1. Lattice polytopes, namely convex polytopes with vertices in the lattice $M := \mathbb{Z}^n$, can be used to construct local/non-compact toric Calabi-Yau $(n+1)$-folds (or more precisely, Gorenstein singularities) and encode certain combinatorial data of $\mathrm{CY}_{n+1}$. Hence, they are also known as toric diagrams[1].

   (a) Given a polytope $P$ in the lattice $M$, the Ehrhart polynomial is

   $$\mathrm{ehr}_P(k) := |kP \cap M| \;, \tag{1}$$

   which counts the number of lattice points within the $k$-dilation of $P$. Then we can define the generating function called the Ehrhart series:

   $$\mathrm{Ehr}_P(t) = \sum_{k\geq 0} \mathrm{ehr}_P(k) t^k, \tag{2}$$

   for a formal variable $t$.

   Let us consider the 2d polygon $P$ whose vertices are $(1,0)$, $(0,1)$, $(-1,-1)$ as an example. Then $\mathrm{ehr}_P(0) = 1$ and $\mathrm{ehr}_P(1) = 4$ etc. Compute its Ehrhart series. What is the area of $P$?

   (b) The (polar) dual of a lattice polytope is defined as

   $$P^\circ := \{\boldsymbol{v} \in \mathbb{R}^n | \boldsymbol{u} \cdot \boldsymbol{v} \geq -1, \forall \boldsymbol{u} \in P\}. \tag{3}$$

   What is the dual polytope $P^\circ$ for the above example[2]? Compute its Ehrhart series and area.

   From the above two examples, we find that $\mathrm{ehr}_P(k) = c_n k^n + \dots + c_1 k + c_0$ and $\mathrm{Ehr}_P(t) = \frac{g(t)}{(1-t)^{n+1}} = \frac{g_0 + g_1 t + \dots + g_n t^n}{(1-t)^{n+1}}$. They actually hold in general for any lattice polytope. What are the sums of $g_i$ in the above two examples?

   (c) This part is a digression discussing more features of Ehrhart series in mathematics and physics. One may skip this if he/she is more interested in the machine learning part.

   Hilbert series is a generating function that enumerates invariant monomials/holomorphic functions of given degrees. Physically, it counts gauge invariant operators. *In our examples here*, Ehrhart series coincides with the Hilbert series of $\mathrm{CY}_{n+1}$ associated to the dual polytope. We now check this with the above example. For instance, $P$ in the above example is associated to $\mathbb{C}^3/\mathbb{Z}_3$ with action $(1,1,1)$. Compute its (unrefined) Hilbert series.

   (Hint: One way is to count the monomials at each degree directly. In particular, the action acts on the coordinates $z_{1,2,3}$ as $z_i \sim \omega z_i$ where $\omega$ can be chosen as the primitive root such that $\omega^3 = 1$. The invariant monomials are then of form $z_1^{a_1} z_2^{a_2} z_3^{a_3}$ with $a_1 + a_2 + a_3 \equiv 0 \pmod 3$. The HS is then the sum of $t_1^{a_1} t_2^{a_2} t_3^{a_3}$ running over all possible $a_i$. To compute the sum, one method is to consider the map $t_1 = t x_1, t_2 = x_2/x_1, t_3 = t/x_2$ and compare the

[1] By local, we mean that the local CY is a open neighbourhood in some compact CY. For more details on how lattice polytopes represent toric CY cones, see for example [1]. In this question, we shall not worry about this.

[2] In this case, the dual polytope is again an integral polytope on the lattice. Such polytopes are called reflexive, and there are only 16 of them for 2d polygons (including self-dual ones). In 2d, a polygon is reflexive if and only if it has precisely one interior point. However, this is not true for higher dimensions (reflexivity implies one interior point, but not vice versa). In fact, one may also associate a compact Fano $n$-fold to each polytope and reflexive duals give mirror pairs of the Fano varieties. For non-reflexive lattice polytopes, their dual are rational polytopes and one may also define certain Ehrhart series for them.

terms with characters of SU(3) irreps[3]. The unrefinement ($x_{1,2} = 1$) then has dimension of the irrep as the coefficient at each order of $t$. Do these coefficients look familiar?

The approach to compute this HS which makes use of the lattice polytope can be found in [3]. There is also another method for this HS as discussed in [4].)

