Distance problems and their many variants

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Modern trends in Harmonic Analysis
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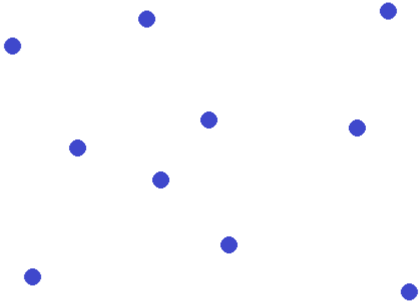
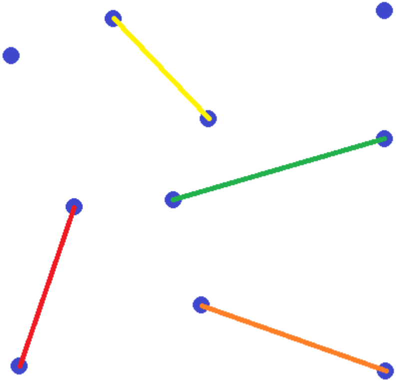
Patterns?

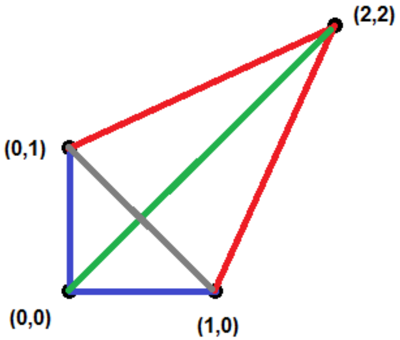


Distances



$$\Phi(x_1, x_2) = |x_1 - x_2|$$

Example



- Distances: 1, 1, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{5}$, $\sqrt{8}$
- Distinct distances: 1, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{8}$

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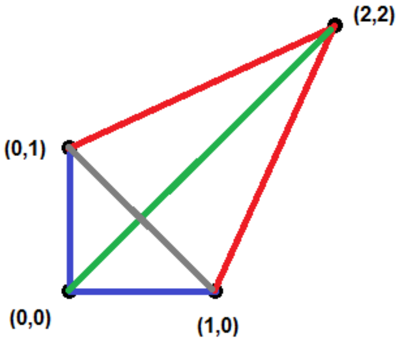
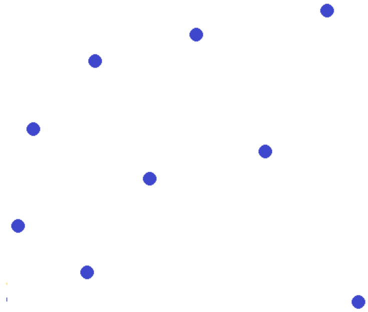


- Distances: 1, 1, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{5}$, $\sqrt{8}$
- Distinct distances: 1, $\sqrt{2}$, $\sqrt{5}$, $\sqrt{8}$
- How many distinct distances are there in general?

It is easy to have many distinct distances



- N points in the plane.
- Upper bound $\binom{N}{2} = \frac{N(N-1)}{2} \sim N^2$.

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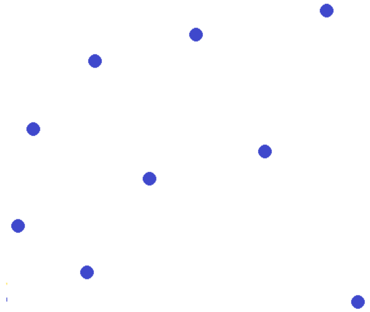

- N points in the plane.
- Upper bound $\binom{N}{2} = \frac{N(N-1)}{2} \sim N^2$.
- If randomly selected obtain $\binom{N}{2} \sim N^2$.

The Erdős distinct distance problem

- What is the least number of distinct distances determined by N points in the plane?

