Distance problems and their many variants

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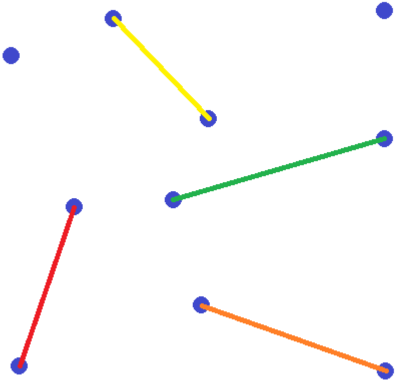
July 4, 2023

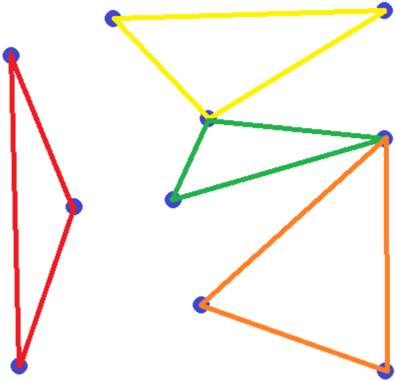
Modern trends in Harmonic Analysis
International Centre for Theoretical Sciences
Bengaluru, India

Distances



$$\Phi(x_1, x_2) = |x_1 - x_2|$$

Triangles



$$\Phi(x_1, x_2, x_3) = (|x_1 - x_2|, |x_1 - x_3|, |x_2 - x_3|)$$

An Erdős type problem for triangles

- What is the least number of distinct triangles determined by N points in the plane?

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- What is the least number of distinct triangles determined by N points in the plane?
- Rudnev obtained N^2.
- Achieved by the regular N-gon.
- Wide open in higher dimensions and not clear what the conjecture should be.

A Falconer type problem for triangles

- How large does $\dim_{\mathcal{H}}(E)$, for $E \subset \mathbb{R}^d$ compact, need to be to ensure that the set of triangles

$$D_\Delta(E) = \{(|x_1 - x_2|, |x_1 - x_3|, |x_2 - x_3|) : x_1, x_2, x_3 \in E\}$$

has positive three-dimensional Lebesgue measure?

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- Erdoğan and Iosevich conjecture for triangles in the plane

$$dim_{\mathcal{H}}(E) > \frac{3}{2} \text{ in } \mathbb{R}^2$$

- Only know the trivial restriction $\dim_{\mathcal{H}}(E) > \frac{d}{2}$ for $d \geq 3$.

Progress on the Falconer type problem for triangles

- Grafakos, Greenleaf, Iosevich, P.
  - $\dim_{\mathcal{H}}(E) > \frac{3}{4}d + \frac{1}{4}$ in $\mathbb{R}^d$
- Greenleaf, Iosevich, Liu, P.
  - $dim_{\mathcal{H}}(E) > \frac{8}{5}$ in $\mathbb{R}^2$
  - $dim_{\mathcal{H}}(E) > \frac{2}{3}d + \frac{1}{3}$ in $\mathbb{R}^d$ when $d \geq 3$

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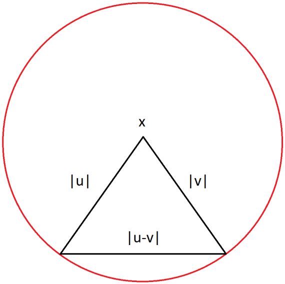
- Erdoğan, Hart, Iosevich
  - $\dim_{\mathcal{H}}(E) > \frac{1}{2}d + \frac{3}{2}$ in $\mathbb{R}^d$
- Iosevich, Pham, Pham, Shen
  - $\dim_{\mathcal{H}}(E) > \frac{1}{2}d + 1$ in $\mathbb{R}^d$

The triangle averaging operator



- The triangle averaging operator

$$A_{\Delta}(f,g)(x) = \int_M f(x-u)g(x-v)d\sigma_{\Delta}(u,v)$$

where $d\sigma_{\Delta}(u,v)$ is the normalized surface measure on

$$\{(u,v) \in \mathbb{R}^d \times \mathbb{R}^d : |u| = t_1, |v| = t_2, |u-v| = t_3\}$$

Properties of the equilateral triangle averaging operator

- The equilateral triangle averaging operator

$$A_\Delta(f,g)(x) = \int_{|u|=|v|=|u-v|=1} f(x-u)g(x-v)d\sigma_\Delta(u,v)$$

becomes on the Fourier side (with some multilinear complications)

$$\widehat{f}(\xi)\widehat{g}(\eta)\widehat{\sigma_\Delta}(\xi,\eta)$$

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- Through stationary phase estimates (Iosevich-Liu)

$$|\widehat{\sigma_\Delta}(\xi,\eta)| \lesssim \begin{cases} (1+\min(|\xi|,|\eta|)|\sin(\theta)|)^{-\frac{d-2}{2}}(1+|(\xi,\eta)|)^{-\frac{d-2}{2}} \\ |\xi + g_{\frac{\pi}{3}}\eta|^{-\frac{1}{2}}|\xi|^{-\frac{d-2}{2}}|\eta|^{-\frac{d-2}{2}}|sin(\theta)|^{-\frac{d-2}{2}} \end{cases}$$

where θ is the angle between ξ and η.