(d) As we can see, the Ehrhart series encodes the dimension of the polytope in both the denominator and the numerator. Given a generating function, we sometimes do not have its exact closed form but only have its perturbative expansion to some order. In the attached file (`data.db`), we have data for randomly generated polytopes with "hilb", "dim" and "deg", which are their Ehrhart series expansions up to order 30, dimensions and (normalized) volumes respectively[4]. Now, use the Ehrhart expansions to predict the dimensions. Try this with NN, SVM and random forest. How do you find the results? Try to explain the performance.

(Hint: The data contains polytopes of dimensions from 1 to 6. One may use a classifier with 6 classes. One may consult [5] for the reasoning of the performance.)

(e) The normalized volumes range from 1 to 6717479. Use linear regression to predict the volume from Ehrhart expansions. Explain the result.

(Hint: Train an independent regressor for each dimension. Then check the coefficients in the linear regressor and consider the observation in (1b).)

2. An amoeba $\mathcal{A}_P$ is the set of points $\{(\log|z|, \log|w|) | P(z,w) = 0\}$ for some Laurent polynomial $P(z,w) \in \mathbb{C}[z^{\pm1}, w^{\pm1}]$ [5]. The Laurent polynomial (aka Newton polynomial) can be associated to lattice polygons (aka Newton polygons) as follows. For each monomial $z^m w^n$ in $P(z,w)$, it corresponds to the lattice point $(m,n)$ in the Newton polygon. For instance, the example considered in (1a) yields the Newton polynomial $c_1 z + c_2 w + c_3$ for some coefficients $c_{1,2,3}$. Amoeba has interesting applications in physics. See for example [6, 7, 8, 9].

   (a) Consider the toric diagram with vertices $(\pm1, 0)$, $(0, \pm1)$. Choose several sets of coefficients $c_i$ for the Newton polynomial and plot the amoebae. Check that the shape of the amoeba is always the thickening of the dual graph of the toric diagram[6]. What is the genus $g$ (i.e., the number of bounded complementary regions) of the amoeba?

   (Hint: You may write an algorithm in `Mathematica` and use Monte Carlo method.)

   (b) Using the columns "coeffs" and "genus" in the file `dataf0.csv`, build a machine learning model with input $\{c_i\}$ and output $g$.

   (c) Let us try to see why the model can have good performance. Apply principal component analysis (PCA) and multi-dimensional scaling (MDS) manifold projection to plot the distribution of the data points. What do the plots look like?

   (Hint: You may find the `Yellowbrick` package in `Python` useful for MDS projection and also spectral embedding in (2e) below[7].)

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[3] The SU(3) here should not be a coincidence. In particular, $\mathbb{Z}_3$ is a discrete subgroup of it. See [2] for more details.

[4] It is explained in (1c) why the Ehrhart expansion is named "hilb". The normalized volume column is named "deg" since it is equal to the degree of the toric variety polarized by the Cartier divisor $D_P$.

[5] Most of the discussions in this question can be naturally extended to any dimension though we will only focus on 2-dimensional amoebae here.

[6] Such dual graph is known as the spine of the amoeba

[7] The webpage for `Yellowbrick` has some explanations of different manifold learnings.

(d) Given a set of non-negative numbers, we say that it is lopsided if one of the numbers is greater than the sum of all the others. Write the Newton polynomial as $P(z,w) = m_1(z,w) + m_2(z,w) + \dots$, where $m_i(z,w)$ are monomials (whose coefficients are not necessarily 1). Consider the set $P\{z,w\} := \{|m_i(z,w)|\}$. Then we can define the lopsided amoeba as $\mathcal{LA}_P := \{(\log(z), \log(w)) | P\{z,w\} \text{ is not lopsided}\}$. It was shown in [10] that a point $(\log|z_0|, \log|w_0|)$ is in $\mathcal{LA}_P$ iff $P\{z_0,w_0\}$ is not lopsided. Moreover, $\mathcal{A}_P \subseteq \mathcal{LA}_P$ [8]. Use this fact to show that[9]

$$g = \begin{cases} 0, & |c_5| \leq 2|c_1c_3|^{1/2} + 2|c_2c_4|^{1/2} \\ 1, & |c_5| > 2|c_1c_3|^{1/2} + 2|c_2c_4|^{1/2} \end{cases}, \tag{4}$$

where $P(z,w) = c_1z + c_2w + c_3z^{-1} + c_4w^{-1} + c_5$. Use this to explain the PCA and MDS plots. Verify that the hole emerges from the centre of the (lopsided) amoeba.

(e) Try spectral embedding using `Yellowbrick`, and try to explain the distribution of the plot.

