# Interpretations of numerical results for MBL

Piotr Sierant[1]

[1]ICFO - The Institute of Photonic Sciences, Castelldefels, Barcelona, Spain

Stability of Quantum Matter in and out of Equilibrium at Various Scales
ICTS, Bengaluru

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# This talk:

1. Many-body localization

- Motivation: "Does MBL exist?"
- POLFED algorithm
- Numerical results for ETH-MBL crossover in disordered spin chains

2. Error-resilience phase transitions in encoding-decoding circuits

X. Turkeshi, PS, arXiv:2308.06321

# Slow dynamics due to interactions

Random field XXZ spin-1/2 chain: $H = \sum_{i=1}^{L} J\left(S_i^x S_{i+l}^x + S_i^y S_{i+l}^y + \Delta S_i^z S_{i+l}^z\right) + \sum_{i=1}^{L} h_i S_i^z$

Imbalance: $I(t) = \frac{4}{L}\sum_{i=1}^{L}\langle S_i^z(t) S_i^z(0)\rangle$ $\quad J = 1,\ h_i \in [-W, W]$

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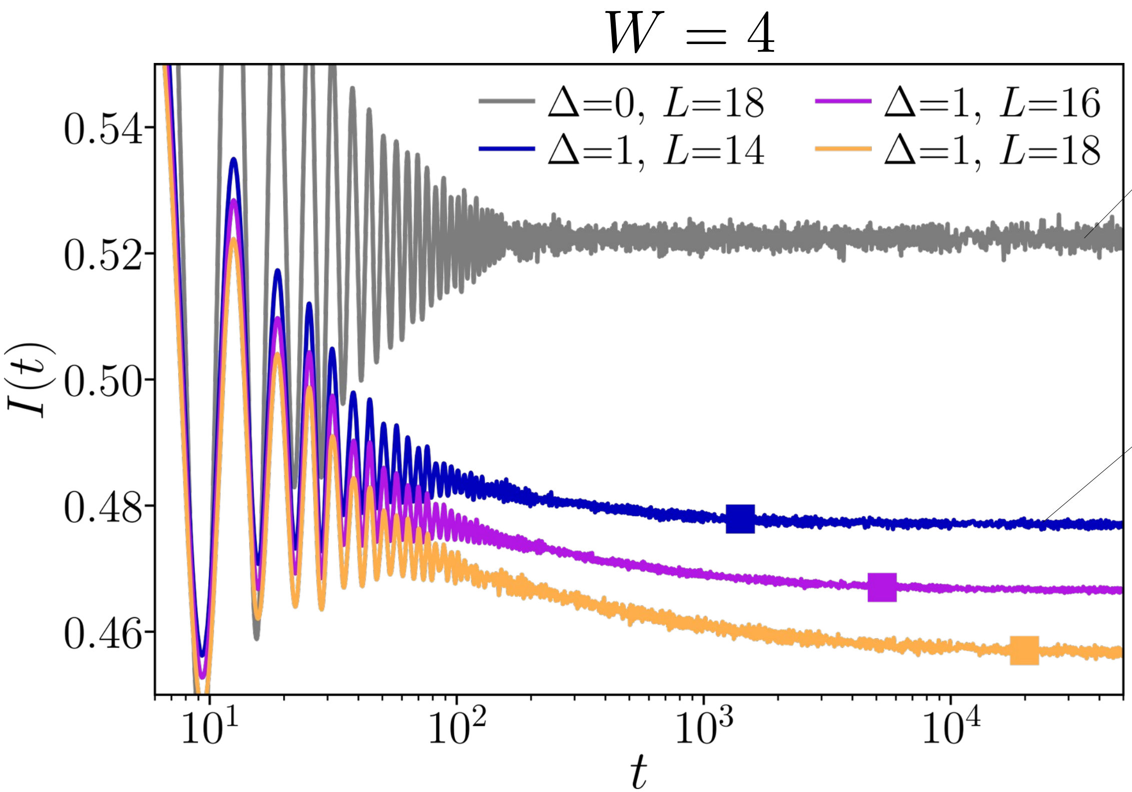


Non-interacting system, Anderson localized

Interacting system, strong disorder, non-zero “stationary” imbalance

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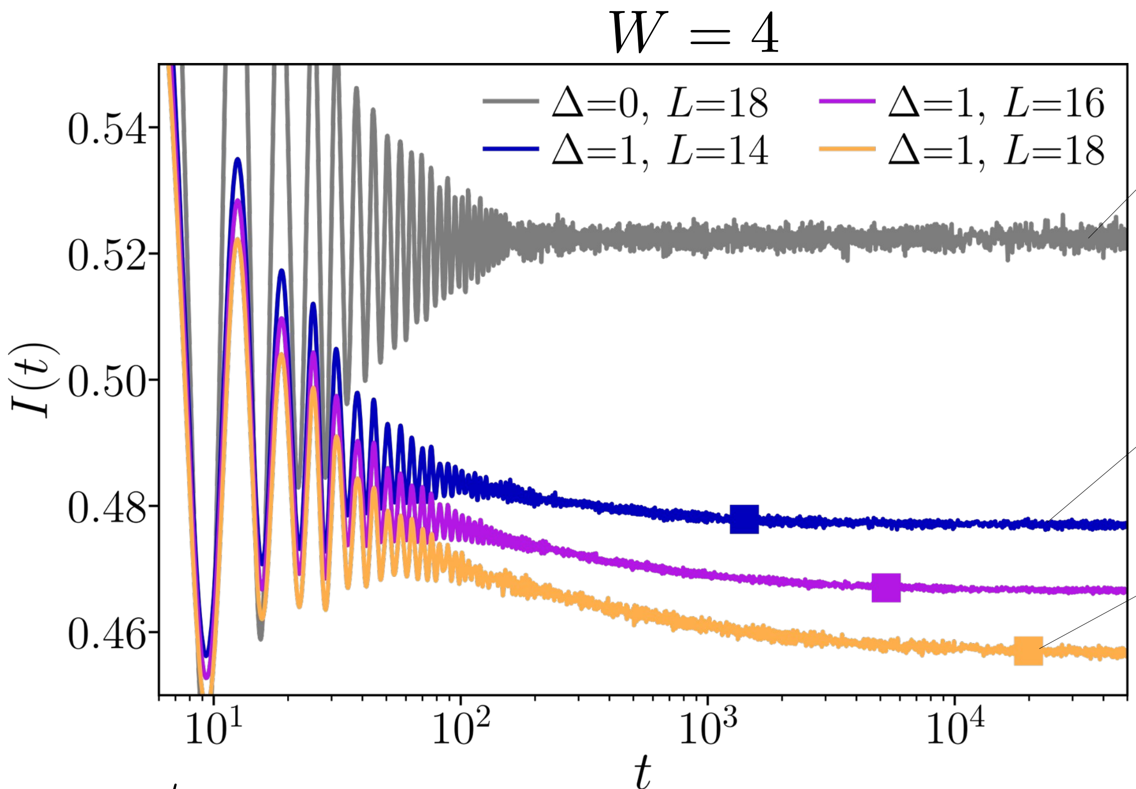



Non-interacting system, Anderson localized

Interacting system, strong disorder, non-zero "stationary" imbalance

But the decay persist nearly to the Heisenberg time

- $\overline{A} = \lim_{t_0\to\infty} \frac{1}{t_0}\int_0^{t_0} dt A(t) = \sum_m |C_m|^2 A_{mm} \overset{?}{=} \frac{1}{\mathcal{N}_{\langle E\rangle,\delta E}} \sum_{|E_m - \langle E\rangle| < \delta E} A_{mm}$

J. M. Deutsch, PRA **43**, 2046 (1991)
M. Srednicki, PRE **50**, 888, (1994)
M.Rigol, V. Dunjko, M. Olshanii, Nature **452**, 854-858 (2008):

- Double limit $\lim_{L\to\infty}\ \lim_{t\to\infty}$ to decide between ETH and *MBL phase*

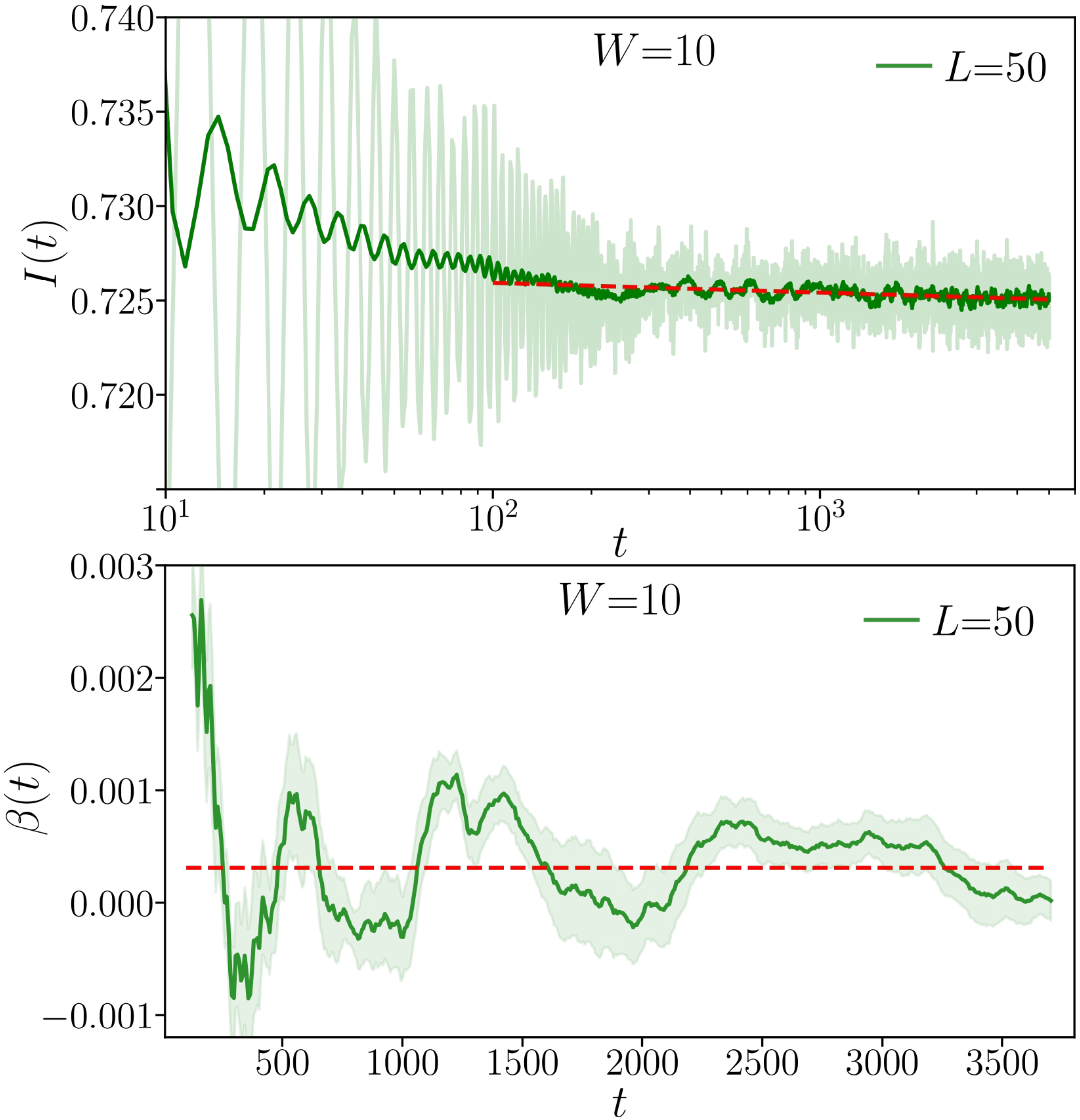
"Does MBL exist?"

# Strong disorder and interactions (W=10)

Random field XXZ spin-1/2 chain: $H_{XXZ} = \sum_{i=1}^{L} J\left(S_i^x S_{i+l}^x + S_i^y S_{i+l}^y + \Delta S_i^z S_{i+l}^z\right) + \sum_{i=1}^{L} h_i S_i^z$

we set $\Delta = 1$

- Tensor networks (TDVP) simulation of time evolution: PS, J. Zakrzewski, PRB **105**, 224203 (2022)



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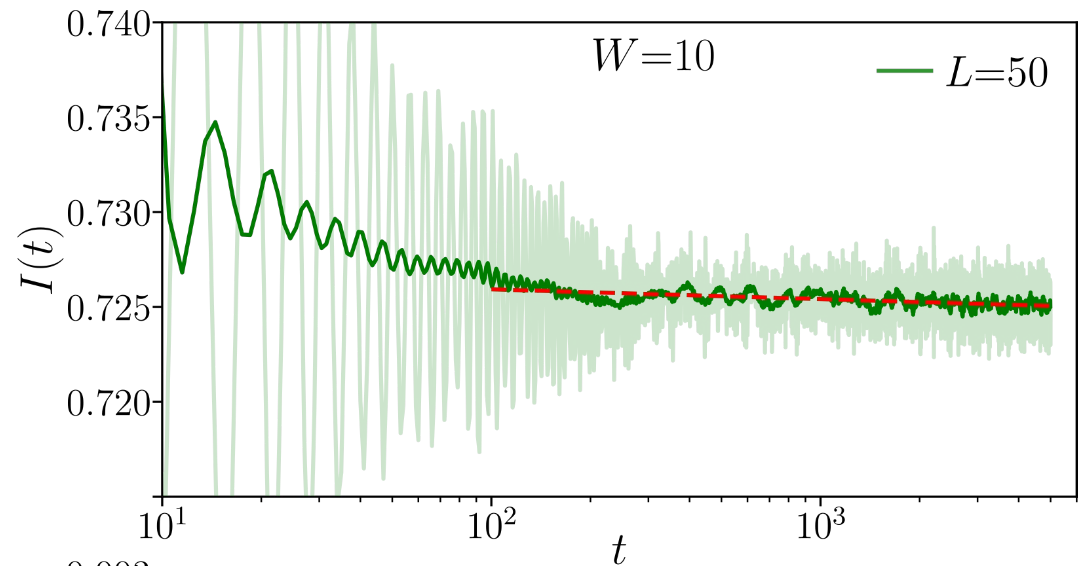
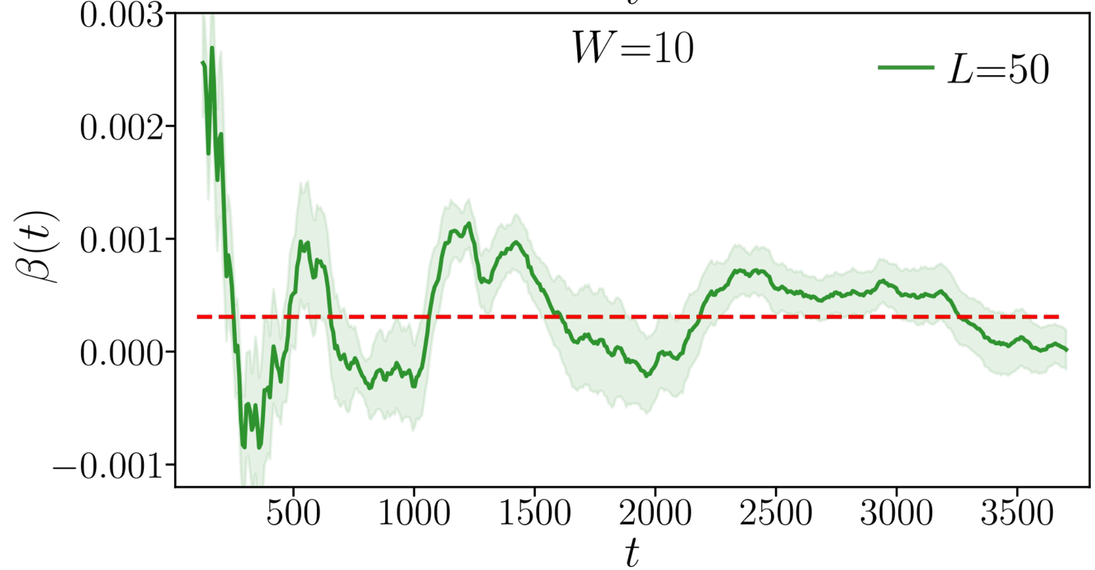
| | $t_{\max}$ | $\chi$ | $n_{\text{real}}$ | $\overline{\beta}$ |
|---|---|---|---|---|
| $L$=50 | 1500 | 128 | 4000 | $(3.93 \pm 0.82) \cdot 10^{-4}$ |
| $L$=100 | 1500 | 128 | 2000 | $(3.60 \pm 0.53) \cdot 10^{-4}$ |
| $L$=200 | 1200 | 160 | 1000 | $(3.50 \pm 0.87) \cdot 10^{-4}$ |
| $L$=50 | 5000 | 192 | 2000 | $(3.08 \pm 0.51) \cdot 10^{-4}$ |