The Erdős distinct distance problem

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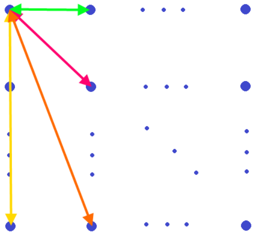
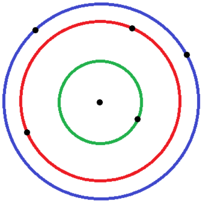
- Conjecture $\frac{N}{\sqrt{\log(N)}}$ as $N \to \infty$.

- Distances squared
  $1^2 + 0^2 = 1$
  $1^2 + 1^2 = 2$
  $\vdots$
  $(\sqrt{N})^2 + (\sqrt{N})^2 = 2N$



Circles encode distances



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- Progress through the decades by Chung, Katz, Moser, Solymosi, Szekely, Szemeredi, Toth, Trotter and so on.

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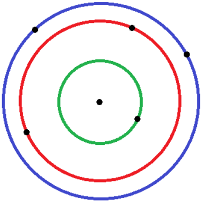

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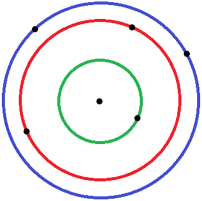

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- Guth and Katz obtained $\frac{N}{\log(N)}$ in 2015.
- Open in $\mathbb{R}^d$ when $d \geq 3$ with the conjecture being $N^{\frac{2}{d}}$.

Quote by Erdős

"My most striking contribution to geometry is, no doubt, my problem on the number of distinct distances. This can be found in many of my papers on combinatorial and geometric problems."

The distance set

- The *distance set* of $E \subseteq \mathbb{R}^d$ is

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- If $E \subseteq \mathbb{R}^d$ such that $\#E = \infty$ then

$$\#(D(E)) = \infty$$

The Steinhaus theorem

- Idea

$$E \subseteq \mathbb{R}^d \text{ large} \implies D(E) \text{ large (\& structured)}$$

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Theorem (Steinhaus 1920)
For $E \subseteq \mathbb{R}^d$ with $\mathcal{L}^d(E) > 0$

$$E - E := \{x - y : x, y \in E\}$$

contains a neighborhood of the origin.

- Immediately implies

$$\mathcal{L}^d(E) > 0 \implies \mathcal{L}(D(E)) > 0$$

and further $D(E)$ contains an interval.

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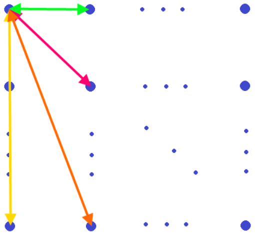
- Can construct $E \subset \mathbb{R}$ with $\dim_{\mathcal{H}}(E) = 1$ such that $\mathcal{L}(D(E)) = 0$.
- Falconer's conjecture $\dim_{\mathcal{H}}(E) > \frac{d}{2}$



First results

For a compact set $E \subset \mathbb{R}^d$, $d \geq 2$.

- Falconer in 1985

$$\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{2} \implies \mathcal{L}(D(E)) > 0$$

First results

For a compact set $E \subset \mathbb{R}^d$, $d \geq 2$.

- Falconer in 1985

$$\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{2} \implies \mathcal{L}(D(E)) > 0$$

- Mattila, Sjölin in 1999

$$\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{2} \implies D(E) \text{ contains an interval}$$

- Iosevich, Mourgoglou and Taylor extended this to a wide range of distance metrics in 2011.

Encode dimension with measures

- For a compact set $E \subset \mathbb{R}^d$ and $0 < s < \dim_{\mathcal{H}}(E)$ there is a probability measure μ supported on E with

$$\mu(B_r) \lesssim r^s$$

for any ball B_r of radius r.

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- Taking s arbitrarily smaller

$$I_s(\mu) = \iint |x-y|^{-s} d\mu(x) d\mu(y) = c_{s,d} \int |\widehat{\mu}(\xi)|^2 |\xi|^{s-d} d\xi < \infty$$

Call $I_s(\mu)$ the *energy integral* of μ.