L^p bounds for the triangle averaging operator

- Trivially $A_\Delta : L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ when $r \geq 1$ by Young's convolution inequality.

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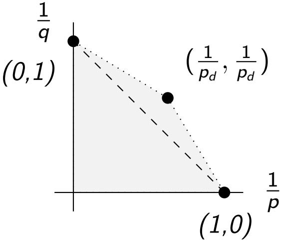
Theorem (P, Sovine in 2019)

For $d \geq 7$ the operator A_Δ is bounded $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ where $\left(\frac{1}{p}, \frac{1}{q}\right)$ come from the following region where $p_d = \frac{19d-4}{11d-12}$.



Geometric information leads to better bounds

- If σ_u is the natural measure on a lower dimensional sphere

$$A_\Delta(f,g)(x) = \int_{|u|=|v|=|u-v|=1} f(x-u)g(x-v)d\sigma_u(v)d\sigma(u)$$

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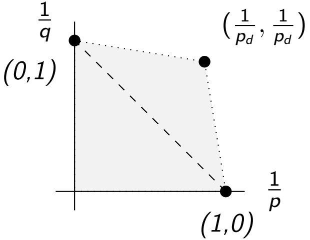
Theorem (Iosevich, P, Sovine in 2021)

For $d \geq 2$ the operator A_Δ is bounded $L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \to L^r(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ where now $p_d = \frac{d+1}{d}$.



L^p improving bounds

From the work of Stovall as well as Greenleaf, Iosevich, Krause and Liu can obtain the following sharp L^p improving bounds for A_Δ.

(a) $L^{\frac{3}{2}}(\mathbb{R}^2) \times L^{\frac{3}{2}}(\mathbb{R}^2) \to L^1(\mathbb{R}^2)$

(b) $L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ restricted strong type

as well as bounds coming from the linear setting and interpolated bounds with trivial estimates.

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Theorem (Iosevich, P, Sovine in 2021)

For $d \geq 2$ the operator A_Δ satisfies

(a) $L^{\frac{d+1}{d}}(\mathbb{R}^d) \times L^{\frac{d+1}{d}}(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$

(b) $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *restricted strong type*

(c) $L^{\frac{m(d+1)}{d}}(\mathbb{R}^d) \times L^{\frac{m(d+1)}{d}}(\mathbb{R}^d) \to L^{\frac{m(d+1)}{2}}(\mathbb{R}^d)$, $d \geq 2m, m \geq 2$

and the first of those bounds is sharp.

Maximal equilateral triangle averaging operator

- The maximal equilateral triangle averaging operator

$$M_\Delta(f,g)(x) = \sup_{r>0} \left| \int_{|u|=|v|=|u-v|=1} f(x-ru)g(x-rv)d\sigma_\Delta(u,v) \right|$$

- Expect mapping properties of the type

$$M_\Delta : L^p(\mathbb{R}^d) \times L^q(\mathbb{R}^d) \to L^r(\mathbb{R}^d), \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r}$$

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- $r > \frac{d}{d-1}$ trivial and optimal if either $p = \infty$ or $q = \infty$.

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- P, Sovine conjecture restricted type bounds

$$M_\Delta : L^{\frac{d}{d-1}}(\mathbb{R}^d) \times L^{\frac{d}{d-1}}(\mathbb{R}^d) \to L^{\frac{d}{2d-2},\infty}(\mathbb{R}^d)$$

Some positive results

- Cook, Lyall, Magyar established

$$M_\Delta : L^{\frac{m}{m-1}\,\frac{d}{d-1}}(\mathbb{R}^d) \times L^{\frac{m}{m-1}\,\frac{d}{d-1}}(\mathbb{R}^d) \to L^{\frac{m}{m-1}\,\frac{d}{2d-2}}(\mathbb{R}^d)$$

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- P, Sovine in a recent paper established sparse bounds for M_Δ in the Banach range.
  - (a) Generalizes bounds obtained by Lacey for M_S.
  - (b) Builds on techniques of Roncal, Shrivastava, and Shuin for a maximal bilinear product spherical averaging operator.