- *Assuming that* $I(t) = I_0 t^{-\bar{\beta}}$ *persist, the imbalance decays to 10% of its initial value after* *$10^{3000}$* *tunneling times*



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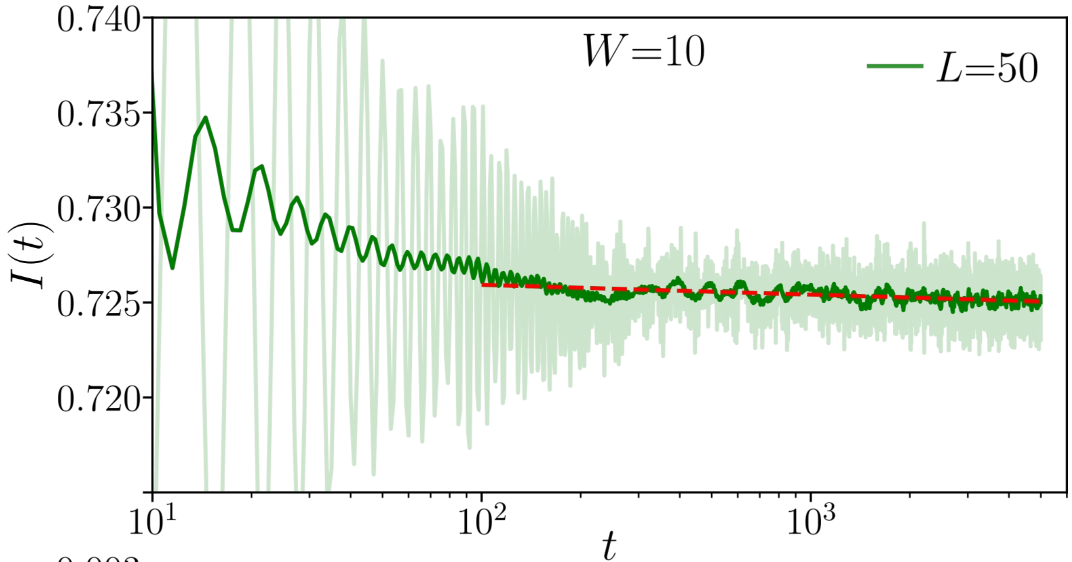


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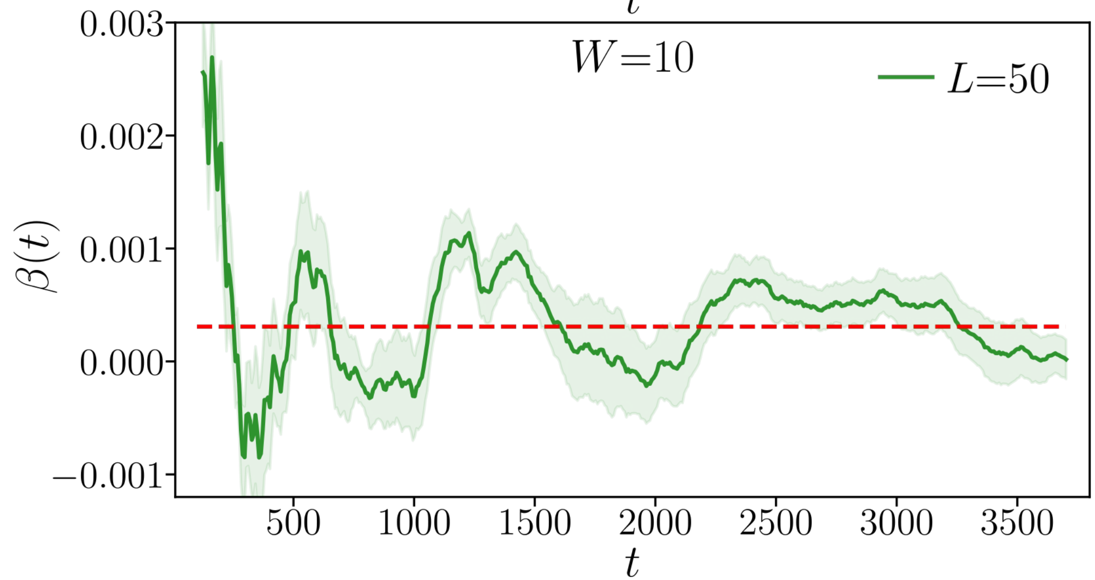


- Is the question of *MBL phase* relevant?

*Anderson model in 2D with lattice of size of the earth (with lattice spacing $10^{-10}$m) is in delocalized regime for $W^* < 0.8$*

J. Šuntajs, T. Prosen, L. Vidmar, Phys. Rev. B 107, 064205 (2023)

# Hamiltonian matrix of many-body system

- Random field XXZ spin-1/2 chain

$$H_{XXZ} = \sum_{i=1}^{L} \left( S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right) + \sum_{i=1}^{L} h_i S_i^z, \quad h_i \in [-W, W] \ \ (i.i.d.)$$

- Hilbert space dimension (total $S^z = 0$ sector)

$$\mathcal{N} = \binom{L}{L/2} \approx e^{L \ln 2}/\sqrt{L}$$

- Full exact diagonalization: $\mathcal{N} = \mathcal{O}(10^5)$

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- But the $H_{XXZ}$ matrix is sparse in $S_i^z$ eigenbasis:

$$\langle \downarrow\uparrow \uparrow\uparrow\downarrow\downarrow | H_{XXZ} | \uparrow\downarrow\uparrow\uparrow\downarrow\downarrow \rangle \neq 0$$

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Each spin configuration coupled to at most L states by $H_{XXZ}$

# The idea of POLFED

- Lanczos algorithm: an iterative method to obtain exterior eigenpairs

  C. Lanczos, Journal of Research of the National Bureau of Standards 45 (1950)

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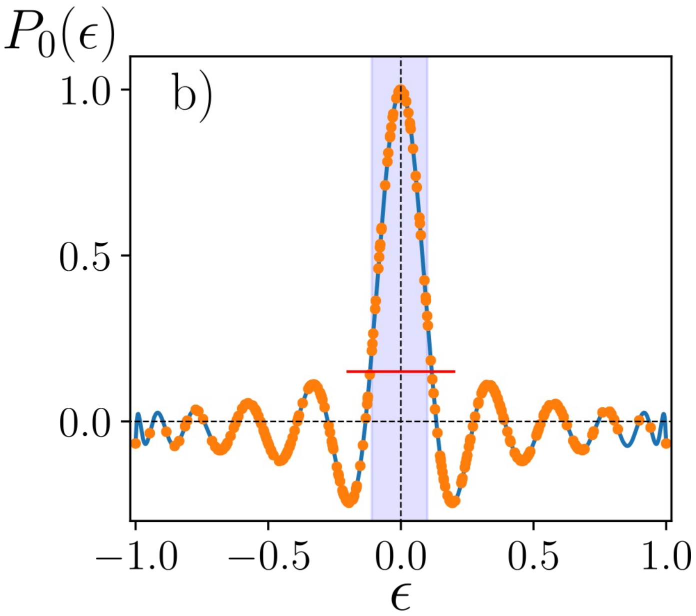


*Polynomial spectral transformation:*

$$H \to P_\sigma^K(H) = \frac{1}{D}\sum_{n=0}^{K} c_n^\sigma T_n(H)$$

$T_n(x)$ : n-th Chebyshev polynomial,
$c_n^\sigma$ : from expanding a Dirac delta function centered at $\sigma$

PS, M. Lewenstein, J. Zakrzewski, Phys. Rev. Lett. **125**, 156601 (2020)

# POLFED vs shift-and-invert

- POLFED: Lanczos algorithm + polynomial spectral transformation
- Shift-and-invert (SIMED) data from:

F Pietracaprina et al., SciPost Phys. 5, 045 (2018)
D. Luitz, N. Laflorencie, F. Alet, Phys. Rev. B 91, 081103(R) (2015)

| | $L$ | $t_{CPU}[h]$ | $N_{cores}$ | $t_W[h]$ | $RAM[GB]$ | $N_{ev}$ |
|---|---|---|---|---|---|---|
| POLFED | 20 | 3.1 | 1 | 3.1 | 3.9 | 1000 |
| | 22 | 62.2 | 4 | 15.5 | 21.2 | 1400 |
| | 24 | 1503 | 24 | 62.6 | 114 | 2000 |
| | 26 | 19870 | 24 | 828 | 488 | 2000 |
| SIMED | 20 | 0.5 | 20 | 0.026 | 22 | 100 |
| | 22 | 20.2 | 120 | 0.17 | 244 | 100 |
| | 24 | 840 | 2880 | 0.23 | 12288 | 50 |
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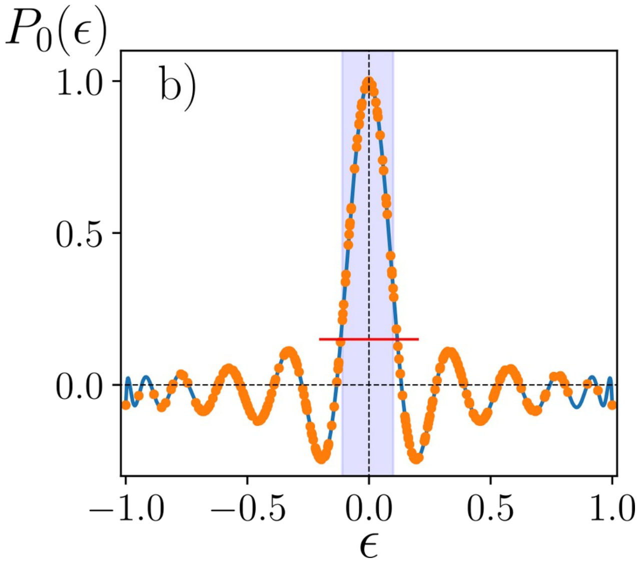
Table I. POLFED vs SIMED for $XXZ$ spin chain.



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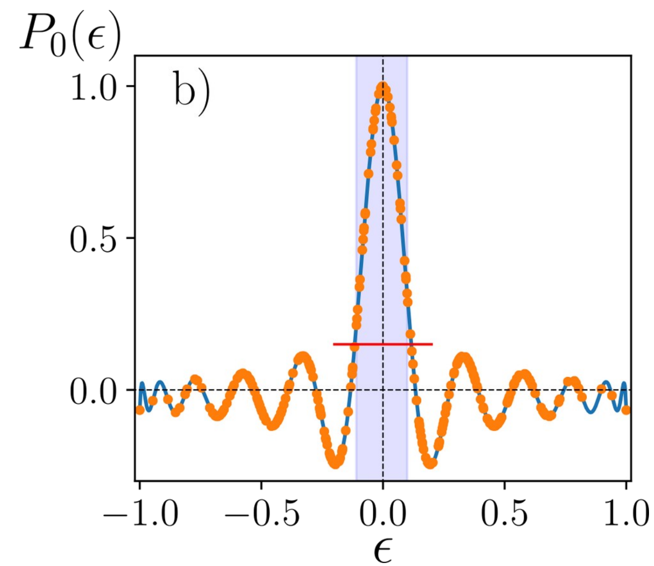
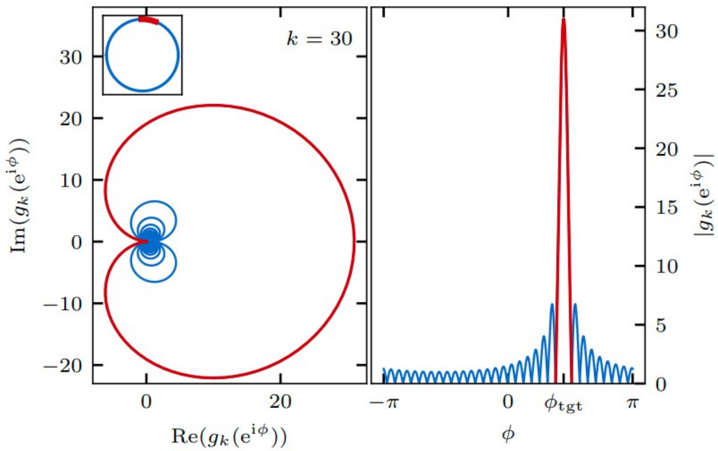



- POLFED can be used for Floquet systems:

$$U = U_1 U_2 = e^{-iH_1} e^{-iH_2}$$

D. Luitz, SciPost Phys. 11, 021 (2021)



# System size drifts at MBL crossover

$$H_{J_1-J_2} = \sum_{i=1}^{L}\sum_{l=1}^{2}\left(S_i^x S_{i+l}^x + S_i^y S_{i+l}^y + \Delta_l S_i^z S_{i+l}^z\right) + \sum_{i=1}^{L} h_i S_i^z$$

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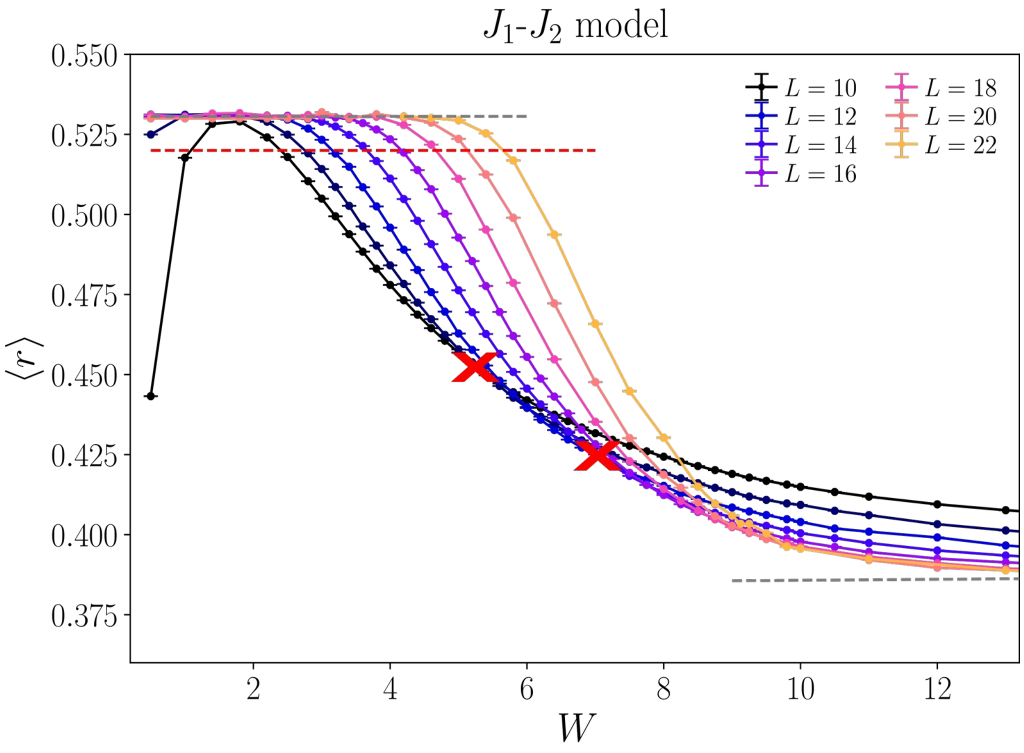