(f) By virtue of (4), we can use the absolute values of coefficients as input (the column "coeffsabs" in `dataf0.csv`). Train a new model and check that this improves the behaviour of neural network. Notice that this is not the case for more general polygons.

(g) A set of two-dimensional points $(x,y)$ from MDS projection of the input vectors are given in the column "reduced". What does the $c_5$-$x$ plot look like? Is there a similar plot for $y$ (with a different horizontal axis)?

(Hint: One may try to find a fit of $y$ in terms of $|c_{1,2,3,4}|$. It is also a useful fact that $\sqrt{m} \approx (0.1k + 1.2) \times 10^n$ for any real $k \in [1, 100)$ and $n \in \mathbb{Z}$ such that $m = k \times 10^{2n}$.)

3. For interested readers, various neural networks may have correspondences with different contexts in mathematics and physics. For instance, optimal transport was applied to GAN in [11]. Its relation to Hessian manifolds can be found in the handout. NN was studied using the context of Wilsonian effective field theory in [12]. The holographic duality was related to NN in [13]. Category theory has also been applied to machine learning in [14]. This list is never exhaustive, and more examples can be found in literature.

# References

[1] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, vol. 124. American Mathematical Soc., 2011.

[2] S. Benvenuti, B. Feng, A. Hanany, and Y.-H. He, "Counting BPS Operators in Gauge Theories: Quivers, Syzygies and Plethystics," *JHEP* **11** (2007) 050, `arXiv:hep-th/0608050`.

[3] Y.-H. He, R.-K. Seong, and S.-T. Yau, "Calabi–Yau Volumes and Reflexive Polytopes," *Commun. Math. Phys.* **361** no. 1, (2018) 155–204, `arXiv:1704.03462 [hep-th]`.

[8] In some special cases, such as $P = z+w+1$, the amoeba coincides with the lopsided amoeba. However, the amoeba is a proper subset of $\mathcal{LA}_P$. One may also consider the lopsidedness of the sets corresponding to the cyclic resultants of $P(z,w)$. The higher order the cyclic resultant has, the better approximation of the boundaries the lopsided amoeba can give.

[9] Strictly speaking, this should be the genus for the lopsided amoeba. However, it suffices for our purpose here.

[4] A. Hanany and R.-K. Seong, "Brane Tilings and Reflexive Polygons," *Fortsch. Phys.* **60** (2012) 695–803, arXiv:1201.2614 [hep-th].

[5] M. Beck, J. D. Loera, M. Develin, J. Pfeifle, and R. Stanley, "Coefficients and roots of ehrhart polynomials," *Contemporary Mathematics* **374** (2005) 15–36, arXiv:math/0402148 [math.CO].

[6] R. Kenyon, A. Okounkov, and S. Sheffield, "Dimers and amoebae," arXiv:math-ph/0311005.

[7] B. Feng, Y.-H. He, K. D. Kennaway, and C. Vafa, "Dimer models from mirror symmetry and quivering amoebae," *Adv. Theor. Math. Phys.* **12** no. 3, (2008) 489–545, arXiv:hep-th/0511287.

[8] H. Ooguri and M. Yamazaki, "Emergent Calabi-Yau Geometry," *Phys. Rev. Lett.* **102** (2009) 161601, arXiv:0902.3996 [hep-th].

[9] H. Ooguri, "Topological strings and crystal melting revisited." https://member.ipmu.jp/yuji.tachikawa/stringsmirrors/2009/ooguri.pdf.

[10] J. Forsgård, L. Matusevich, N. Mehlhop, and T. De Wolff, "Lopsided approximation of amoebas," *Mathematics of Computation* **88** no. 315, (2019) 485–500, arXiv:1608.08663 [cs.SC].

[11] N. Lei, K. Su, L. Cui, S. Yau, and X. D. Gu, "A geometric view of optimal transportation and generative model," *CoRR* (2017) , arXiv:1710.05488.

[12] J. Halverson, A. Maiti, and K. Stoner, "Neural Networks and Quantum Field Theory," *Mach. Learn. Sci. Tech.* **2** no. 3, (2021) 035002, arXiv:2008.08601 [cs.LG].

[13] K. Hashimoto, S. Sugishita, A. Tanaka, and A. Tomiya, "Deep learning and the AdS/CFT correspondence," *Phys. Rev. D* **98** no. 4, (2018) 046019, arXiv:1802.08313 [hep-th].

[14] D. Shiebler, B. Gavranović, and P. Wilson, "Category theory in machine learning," arXiv:2106.07032 [cs.LG].