The discrete maximal triangle averaging operator

- With Theresa Anderson and Angel Kumchev we studied

$$\mathcal{T}(f,g)(k) = \sup_{\lambda>0} \left| \frac{1}{\#\mathcal{U}_\lambda} \sum_{u,v\in\mathcal{U}_\lambda} f(k-u)g(k-v) \right|$$

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- Can see $\#\mathcal{U}_\lambda \approx \lambda^{d-3}$ for d large enough for example through modular forms or the circle method.
- We obtain a wide range of estimates of the type $\ell^p(\mathbb{Z}^d) \times \ell^q(\mathbb{Z}^d) \to \ell^r(\mathbb{Z}^d)$ when $d \geq 9$ where $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{r}$, $r > \max(\frac{32}{d+9}, \frac{d+4}{d-2})$ and $p, q > 1$.

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- Improvements in high dimensions and certain ranges by Cook, Lyall and Magyar.

The Mattila-Sjölin theorem

- How large does $\dim_{\mathcal{H}}(E)$, for $E \subset \mathbb{R}^d$ compact, need to be to ensure that the distance set

$$D(E) = \{|x - y| : x, y \in E\}$$

has non-empty interior and thus contains an interval?

- Sets of positive measure need not have non-empty interior!

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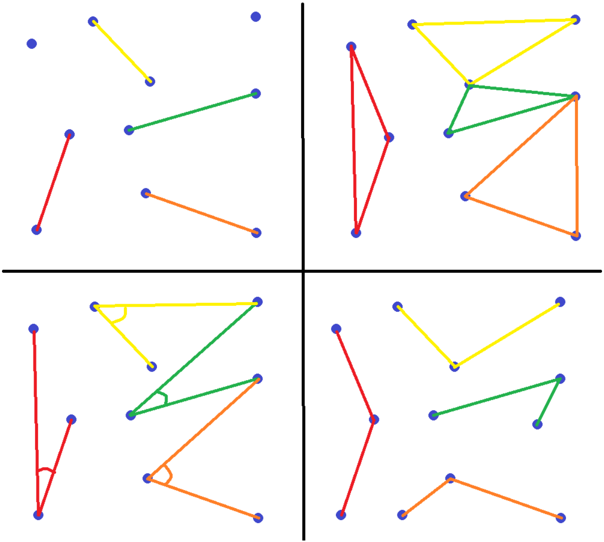
Theorem (Mattila, Sjölin in 1999)

Let $E \subset \mathbb{R}^d$, $d \geq 2$, be compact. If $\dim_{\mathcal{H}}(E) > \frac{d+1}{2}$ then $D(E)$ has non-empty interior.

- Iosevich, Mourgoglou and Taylor extended this to a wide range of distance metrics in 2011.

Many interesting point configurations



More complicated configurations

- Greenleaf, Iosevich and Taylor showed Mattila-Sjölin type theorems for various k-point configurations.

- One example is that if $E \subset \mathbb{R}^2$ is compact with $\dim_{\mathcal{H}}(E) > \frac{5}{3}$ then the set of areas of triangles determined by triples of points of E

$$\left\{ \frac{1}{2} |\det[x - z, y - z]| : x, y, z \in E \right\} \subset \mathbb{R}$$

contains an open interval.

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- Their FIO method did not originally apply to the triangle set.

Mattila-Sjölin theorems for triangles

Theorem (P, Romero Acosta in 2021)

Let $E \subset \mathbb{R}^d$, $d \geq 4$, *be compact. If* $\dim_{\mathcal{H}}(E) > \frac{2}{3}d + 1$ *then* $D_{\Delta}(E)$ *has non-empty interior.*

- View $D_{\Delta}(E)$ from side-angle-side.
- Builds on work of Iosevich and Liu.
- Later matched by Greenleaf, Iosevich and Taylor.

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Theorem (P, Romero Acosta in 2022)

Let $E \subset \mathbb{R}^3$ *be compact. If* $\dim_{\mathcal{H}}(E) > \frac{23}{8}$ *then* $D_\Delta(E)$ *has non-empty interior.*

- Classic side-side-side viewpoint.
- Builds on work of Iosevich and Magyar.
- Extends to simplexes in higher dimensions.

The L^2 approach

- Define a measure $\delta(\mu)(\mathbf{t})$ on $D_\Delta(E)$ by the relation

$$\int f(\mathbf{t})d\delta(\mu)(\mathbf{t}) = \iiint f\left(|x_1 - x_2|, |x_1 - x_3|, |x_2 - x_3|\right) d\mu(x_1)d\mu(x_2)d\mu(x_3)$$

where μ is a Frostman measure supported on E.