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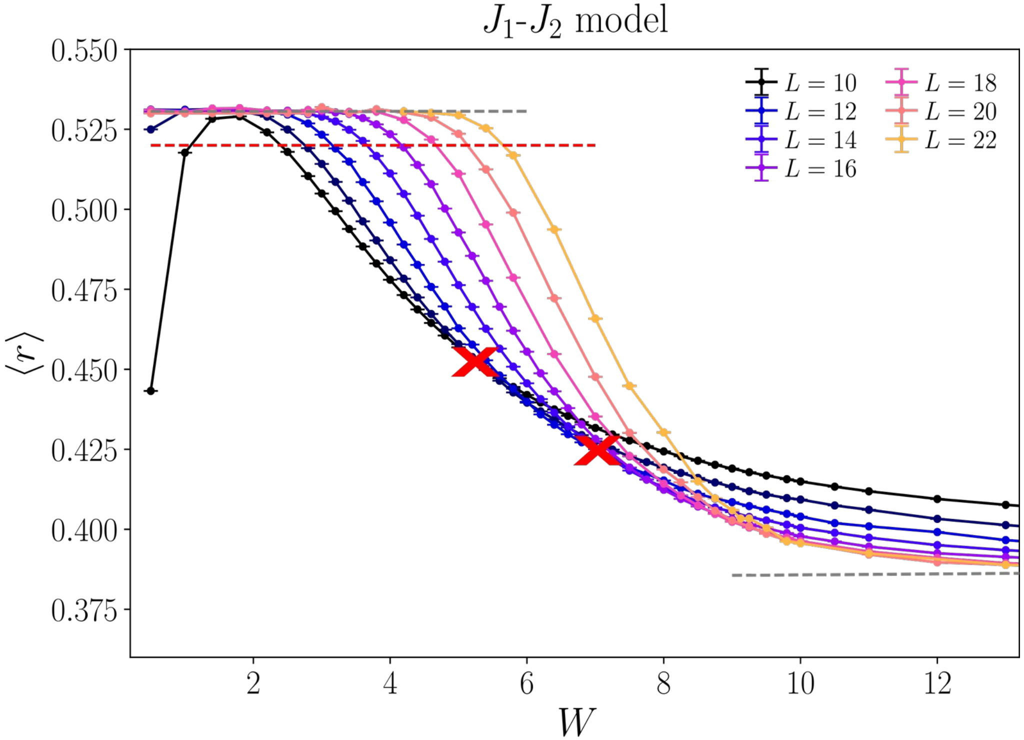
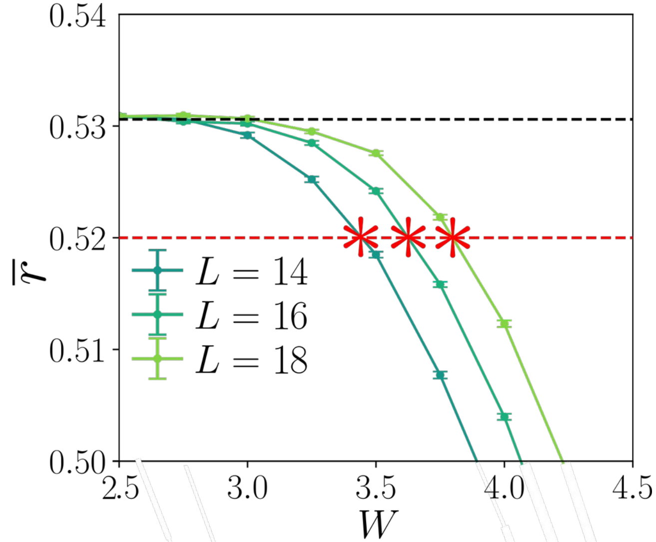



- $W_T(L)$ : deviation from ergodic behavior

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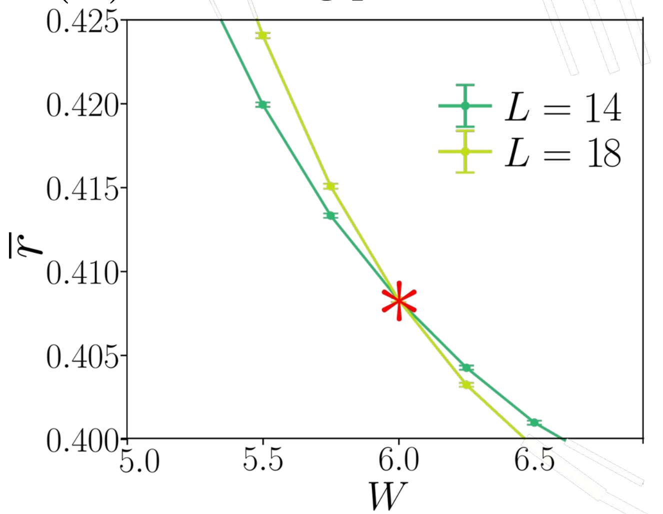

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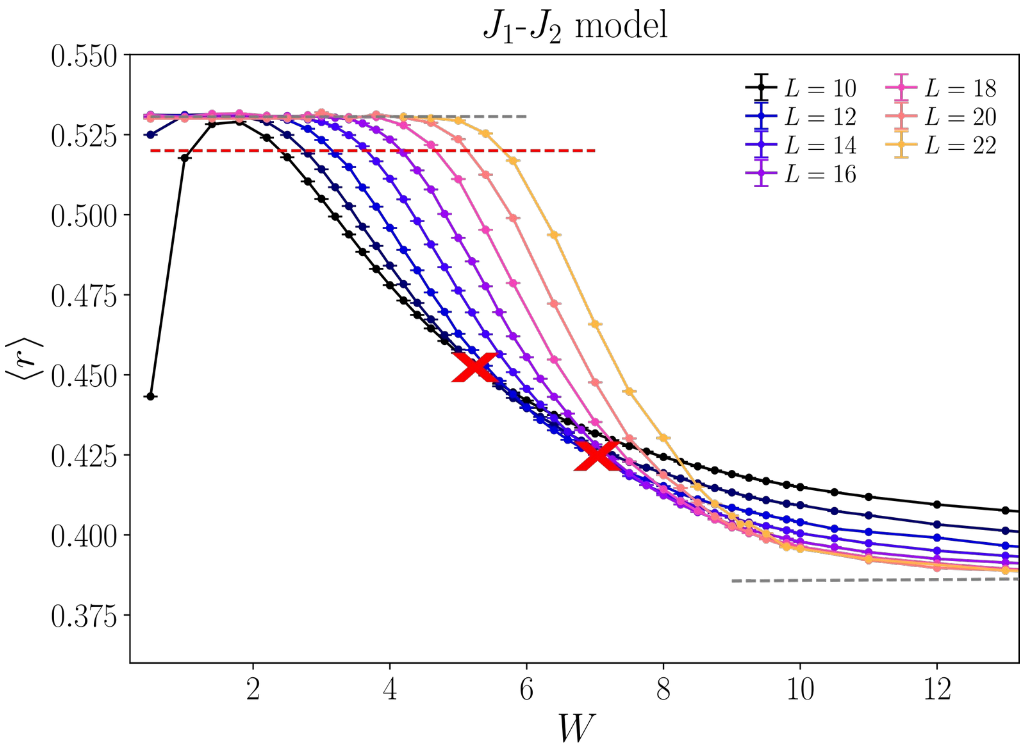


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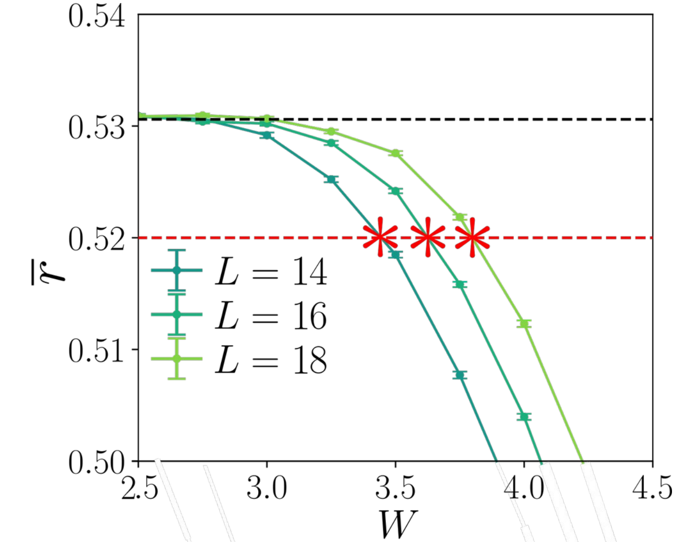


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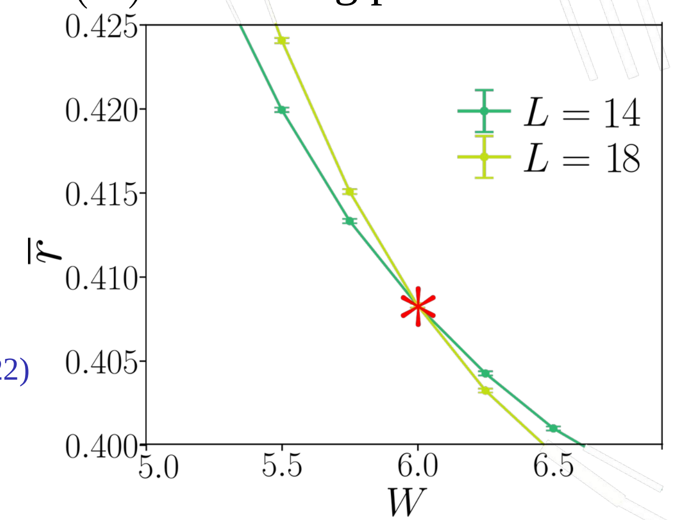


Check 1: Anderson model in 3D — J. Šuntajs, T. Prosen, L. Vidmar, Annals of Physics **435**, 168469 (2021)

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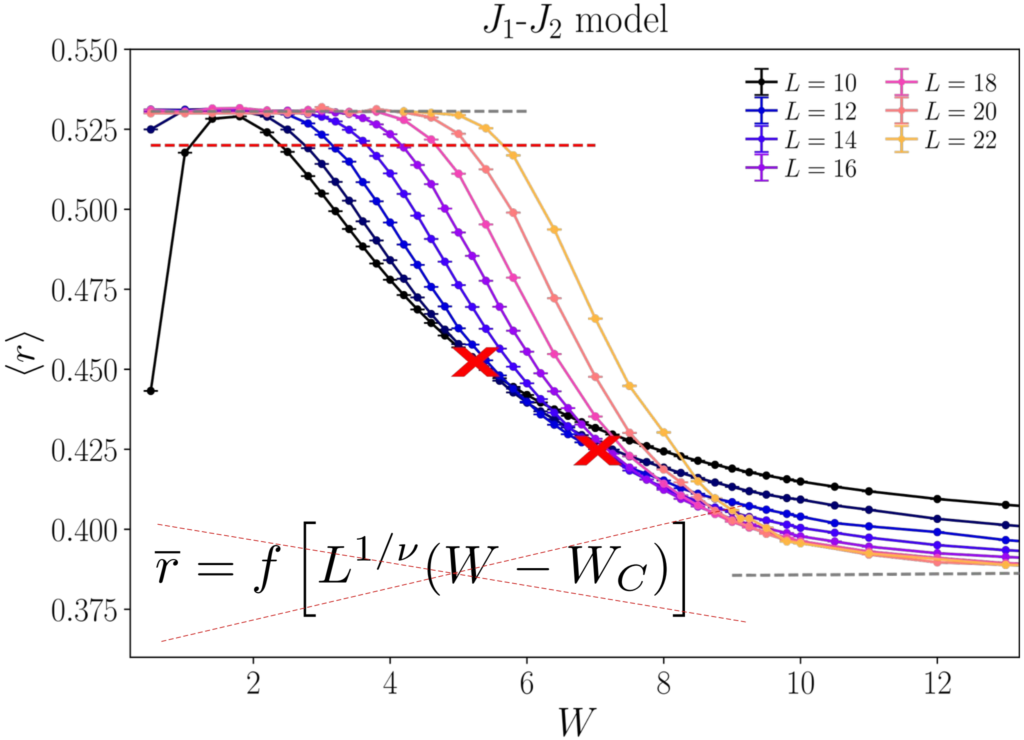


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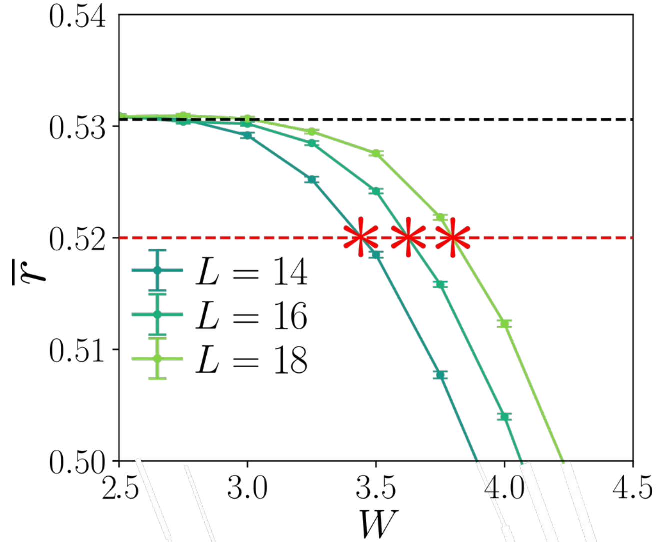


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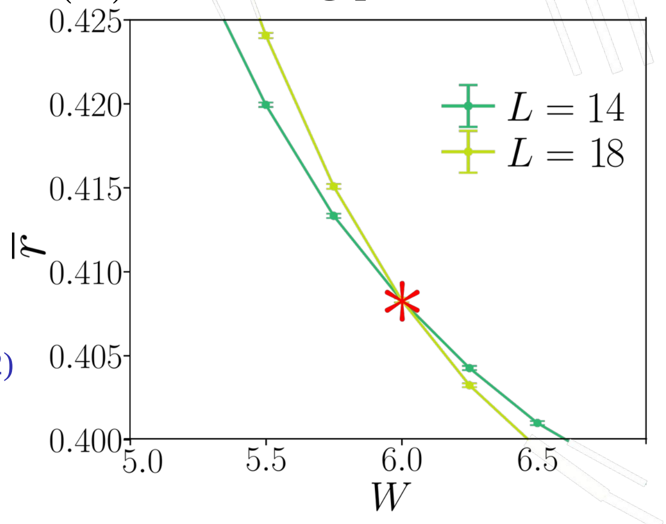


