_{min}$ resp.
  - *This condition is satisfied if the social network is strongly connected.*

Consensus

Theorem (Existence of Consensus Equilibrium)

For OD, $\alpha\mathbf{1} \in \mathcal{E}$ if and only if $m_i = \alpha$, $\forall i \in \mathcal{A}$.

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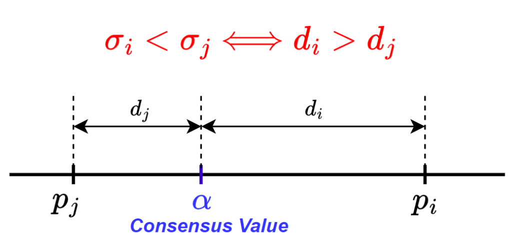
Lemma (Dominance at Consensus)

For each $i \in \mathcal{A}$, define (dominance weight) $\sigma_i := w_i r_i$. Then for any $i, j \in \mathcal{A}$,



Opinion Game: $\mathcal{G} = \langle \mathcal{A}, (\mathbb{R})_{i \in \mathcal{A}}, (U_i)_{i \in \mathcal{A}} \rangle$

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Definition (Nash Equilibrium or NE)

$\mathbf{z}^* = (z_i^*, \mathbf{z}_{-i}^*) \in \mathbb{R}^n$ is a NE if $\forall i \in \mathcal{A}$,

$$U_i(z_i^*, \mathbf{z}_{-i}^*) = \max_{z_i \in S_i} U_i(z_i, \mathbf{z}_{-i}^*)$$

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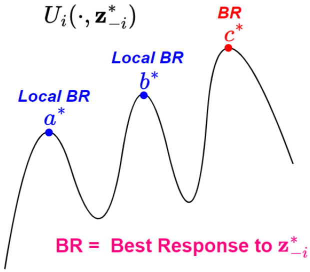

Definition (Local Nash Equilibrium or LNE)

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$$U_i(z_i^*, \mathbf{z}_{-i}^*) \geq U_i(z_i, \mathbf{z}_{-i}^*), \;\forall\, z_i \text{ s.t. } |z_i^* - z_i| \leq \rho_i$$

- $\mathcal{NE}$: Set of NE of $\mathcal{G}$.
- $\mathcal{NE}_\ell$: Set of LNE of $\mathcal{G}$.

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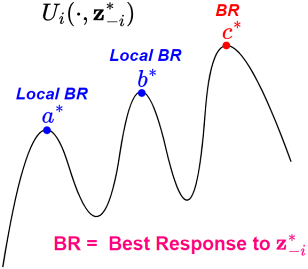

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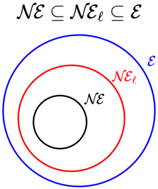

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Nash Equilibria

Theorem

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$$\tau_i := \frac{-r_i}{3}\left[w_i + \sum_{k \in \mathcal{A}/\{i\}} a_{ik}\right] =: \frac{-r_i d_i^{in}}{3}. \text{ Then,}$$

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$$\tau_i := \frac{-r_i}{3}\left[w_i + \sum_{k \in \mathcal{A}/\{i\}} a_{ik}\right] =: \frac{-r_i d_i^{in}}{3}. \text{ Then,}$$

- $\mathbf{z}^* \in \mathcal{NE}_\ell$ only if $(z_i^*)^2 \geq \tau_i, \ \forall i \in \mathcal{A}$.
- If the above inequality is strict, then $\mathbf{z}^* \in \mathcal{NE}_\ell$.
- If $\exists\, i \in \mathcal{A}$ such that $\tau_i > \eta^2$, then $\mathcal{NE}_\ell = \varnothing$.
- If $d_i^{in} \geq 0$ then $\mathcal{NE} = \mathcal{NE}_\ell = \mathcal{E}$.
- If WAR holds then $\mathcal{NE} = \mathcal{NE}_\ell = \mathcal{E}$ and $|\mathcal{NE}| = 1$.

Nash Equilibria

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  - *Opinions under OD converge to this unique Nash Eq.*

Price of Anarchy (PoA) when $\mathcal{N}_i^e = \varnothing$ and $p_i \neq 0\,,\ \forall i \in \mathcal{A}$

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Definition (Price of Anarchy)

$$\mathsf{PoA} := \frac{\max_{\mathbf{z} \in \mathcal{NE}} \mathsf{Obj}(\mathbf{z})}{\min_{\mathbf{z} \in \mathbb{R}^n} \mathsf{Obj}(\mathbf{z})} \geq 1.$$

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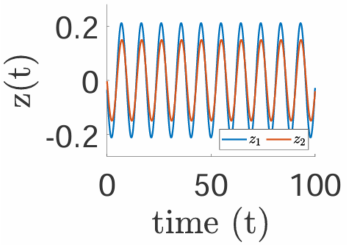
- *Everyone is satisfied with the outcome!*

Periodic Behavior of Opinions: Two Agent Dynamics



- The model could exhibit oscillations in presence of contrarians.

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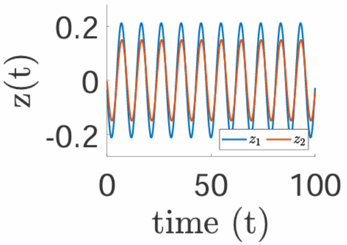

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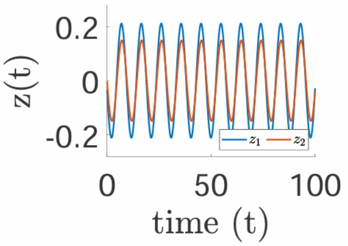

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- A sufficient condition for Hopf Bifurcation to exist.

Summary

- OD derived out of the utility
- Limited resources prevent extreme opinions
- Game theoretic analysis
- Ongoing work: Multi-topic OD under hard resource constraints

Prashil Wankhede, Nirabhra Mandal, Sonia Martinez, PT, IEEE Transactions on Control of Network Systems 2025

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