Distance measure

- Define the distance measure $\delta(\mu)$, supported on $D(E)$, by the relation

$$\int f(r)\, d\delta(\mu)(r) = \iint f\left(|x-y|\right) d\mu(x) d\mu(y)$$

for any continuous function f, where μ is a Frostman measure supported on E.

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- Approximate μ by a smooth function μ_ϵ and get

$$\int f(r)\, d\delta(\mu_\epsilon)(r) = \iint f\left(|x-y|\right) \mu_\epsilon(x)\mu_\epsilon(y) dx dy$$
$$= \int f(r) \left(\int (\sigma_r * \mu_\epsilon)(x)\mu_\epsilon(x) dx \right) dr$$

- Get density

$$\delta(\mu_\epsilon)(r) = \int (\sigma_r * \mu_\epsilon)(x)\mu_\epsilon(x)dx = \int \widehat{\sigma_r}(\xi)|\widehat{\mu_\epsilon}(\xi)|^2 d\xi$$

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$$r^{d-1} \int \widehat{\sigma}(r\xi)|\widehat{\mu}(\xi)|^2 d\xi$$

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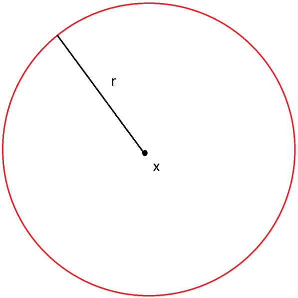
- $1 = \delta(\mu)(D(E)) = \int_{D(E)} \delta(\mu)(r)dr \leq \|\delta(\mu)\|_{L^\infty} \mathcal{L}(D(E))$

Spherical averaging operator



- The spherical averaging operator appeared

$$A_r(f)(x) = \frac{c_d}{r^{d-1}}(\sigma_r * f)(x) = \int_{\mathbb{S}^{d-1}} f(x - ry)d\sigma(y)$$

where $d\sigma$ is the normalized surface measure on $\mathbb{S}^{d-1}$, $d \geq 2$.

Easy bounds

$$A_r(f)(x) = \int_{\mathbb{S}^{d-1}} f(x - ry) d\sigma(y)$$

- Easy

$$\|A_r(f)\|_{L^\infty} \leq \int_{\mathbb{S}^{d-1}} \|f\|_{L^\infty} \, d\sigma(y) = \|f\|_{L^\infty}$$

- By Fubini

$$\|A_r(f)\|_{L^1} \leq \int_{\mathbb{S}^{d-1}} \|f\|_{L^1} \, d\sigma(y) = \|f\|_{L^1}$$

Conclude by interpolation

$$A_r : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d), \quad p \geq 1$$

L^p improving and Sobolev bounds

- L^p improving estimate

$$A_r : L^{\frac{d+1}{d}}(\mathbb{R}^d) \to L^{d+1}(\mathbb{R}^d)$$

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- Full picture

$$A_r : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$$

if and only if

$\left(\frac{1}{p}, \frac{1}{q}\right)$ is within the closed triangle $(0,0)$, $(1,1)$, $\left(\frac{d}{d+1}, \frac{1}{d+1}\right)$.

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- Sobolev bounds

$$A_r : L^2(\mathbb{R}^d) \to L^2_{\frac{d-1}{2}}(\mathbb{R}^d)$$

where $L^2_{\frac{d-1}{2}}(\mathbb{R}^d)$ is a standard homogeneous Sobolev space.

Maximal spherical averaging operator

- The maximal spherical averaging operator

$$M_S(f)(x) = \sup_{r>0} |A_r(f)(x)| = \sup_{r>0} \left| \int_{\mathbb{S}^{d-1}} f(x-ry)d\sigma(y) \right|$$

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- Stein ($d \geq 3$) and Bourgain ($d = 2$) showed

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if $p > \frac{d}{d-1}$. (See also Mockenhaupt, Seeger and Sogge.)