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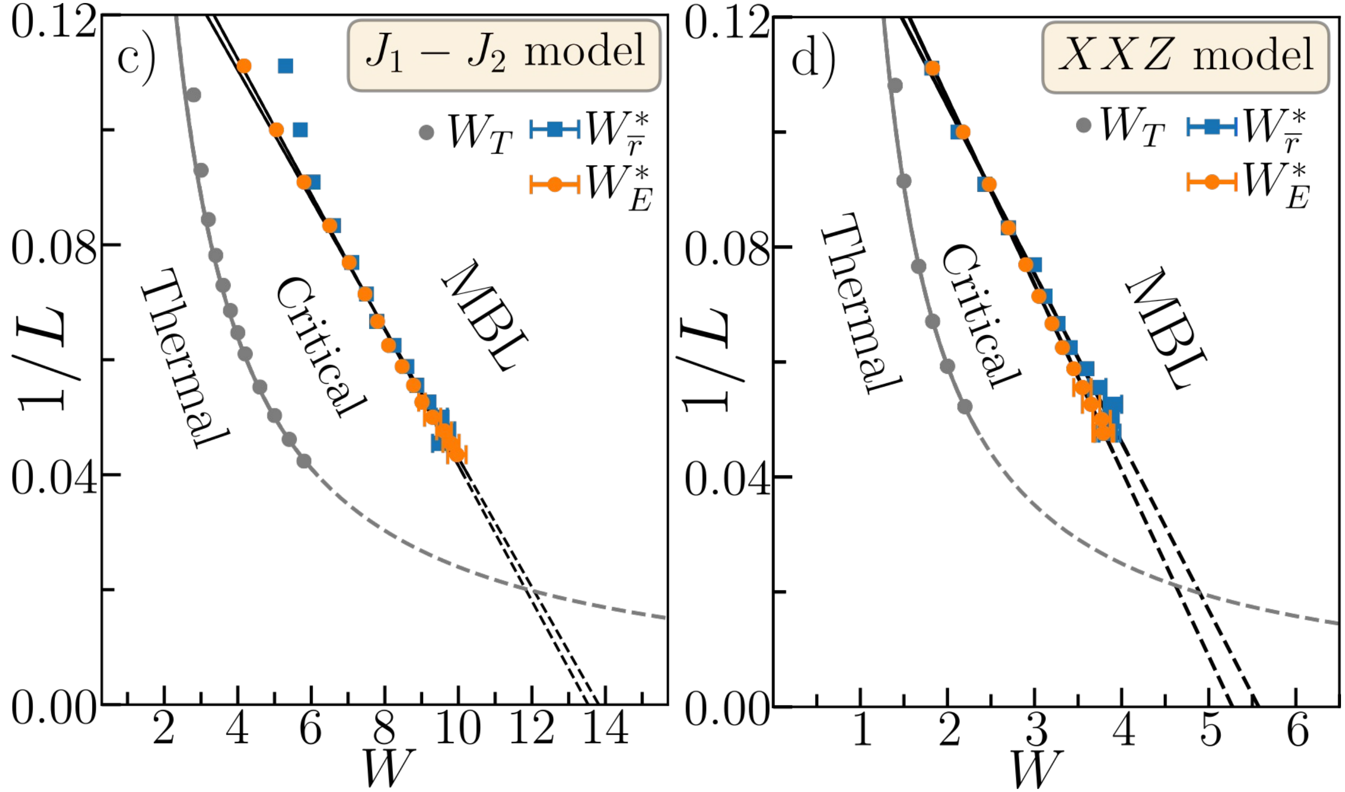


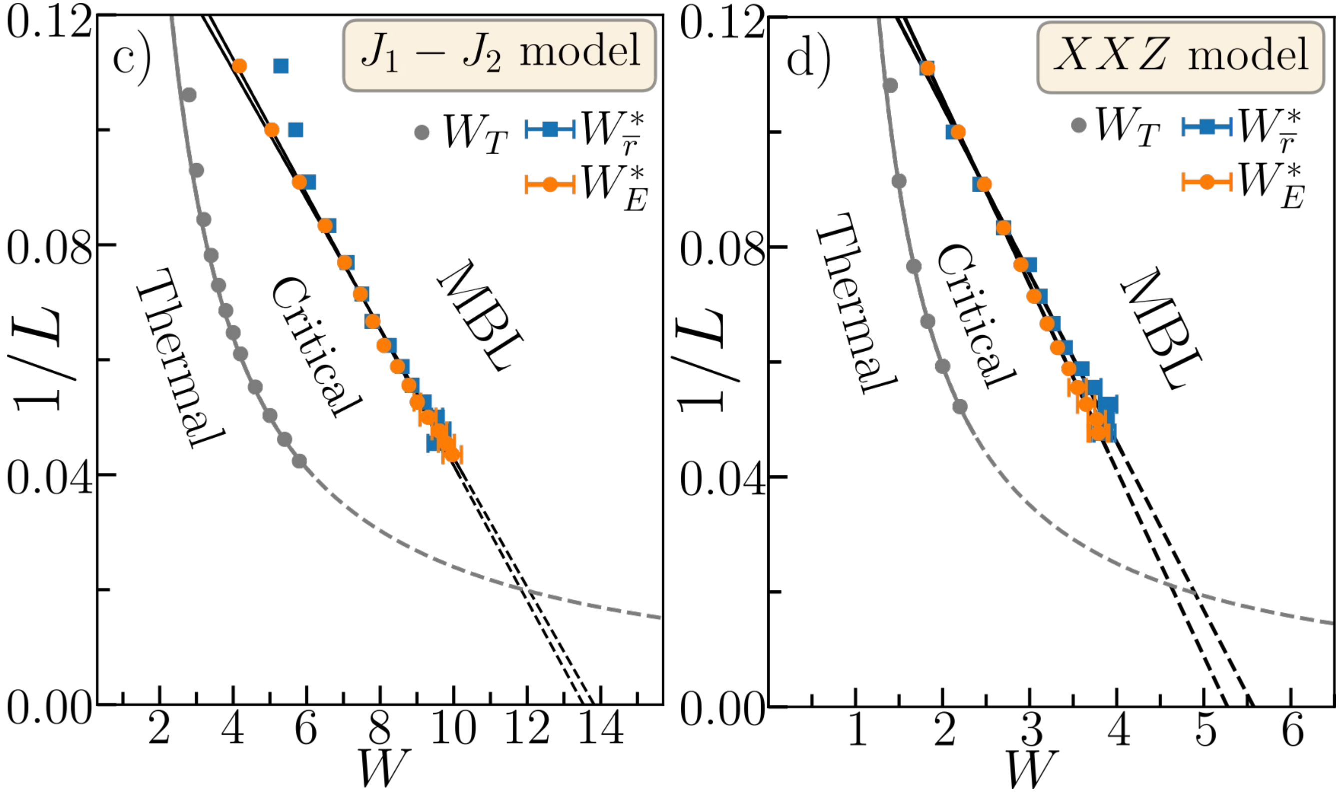

PS, M. Lewenstein, J. Zakrzewski, Phys. Rev. Lett. **125**, 156601 (2020)

# System size drifts at MBL crossover

- $W_T(L) = aL + b$
- $W^*(L) = a/L + W_C$

- The two scalings are incompatible, at least one of them breaks down at $L = L_0 \lesssim 50$
- If the scaling for $W^*(L)$ holds at $L \to \infty$

$$W_C^{J_1-J_2} \approx 13.7 \quad \text{and} \quad W_C^{XXZ} \approx 5.4$$

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# Model dependence of ETH-MBL crossover

- The Rydberg blockade regime (V>>1)

$$\hat{H} = \sum_{i=1}^{L} P_i^{\alpha} S_i^x P_{i+1+\alpha}^{\alpha} + \sum_{i=1}^{L} h_i S_i^z$$

where $P_i^{\alpha} = \prod_{j=i-\alpha}^{i-1} (1/2 + S_j^z)$

The Hilbert space dimension: $\mathcal{N}_{\alpha} = (\Phi_{\alpha})^L$

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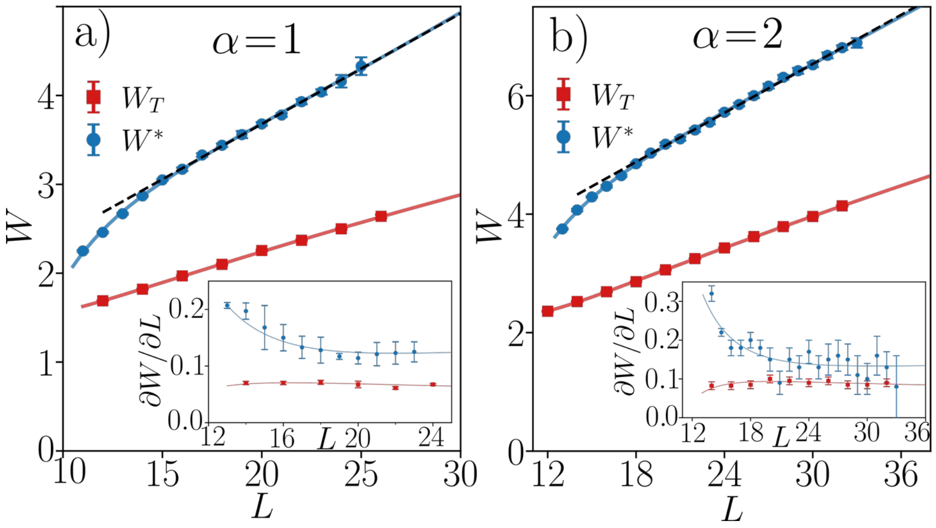


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PS, E. Lazo, M. Dalmonte, A. Scardicchio, J. Zakrzewski, PRL **127**, 126603 (2021)

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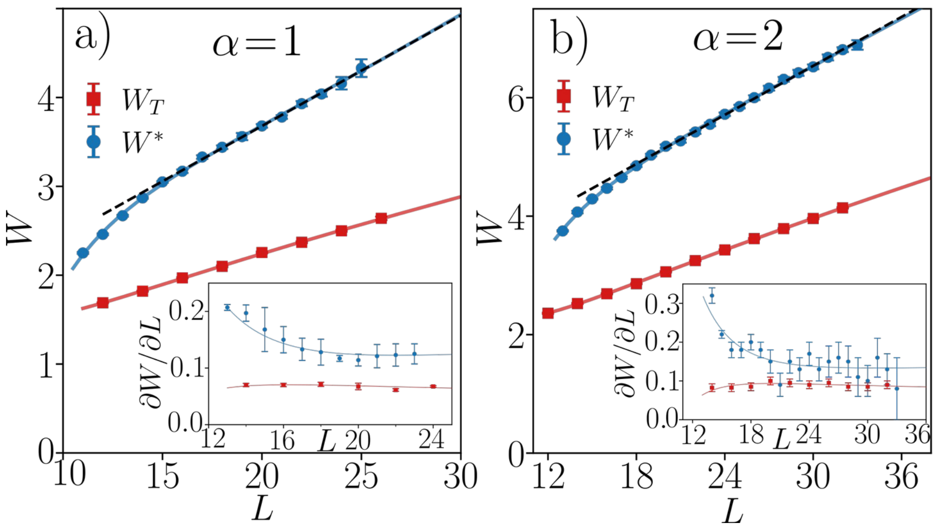


$W_T(L) \sim L$ $\qquad$ $W^*(L) \sim L$

- Kicked Ising model: $U_F =$

$e^{-ig\sum_j X_j} e^{-i\sum_j (gZ_jZ_{j+1}+h_jZ_j)}$

$h_j \in [0, 2\pi] \qquad W = \pi/(4g)$

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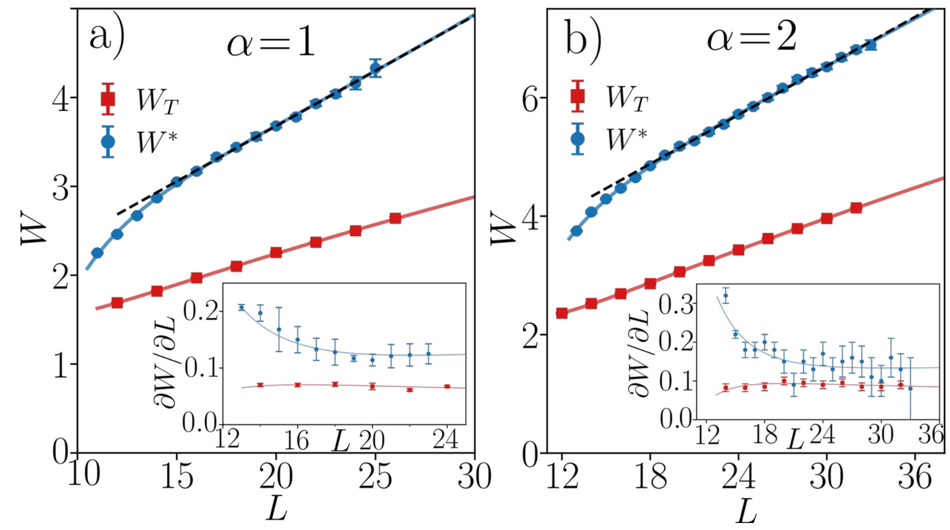


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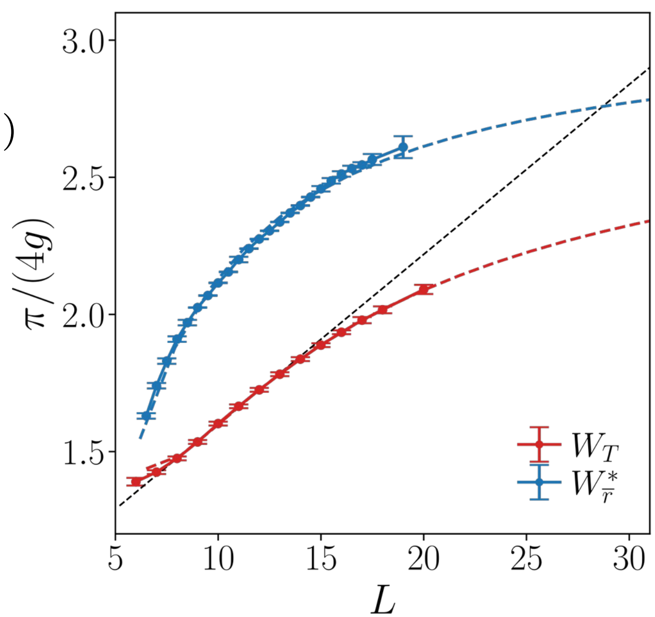
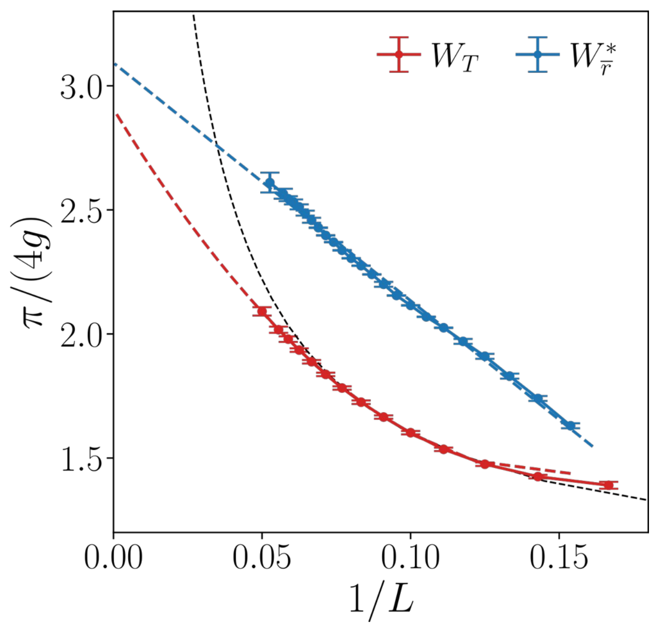
Eigenstates with POLFED
(up to $L \leq 20$)

PS, M. Lewenstein, A. Scardicchio, J. Zakrzewski, Phys. Rev. B **107**, 115132 (2023)