- Try to establish the bound $\int \delta(\mu)^2(\mathbf{t})\, d\mathbf{t} \lesssim 1$.

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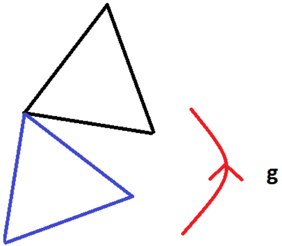
- Idea: $\int \delta(\mu)^2(\mathbf{t})\, d\mathbf{t} = \iint_{\mathbf{s}=\mathbf{t}} \delta(\mu)(\mathbf{t})\delta(\mu)(\mathbf{s})\, d\mathbf{t} d\mathbf{s}$

A group-theoretic point of view

- Leads one to consider (x_1, x_2, x_3) and (y_1, y_2, y_3) that give rise to the same triangle, in other words $|x_i - x_j| = |y_i - y_j|$ for all $1 \le i < j \le 3$.

- Observe that for $x_i \neq x_j$, $|x_i - x_j| = |y_i - y_j|$ if and only if $x_i - x_j = gy_i - gy_j$ for some $g \in \mathbb{O}(d)$, the orthogonal group.



- Using the group-theoretic point of view it follows that

$$\int \delta(\mu)^2(\mathbf{t})\, d\mathbf{t} \leq c \int \mu^6\{(x_1, \ldots, x_3, y_1, \ldots, y_3) : \\ x_i - gy_i = x_j - gy_j,\ 1 \leq i < j \leq 3\} dxdydg$$

where dg denotes the Haar measure on $\mathbb{O}(d)$.

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where dg denotes the Haar measure on $\mathbb{O}(d)$.

- Define a measure $\delta(\mu)_g$ on $E - gE$ by the relation

$$\int f(z) d\delta(\mu)_g(z) := \int \int f(u - gv) d\mu(u) d\mu(v).$$

- Then can write the inequality above as

$$\int \delta(\mu)^2(\mathbf{t})\, d\mathbf{t} \lesssim \int \int \delta(\mu)_g^3(z)\, dz\, dg.$$

A generalized Mattila integral

- From the definition of $\delta(\mu)_g$ one obtains

$$\widehat{\delta(\mu)_g}(\xi) = \widehat{\mu}(\xi)\widehat{\mu}(g\xi).$$

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- Using Plancharel

$$\begin{aligned}
\int \delta(\mu)^2(\mathbf{t})\,d\mathbf{t} &\lesssim \int\int \delta(\mu)_g^3(z)\,dz\,dg \\
&\leq \|\delta(\mu)_g\|_\infty \int\int \delta(\mu)_g^2(z)\,dz\,dg \\
&= \|\delta(\mu)_g\|_\infty \int |\widehat{\mu}(\xi)|^2 \left\{\int |\widehat{\mu}(g\xi)|^2 dg\right\} d\xi \\
&\lesssim \|\delta(\mu)_g\|_\infty \int \left(\int_{S^{d-1}} |\widehat{\mu}(r\omega)|^2 d\omega\right)^2 r^{d-1} dr
\end{aligned}$$

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&\lesssim \|\delta(\mu)_g\|_\infty \int \left(\int_{S^{d-1}} |\widehat{\mu}(r\omega)|^2 d\omega \right)^2 r^{d-1} dr
\end{aligned}$$

- Can we better estimate $\int\int \delta(\mu)_g^3(z)\,dz\,dg$?

The pinned Falconer distance problem

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- A bad example is E is a sphere around x.

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- Can we guarantee $\mathcal{L}(D^x(E)) > 0$?
- A bad example is E is a sphere around x.
- How large does $\dim_{\mathcal{H}}(E)$, for $E \subset \mathbb{R}^d$, $d \geq 2$, need to be to ensure that *there exists* $x \in E$ with $\mathcal{L}(D^x(E)) > 0$?

Group actions and Liu's result

- Peres and Schlag obtained threshold $\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{2}$.

Group actions and Liu's result

- Peres and Schlag obtained threshold $\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{2}$.
- Liu's magic formula

$$\int |\sigma_r * f(x)|^2 r^{d-1} dr = \int |\widehat{\sigma_r} * f(x)|^2 r^{d-1} dr$$

for any $x \in \mathbb{R}^d$ and f a Schwartz function on $\mathbb{R}^d$.

- Builds on the group action viewpoint in continuous setting developed by Greenleaf, Iosevich, Liu and P.
- All thresholds using the Mattila scheme translate directly to the pinned setting due to Liu.

Thank you!

Questions?

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My website: personal.math.vt.edu/palsson/