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- Yields Lebesgue differentiation type theorem

$$\lim_{r \to 0^+} \frac{1}{|\partial B(x,r)|} \int_{\partial B(x,r)} f(y) d\sigma_r(y) = f(x) \text{ for a.e. x}$$

for all $f \in L^p(\mathbb{R}^d)$, $p > \frac{d}{d-1}$.

An improved approach

- Establish

$$\|\delta(\mu)\|_{L^2} < \infty$$

- Idea why sufficient

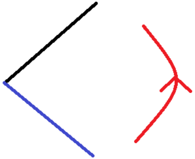
$$1 = \int_{D(E)} \delta(\mu)(r)dr \leq \mathcal{L}(D(E))^{\frac{1}{2}} \|\delta(\mu)\|_{L^2}$$

- Has given rise to all modern improvements.

The classical Mattila integral

- Idea: $\int \delta(\mu)^2(r)\,dr = \iint\limits_{r=s} \delta(\mu)(r)\delta(\mu)(s)\,drds$

- $|x-y| = |x'-y'|$ if and only if $x-y = g(x'-y')$



where $g \in \mathbb{O}(d)$, the orthogonal group.

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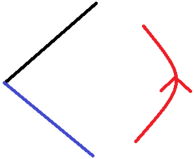

where $g \in \mathbb{O}(d)$, the orthogonal group.

- Using $x - gx' = y - gy'$ and Plancharel we get

$$\int \delta(\mu)^2(r)\,dr \lesssim \int |\widehat{\mu}(\xi)|^2 \left\{ \int |\widehat{\mu}(g\xi)|^2 dg \right\} d\xi$$

$$= C \int \left(\int_{S^{d-1}} |\widehat{\mu}(r\omega)|^2 d\sigma(\omega) \right)^2 r^{d-1} dr$$

Connections to restriction

- Bounding the classical Mattila integral

$$\int \left(\int_{S^{d-1}} |\widehat{\mu}(r\omega)|^2 d\sigma(\omega) \right)^2 r^{d-1} dr$$

requires a weighted restriction estimate.

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- Bounding the classical Mattila integral

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- If μ Frostman measure on $E \subset \mathbb{R}^d$ with $\mu(B_r) \lesssim r^s$ for any ball B_r where $s < dim_{\mathcal{H}}(E)$ then estimates of the form

$$\int_{S^{d-1}} |\widehat{\mu}(r\omega)|^2 d\sigma(\omega) \lesssim_{\epsilon} r^{-\beta_d(s)+\epsilon}$$

hold where $\beta_d(s) \geq \frac{(d-1)s}{d}$ when $\frac{d}{2} < s < d$.

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- Falconer's original threshold
  - $\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{2}$ in $\mathbb{R}^d$

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- Flurry of improvements recently due to Du, Guth, Iosevich, Ou, Ren, Wang, Wilson and Zhang.
  - $\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{4} + \frac{1}{8d-4}$ in $\mathbb{R}^d$

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  - $\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{4} - \frac{1}{8d-4}$ in $\mathbb{R}^d$, $d \geq 3$ (Forthcoming???)

The pinned Falconer distance problem

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Group actions and Liu's result

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- Peres and Schlag obtained threshold $\dim_{\mathcal{H}}(E) > \frac{d}{2} + \frac{1}{2}$.

- Liu's magic formula

$$\int |\sigma_r * f(x)|^2 r^{d-1} dr = \int |\widehat{\sigma_r} * f(x)|^2 r^{d-1} dr$$

for any $x \in \mathbb{R}^d$ and f a Schwartz function on $\mathbb{R}^d$.

- Builds on the group action viewpoint in continuous setting developed by Greenleaf, Iosevich, Liu and P.

- All thresholds using the Mattila scheme translate directly to the pinned setting due to Liu.

Thank you!

Questions?

Contact me: palsson@vt.edu

My website: personal.math.vt.edu/palsson/