# Model dependence of ETH-MBL crossover

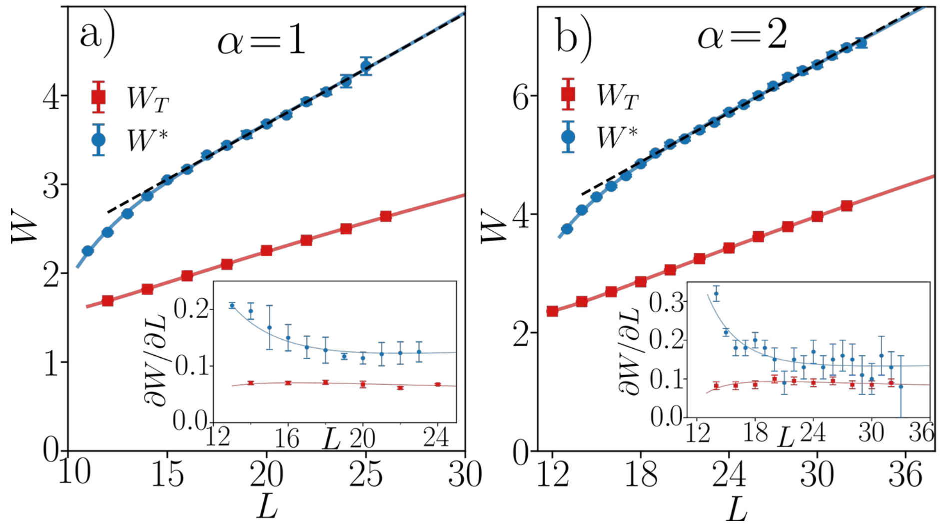
- The Rydberg blockade regime (V>>1)

$$\hat{H} = \sum_{i=1}^{L} P_i^{\alpha} S_i^x P_{i+1+\alpha}^{\alpha} + \sum_{i=1}^{L} h_i S_i^z$$

where $$P_i^{\alpha} = \prod_{j=i-\alpha}^{i-1} (1/2 + S_j^z)$$

The Hilbert space dimension: $\mathcal{N}_\alpha = (\Phi_\alpha)^L$

PS, E. Lazo, M. Dalmonte, A. Scardicchio, J. Zakrzewski, PRL **127**, 126603 (2021)




$W_T(L) \sim L$ $\qquad$ $W^*(L) \sim L$

- Kicked Ising model: $U_F = e^{-ig\sum_j X_j} e^{-i\sum_j (gZ_jZ_{j+1} + h_j Z_j)}$

$h_j \in [0, 2\pi] \qquad W = \pi/(4g)$

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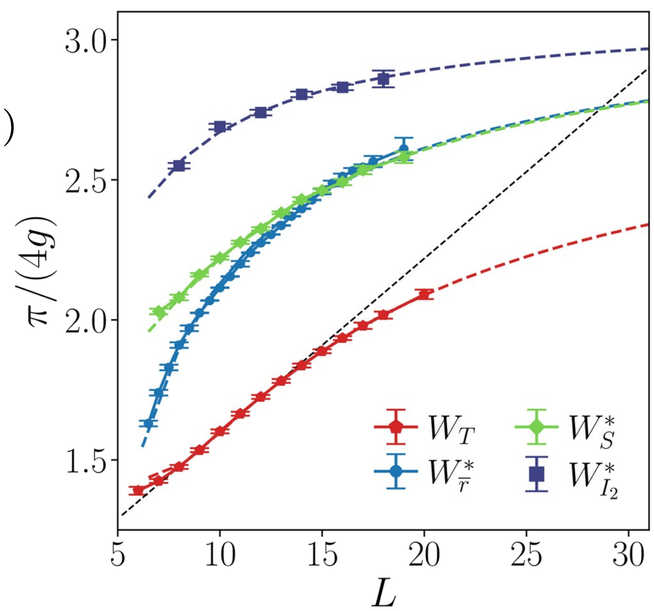
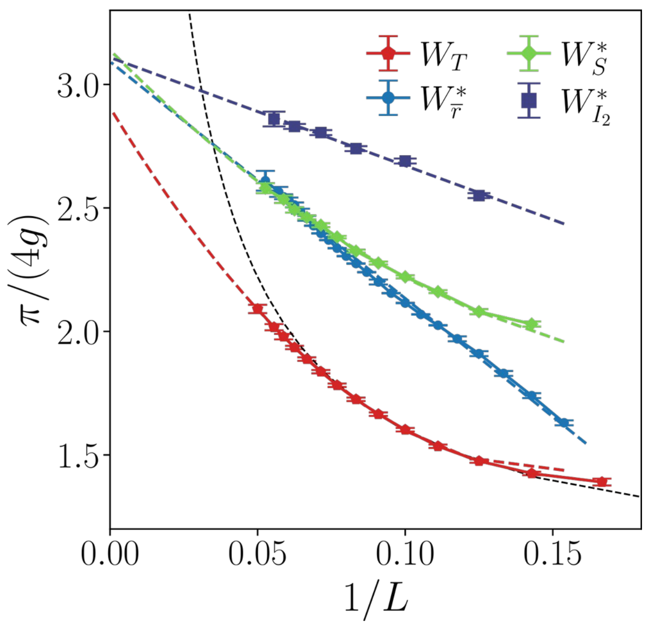


# Interlude – conclusion 1

- POLFED utilizes the sparse structure of Hamiltonian matrix to efficiently obtain highly excited eigenstates

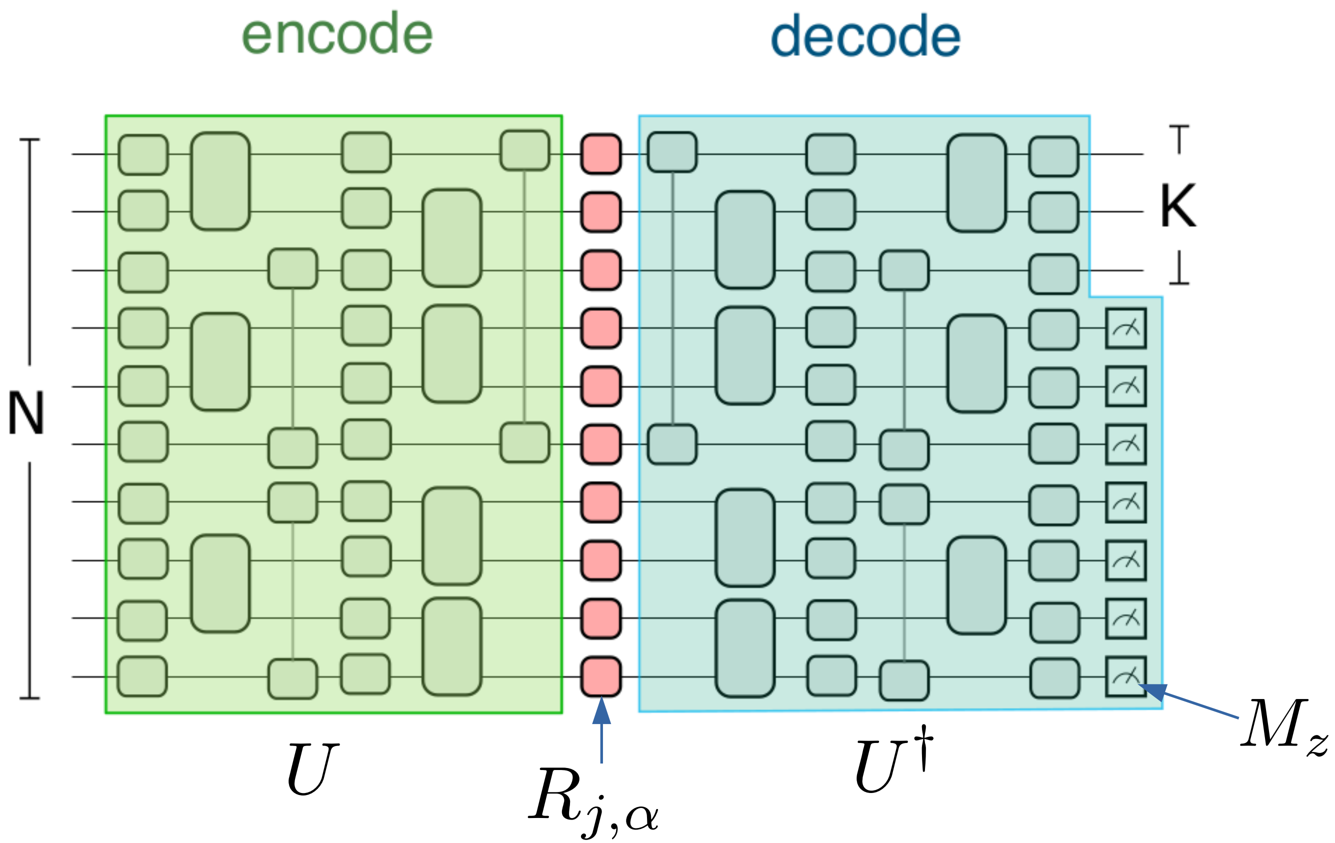
- Studies of ETH/MBL crossover in finite systems

| | |
|---|---|
| 1. XXZ spin chain / $J_1$- $J_2$ model | Incompatible scalings, crossing at $L_0 \approx 50$<br>$W_T(L) = aL + b$<br>$W^*(L) = a/L + W_C, \quad W_C \approx 5.4$ |
| 2. Constrained spin chains<br>*The Hilbert space dimension*<br>$\mathcal{N}_\alpha = (\Phi_\alpha)^L$ | Slow delocalization:<br>$W_T(L) \sim L$<br>$W^*(L) \sim L$ |
| 3. Kicked Ising model | Compatible scalings, crossing at $L_0 \approx 28$<br>$W_T(L)$ deviates from linear<br>$\lim_{L\to\infty} W^*(L) \approx 3.15 \qquad \nu \approx 2$ |

- See also: "Many-Body Localization in the age of classical computing"
  PS, M. Lewenstein, A. Scardicchio, L. Vidmar, J. Zakrzewski

# Encoding-decoding circuits

- “Phase transition in magic with random quantum circuits”, **arXiv:2304.10481**
  P. Niroula, C. D. White, Q. Wang, S. Johri, D. Zhu, C. Monroe, C. Noel, M. J. Gullans



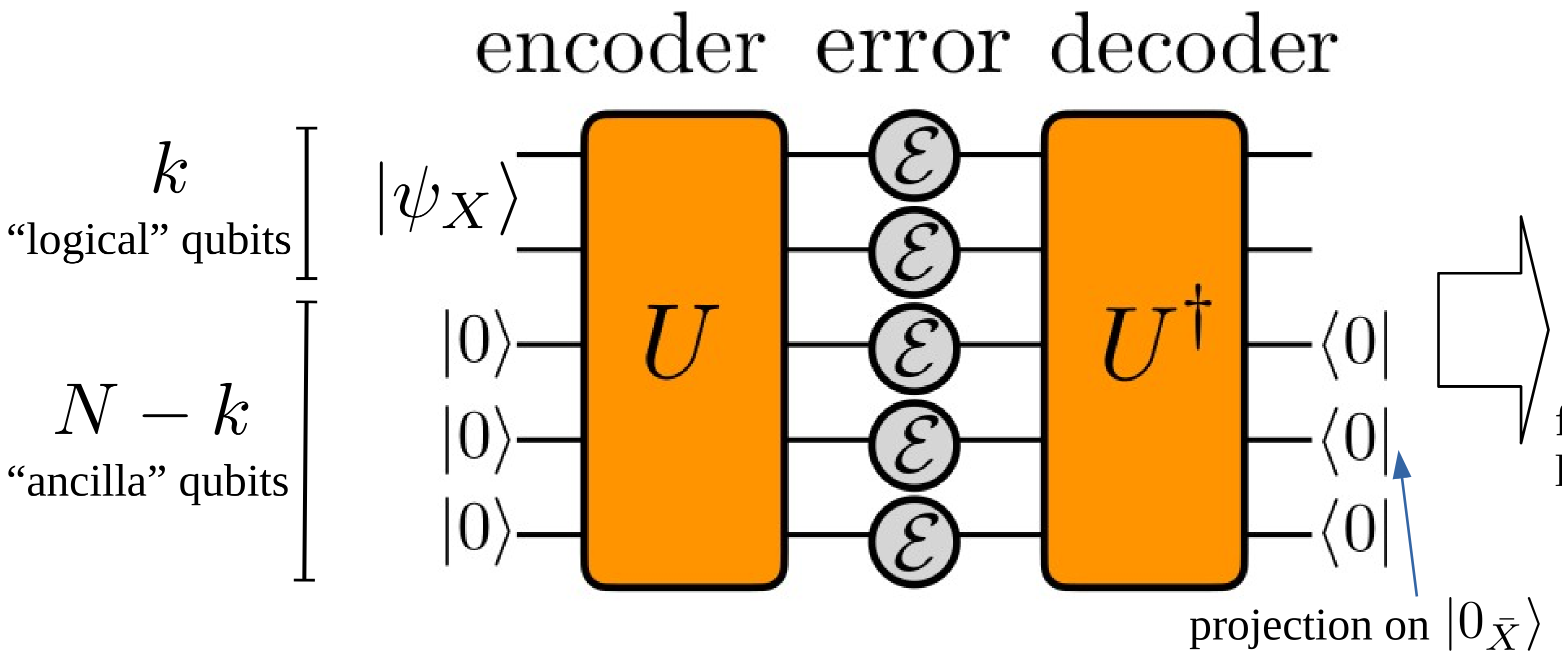

- Coherent “errors” $R_{j,\alpha} = e^{-i\frac{\alpha}{2}\sigma_j^z}$

- Implemented on IonQ’s Aria trapped-ion quantum computer

# Encoding-decoding circuits

X. Turkeshi, PS, arXiv:2308.06321

- We consider the following circuit architecture




$\rho_X$

final state of $k$ logical qubits

# Encoding-decoding circuits

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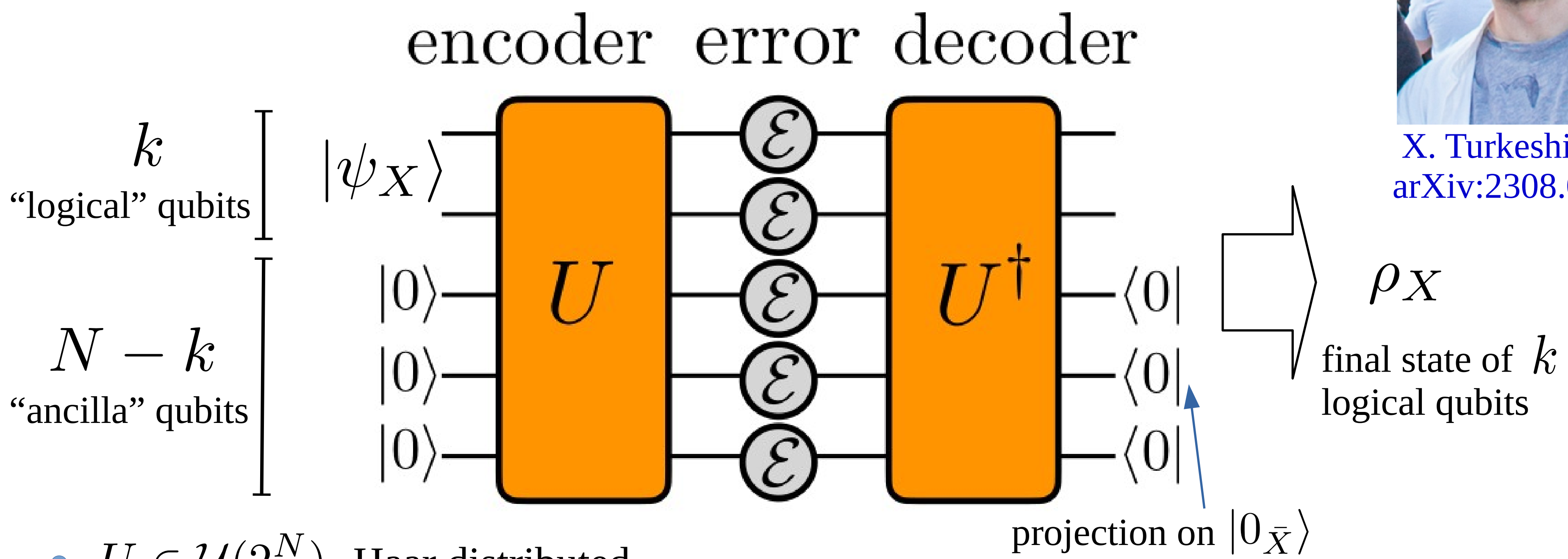


- $U \in \mathcal{U}(2^N)$, Haar distributed
- $\mathcal{E}_j$ local errors: coherent rotations $\mathcal{E}_j(\rho) = R_{j,\alpha}\,\rho\,R_{j,\alpha}^\dagger$

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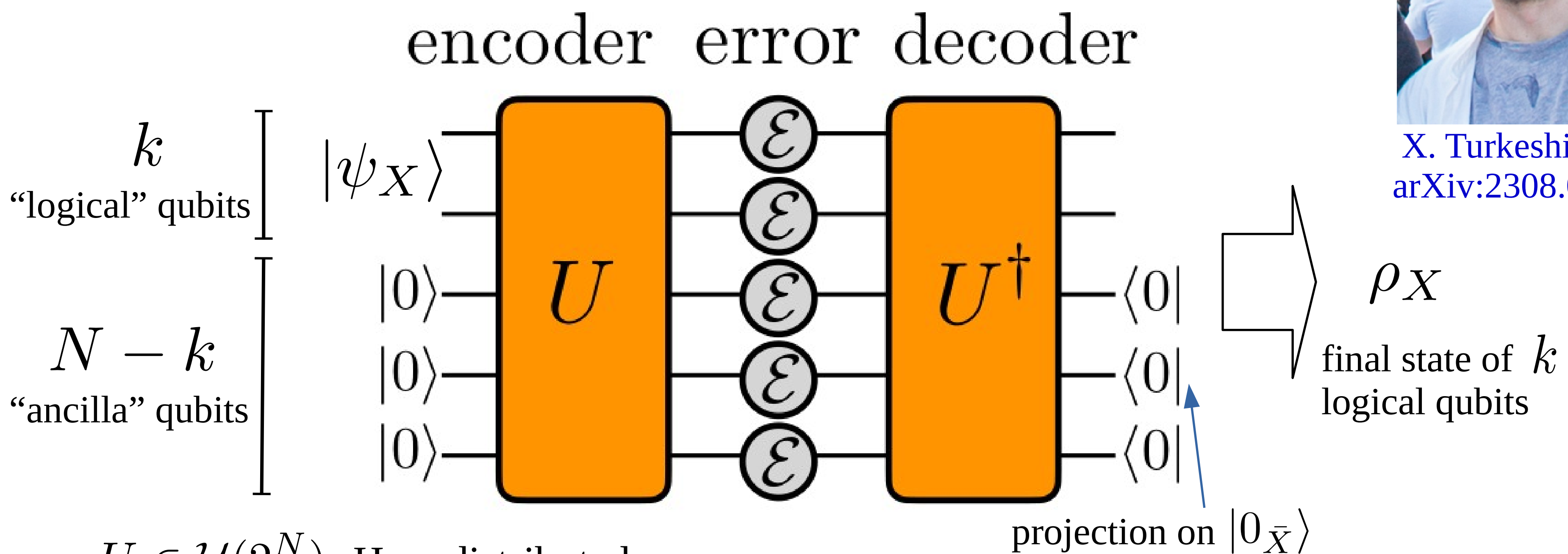



- $U \in \mathcal{U}(2^N)$, Haar distributed
- $\mathcal{E}_j$ local errors: coherent rotations $\mathcal{E}_j(\rho) = R_{j,\alpha}\,\rho\,R^\dagger_{j,\alpha}$

  depolarizing channel $\mathcal{E}_j(\rho) = \sum_{\mu=0}^{3} K_{\mu,j}\rho K^\dagger_{\mu,j}$

  Kraus operators: $K_{0,j} = \sqrt{1-\frac{3\lambda}{4}}$ $\quad K_{1,j} = \sqrt{\frac{\lambda}{4}}\sigma^x_j$ $\quad K_{2,j} = \sqrt{\frac{\lambda}{4}}\sigma^y_j$ $\quad K_{3,j} = \sqrt{\frac{\lambda}{4}}\sigma^z_j$

# Properties of the final state

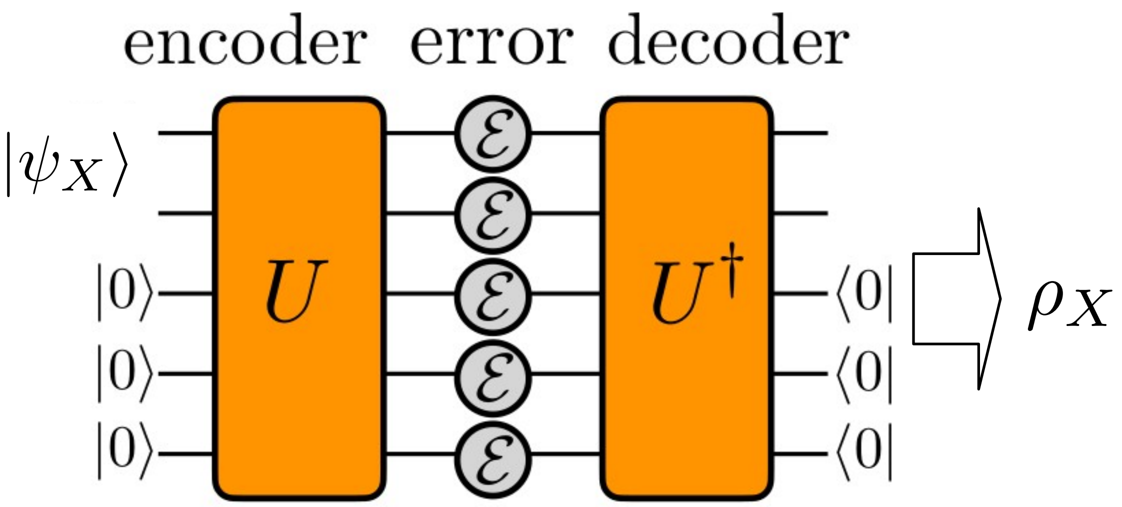
- Fidelity

$$F = \langle \psi_X | \rho_X | \psi_X \rangle$$

- Entropy

$$X = X_1 \cup X_2$$

$$S_q = \frac{1}{1-q} \log_2[\mathrm{tr}_{X_1}(\mathrm{tr}_{X_2}[\rho_X])^q]$$



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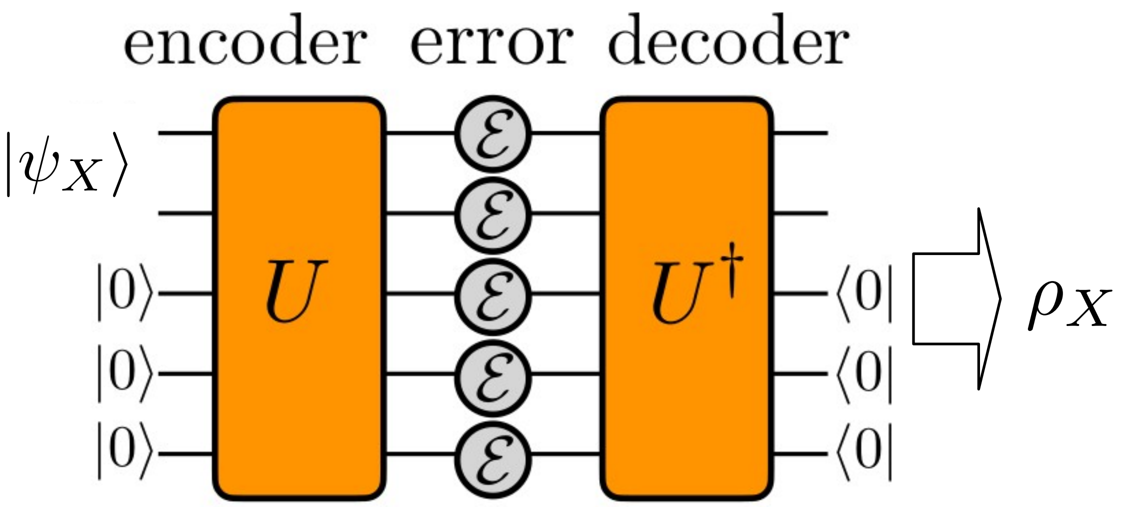


- Final state: $$\rho_X = \frac{\langle 0_{\bar{X}} | U^\dagger \mathcal{E} \left( U \rho_0 U^\dagger \right) U | 0_{\bar{X}} \rangle}{\mathrm{tr} \left[ \langle 0_{\bar{X}} | U^\dagger \mathcal{E} \left( U \rho_0 U^\dagger \right) U | 0_{\bar{X}} \rangle \right]}$$

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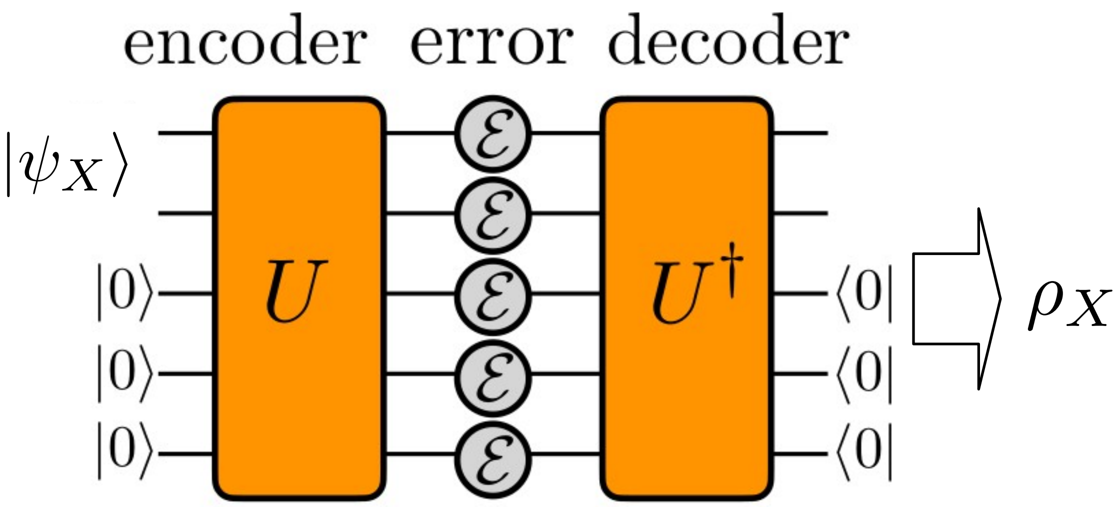


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- Coherent errors: $$R_\alpha \equiv \prod_{j=1}^{N} R_{j,\alpha}$$

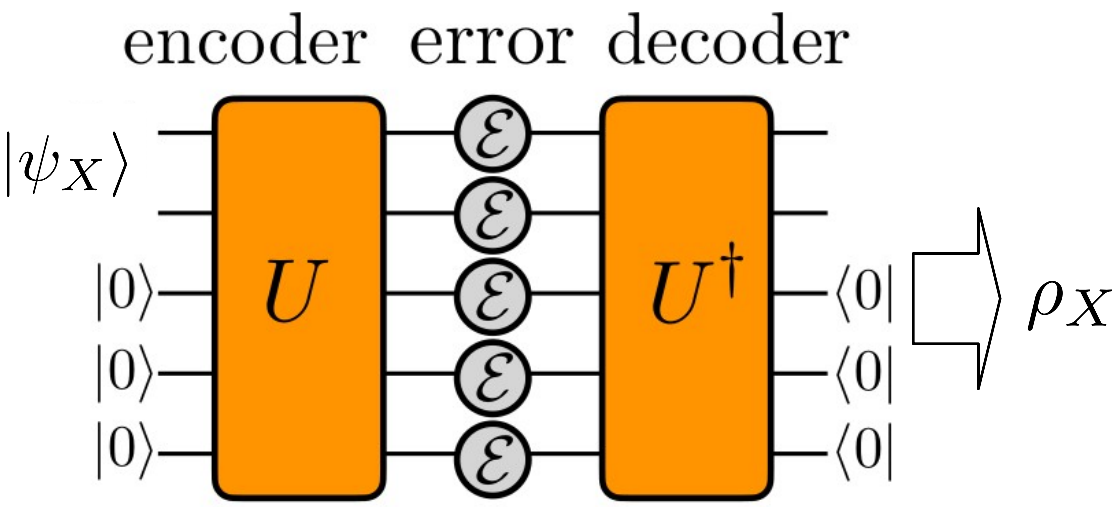
$$\rho_X = \frac{1}{\mathcal{N}_X} \langle 0_{\bar{X}} | U^\dagger R_\alpha U \rho_0 U^\dagger R_\alpha^\dagger U | 0_{\bar{X}} \rangle$$

# The fidelity – replica trick

- Fidelity

$$F = \langle \psi_X | \rho_X | \psi_X \rangle = \frac{1}{\mathcal{N}_X} \langle \psi_X 0_{\bar{X}} | U^\dagger R_\alpha U \rho_0 U^\dagger R_\alpha^\dagger U | \psi_X 0_{\bar{X}} \rangle$$

- $\rho_0 = |\psi_X 0_{\bar{X}}\rangle\langle\psi_X 0_{\bar{X}}|$



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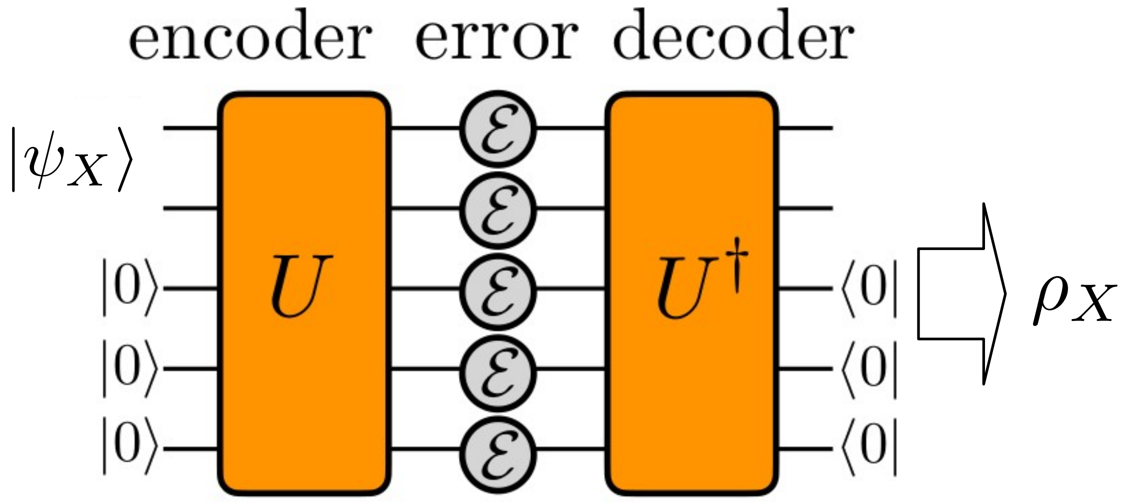


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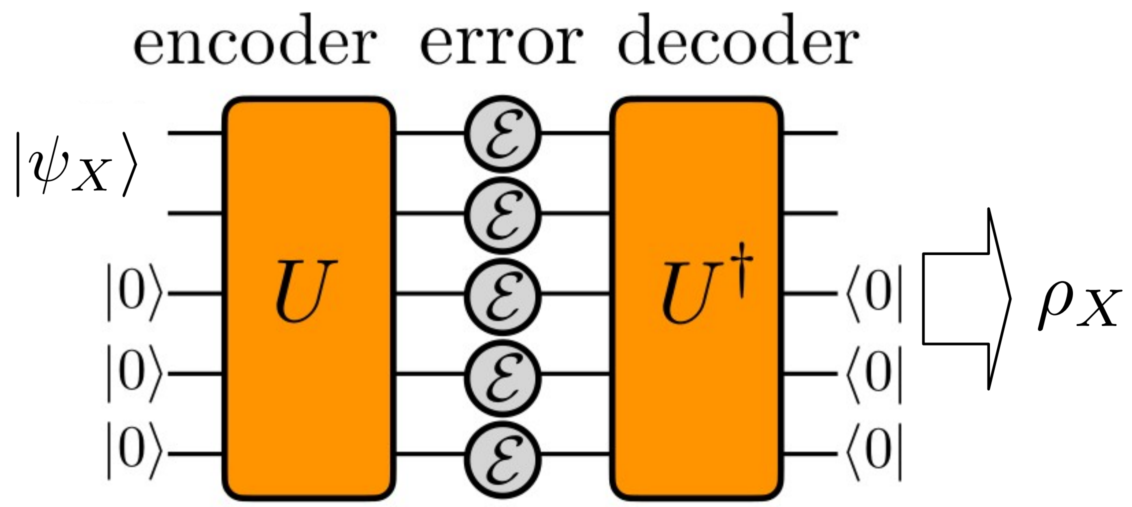
# The fidelity – replica trick

- Fidelity

$$F = \langle\psi_X|\rho_X|\psi_X\rangle = \frac{1}{\mathcal{N}_X}\langle\psi_X 0_{\bar{X}}|U^\dagger R_\alpha U \rho_0 U^\dagger R_\alpha^\dagger U|\psi_X 0_{\bar{X}}\rangle$$

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$$= \frac{1}{\mathcal{N}_X}\langle(\psi_X 0_{\bar{X}})(\psi_X 0_{\bar{X}})|(U^{\dagger\otimes 2})(R_\alpha\otimes R_\alpha^\dagger)(U^{\otimes 2})|(\psi_X 0_{\bar{X}})(\psi_X 0_{\bar{X}})\rangle$$

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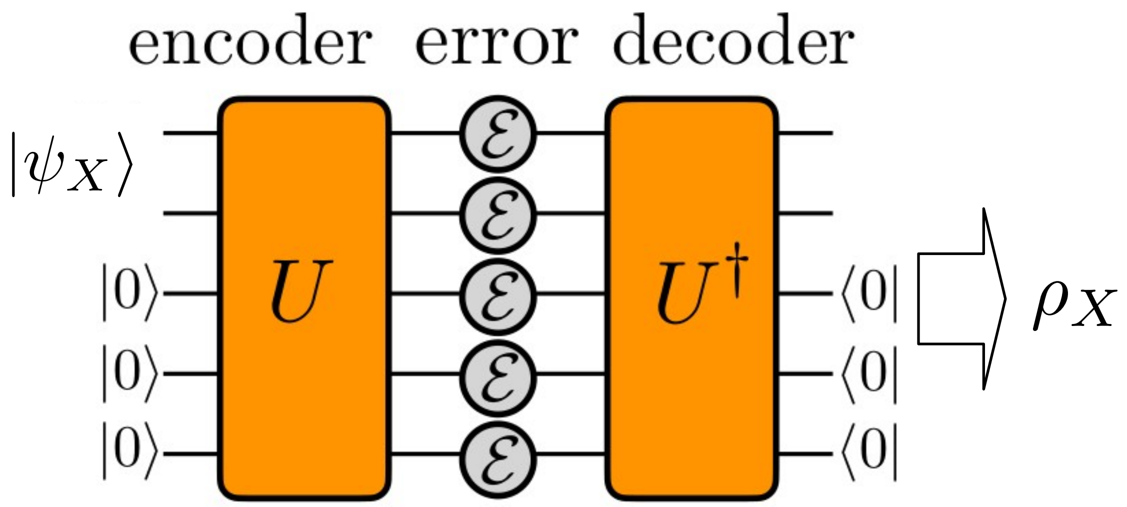


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- Similarly: $$\mathcal{N}_X = \sum_{x=1}^{2^k}\langle(x\,0_{\bar{X}})(\psi_X 0_{\bar{X}})|(U^{\dagger\otimes 2})(R_\alpha\otimes R_\alpha^\dagger)(U^{\otimes 2})|(x\,0_{\bar{X}})(\psi_X 0_{\bar{X}})\rangle$$

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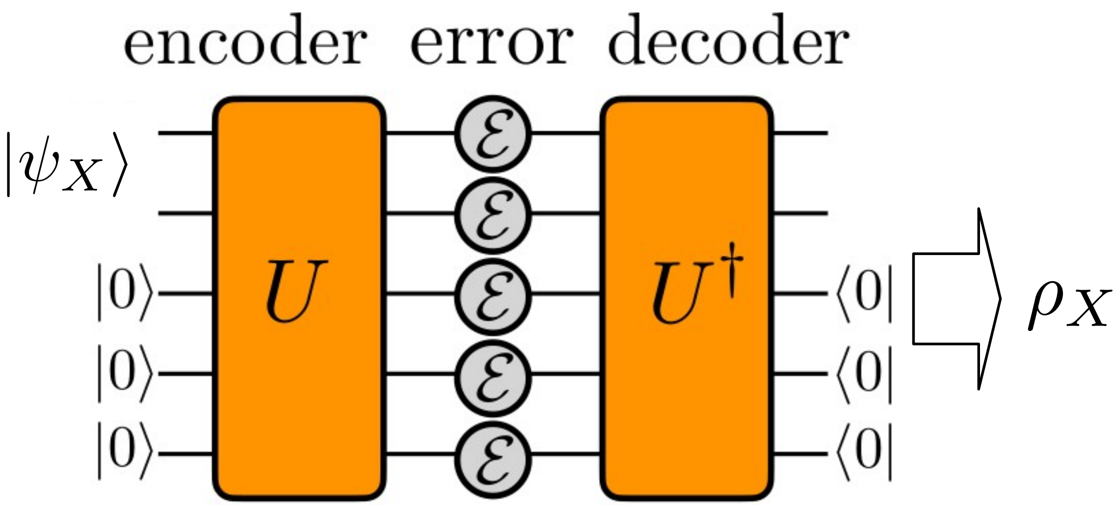


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- All in all: $$F = \frac{\mathrm{tr}(\mathcal{B}_{\mathrm{num}}^{F,(2)} A_U^{(2)})}{\mathrm{tr}(\mathcal{B}_{\mathrm{den}}^{F,(2)} A_U^{(2)})}$$ where $$A_U^{(2)} \equiv (U^{\dagger \otimes 2})(R_\alpha \otimes R_\alpha^\dagger)(U^{\otimes 2})$$

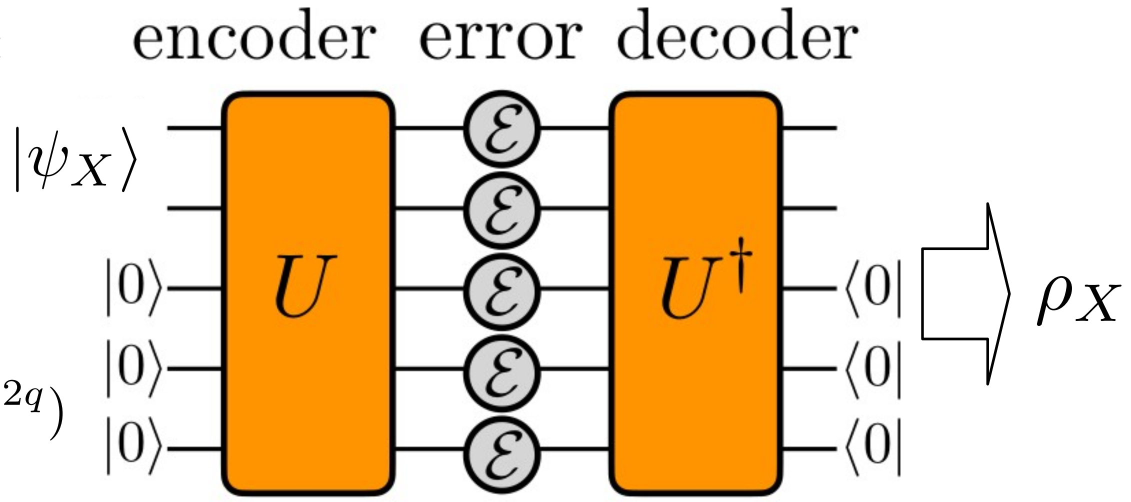
# The replica trick in general

- Fidelity & Entropies can be written as:

$$\Phi(A_U^{(2q)}) = \frac{\mathrm{tr}(\mathcal{B}_{\mathrm{num}}^{\Phi,(2q)} A_U^{(2q)})}{\mathrm{tr}(\mathcal{B}_{\mathrm{den}}^{\Phi,(2q)} A_U^{(2q)})}$$

for appropriate $q$, where

$$A_U^{(2q)} \equiv (U^{\dagger\otimes 2q})(R_\alpha^{\otimes q} \otimes (R_\alpha^\dagger)^{\otimes q})(U^{\otimes 2q})$$



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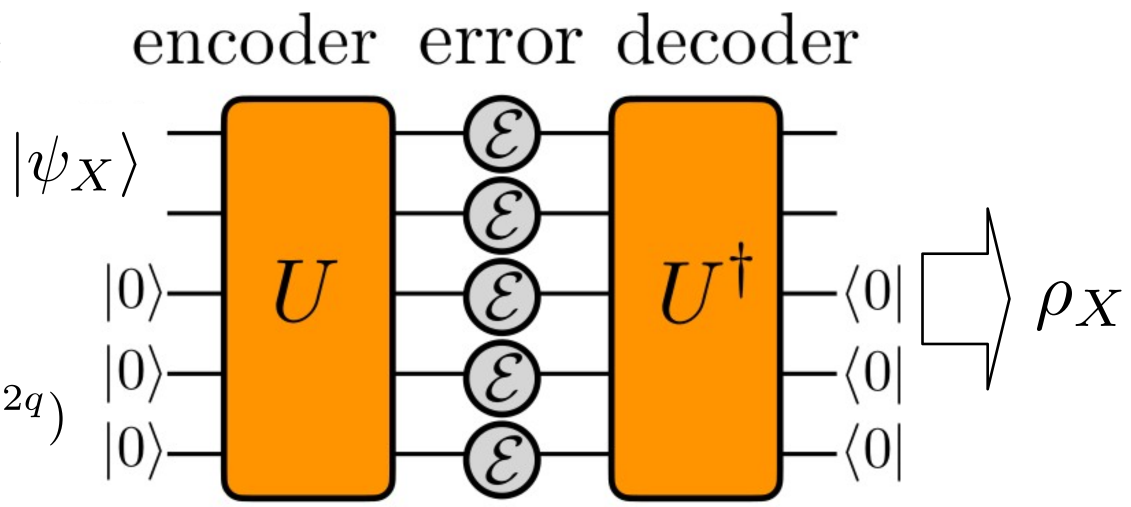


- "Annealed" averages:

$$\overline{\Phi} \equiv \mathbb{E}_{U\in\mathcal{U}(2^N)}\left[\Phi(A_U^{(2q)})\right] = \mathbb{E}_{U\in\mathcal{U}(2^N)}\left[\frac{\mathrm{tr}(\mathcal{B}_{\mathrm{num}}^{X,(2q)} A_U^{(2q)})}{\mathrm{tr}(\mathcal{B}_{\mathrm{den}}^{X,(2q)} A_U^{(2q)})}\right]$$

Numerically

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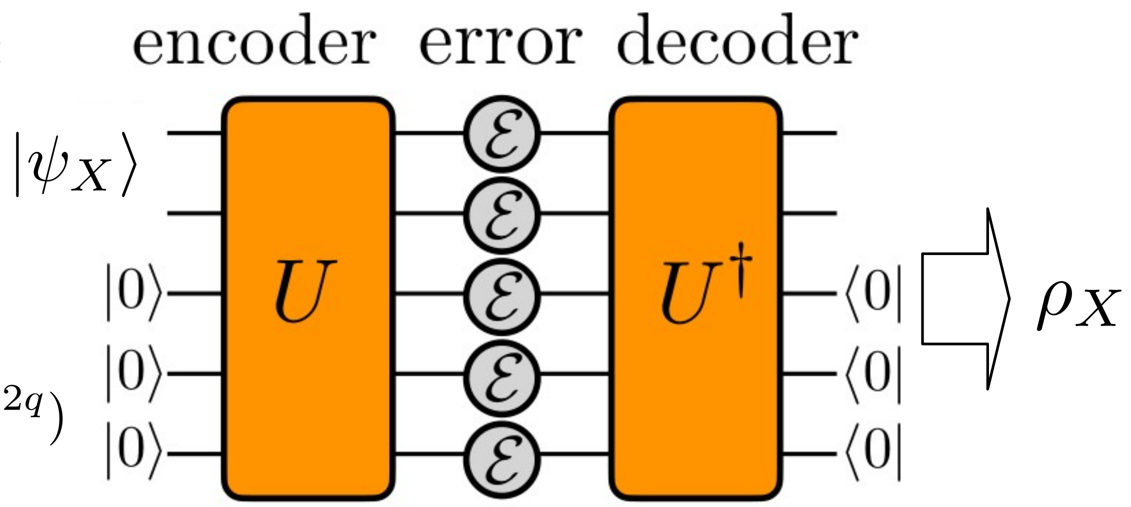


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- “Quenched” averages:

$$\tilde{\Phi} \equiv \Phi(\mathbb{E}_{U\in\mathcal{U}(2^N)} A_U^{(2q)}) = \frac{\mathrm{tr}(\mathcal{B}_{\mathrm{num}}^{X,(2q)} \mathbb{E}_{U\in\mathcal{U}(2^N)}[A_U^{(2q)}])}{\mathrm{tr}(\mathcal{B}_{\mathrm{den}}^{X,(2q)} \mathbb{E}_{U\in\mathcal{U}(2^N)}[A_U^{(2q)}])}$$ Analytically

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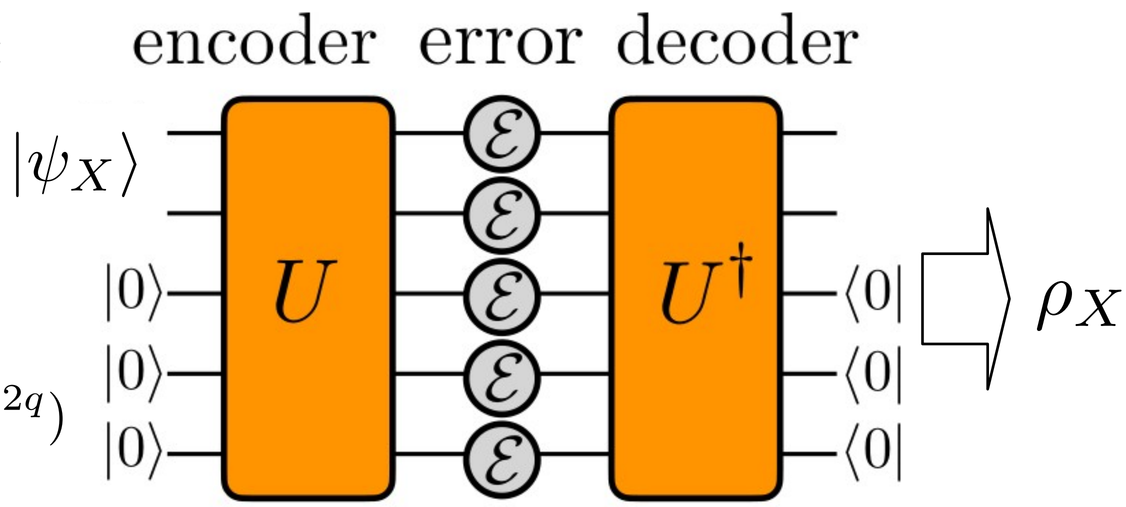


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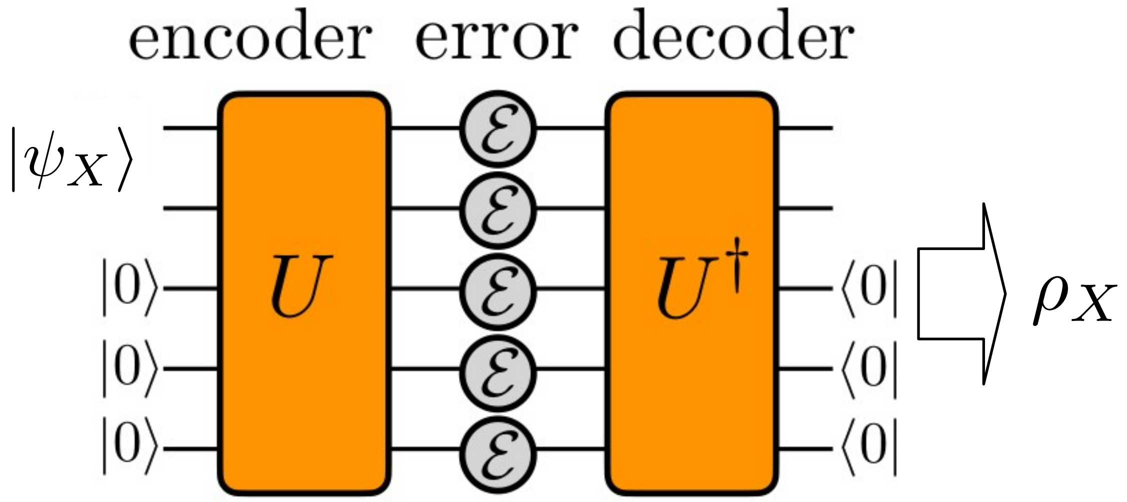
Analytically

- Self-averaging: $\overline{\Phi} = \tilde{\Phi} + \mathcal{O}(2^{-\eta N})$

# Quenched average of Fidelity

$$\tilde{F} = \frac{\mathrm{tr}(\mathcal{B}_{\mathrm{num}}^{F,(2)} \mathbb{E}_{U\in\mathcal{U}(2^N)}[A_U^{(2)}])}{\mathrm{tr}(\mathcal{B}_{\mathrm{den}}^{F,(2)} \mathbb{E}_{U\in\mathcal{U}(2^N)}[A_U^{(2)}])}$$

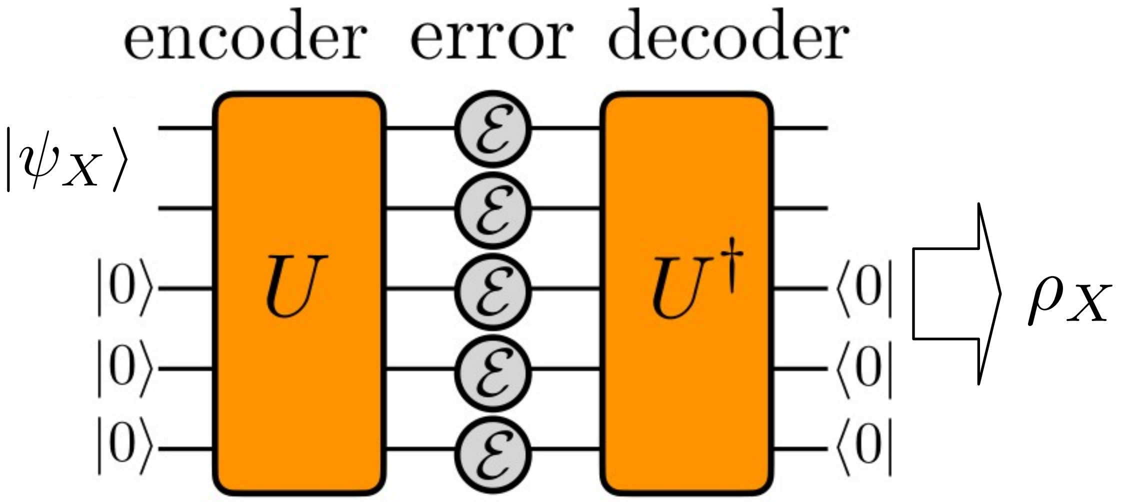



- Recall

$$A_U^{(2)} \equiv (U^{\dagger\otimes 2})(R_\alpha \otimes R_\alpha^\dagger)(U^{\otimes 2})$$

# Quenched average of Fidelity

$$\tilde{F} = \frac{\mathrm{tr}(\mathcal{B}^{F,(2)}_{\mathrm{num}} \mathbb{E}_{U\in\mathcal{U}(2^N)}[A^{(2)}_U])}{\mathrm{tr}(\mathcal{B}^{F,(2)}_{\mathrm{den}} \mathbb{E}_{U\in\mathcal{U}(2^N)}[A^{(2)}_U])}$$




- Recall

$$A^{(2)}_U \equiv (U^{\dagger\otimes 2})(R_\alpha \otimes R^\dagger_\alpha)(U^{\otimes 2})$$

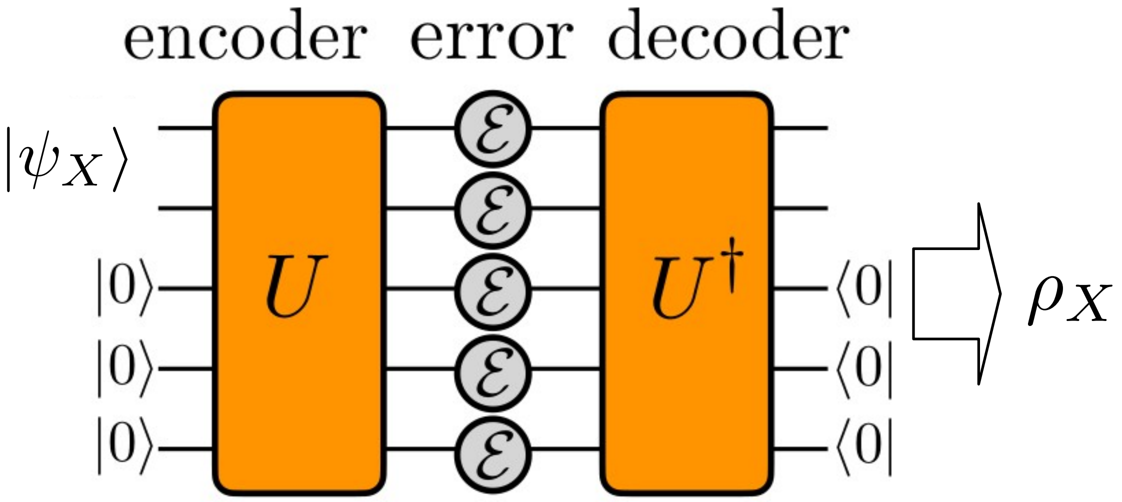
- Schur-Weyl duality - calculation of the unitary group average

$$\mathbb{E}_{U\in\mathcal{U}(2^N)}[A^{(2)}_U] = \sum_{\pi\in S_2} b_\pi T_\pi \quad \text{where} \quad b_\pi = \sum_{\sigma\in S_2} W_{\pi,\sigma} \mathrm{tr}\left[T_\sigma(R_\alpha \otimes R^\dagger_\alpha\right]$$

with $S_2$ permutation group of 2 elements and $T_\sigma$ from its representation over $(\mathbb{C}^{2^N})^{\otimes 2}$ and $W_{\pi,\sigma}$ the Weingarten symbol

# Quenched average of Fidelity

$$\tilde{F} = \frac{\mathrm{tr}(\mathcal{B}^{F,(2)}_{\mathrm{num}} \mathbb{E}_{U\in\mathcal{U}(2^N)}[A^{(2)}_U])}{\mathrm{tr}(\mathcal{B}^{F,(2)}_{\mathrm{den}} \mathbb{E}_{U\in\mathcal{U}(2^N)}[A^{(2)}_U])}$$

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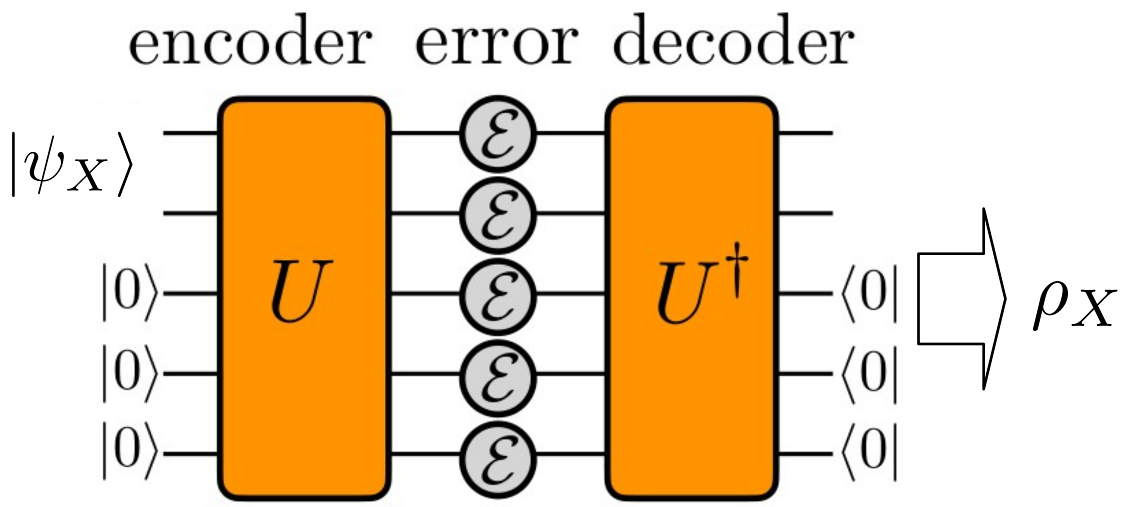
- $S_2$ contains two elements $\{I, S\}$

$$W_{I,I} = W_{S,S} = (4^N - 1)^{-1}$$

$$W_{I,S} = W_{S,I} = -\left(2^N(4^N - 1)\right)^{-1}$$

# Quenched average of Fidelity

$$\tilde{F} = \frac{\mathrm{tr}(\mathcal{B}^{F,(2)}_{\mathrm{num}} \mathbb{E}_{U\in\mathcal{U}(2^N)}[A^{(2)}_U])}{\mathrm{tr}(\mathcal{B}^{F,(2)}_{\mathrm{den}} \mathbb{E}_{U\in\mathcal{U}(2^N)}[A^{(2)}_U])}$$




- Recall

$$A^{(2)}_U \equiv (U^{\dagger\otimes 2})(R_\alpha \otimes R^\dagger_\alpha)(U^{\otimes 2})$$

- Schur-Weyl duality - calculation of the unitary group average

$$\mathbb{E}_{U\in\mathcal{U}(2^N)}[A^{(2)}_U] = \sum_{\pi\in S_2} b_\pi T_\pi \quad \text{where} \quad b_\pi = \sum_{\sigma\in S_2} W_{\pi,\sigma}\mathrm{tr}\left[T_\sigma(R_\alpha \otimes R^\dagger_\alpha\right]$$

with $S_2$ permutation group of 2 elements and $T_\sigma$ from its representation over $(\mathbb{C}^{2^N})^{\otimes 2}$ and $W_{\pi,\sigma}$ the Weingarten symbol

- $S_2$ contains two elements $\{I, S\}$

$$W_{I,I} = W_{S,S} = (4^N - 1)^{-1}$$

$$W_{I,S} = W_{S,I} = -\left(2^N(4^N - 1)\right)^{-1}$$

- The traces are directly evaluated as

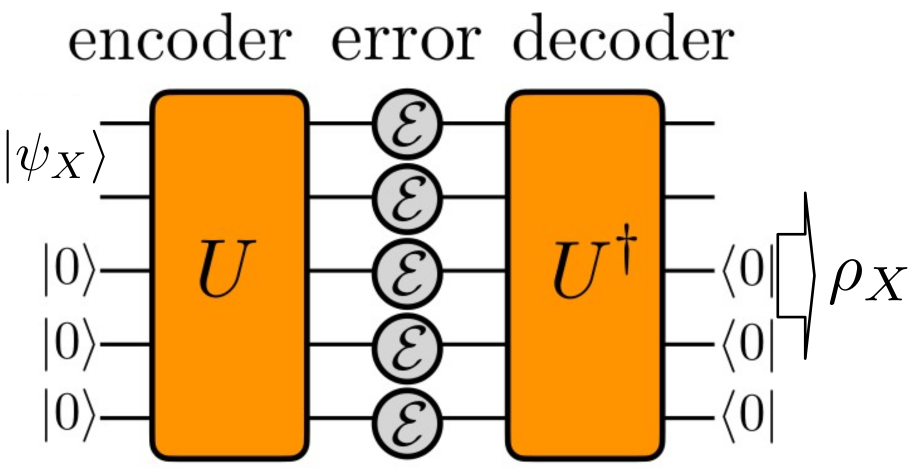
$$\mathrm{tr}\left[T_I(R_\alpha \otimes R^\dagger_\alpha\right] = 4^N \cos^{2N}\left(\frac{\alpha}{2}\right)$$

$$\mathrm{tr}\left[T_S(R_\alpha \otimes R^\dagger_\alpha\right] = 2^N$$

# Averages of Fidelity

- Fidelity - quenched average, coherent errors:

$$\tilde{F} = \frac{\left(2^N - 1\right)\left(2^N \cos^{2N}(\alpha/2) + 1\right)}{2^N \cos^{2N}(\alpha/2)\,(2^N - 2^k) + 2^{N+k} - 1}$$



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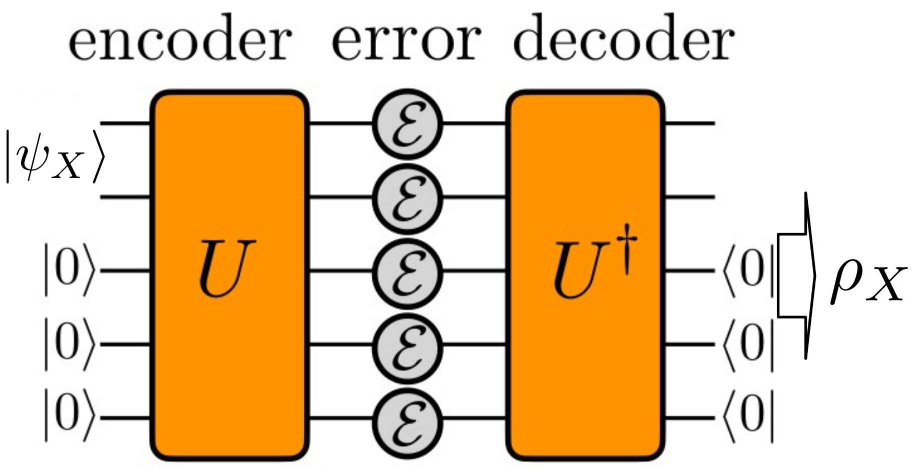
- Fidelity - quenched average, incoherent errors:

$$\tilde{F} = \frac{\left(2^N - 1\right)\left(2^N\left(1 - 3\lambda/4\right)^N + 1\right)}{2^N\left(1 - 3\lambda/4\right)^N\left(2^N - 2^k\right) + 2^{N+k} - 1}$$



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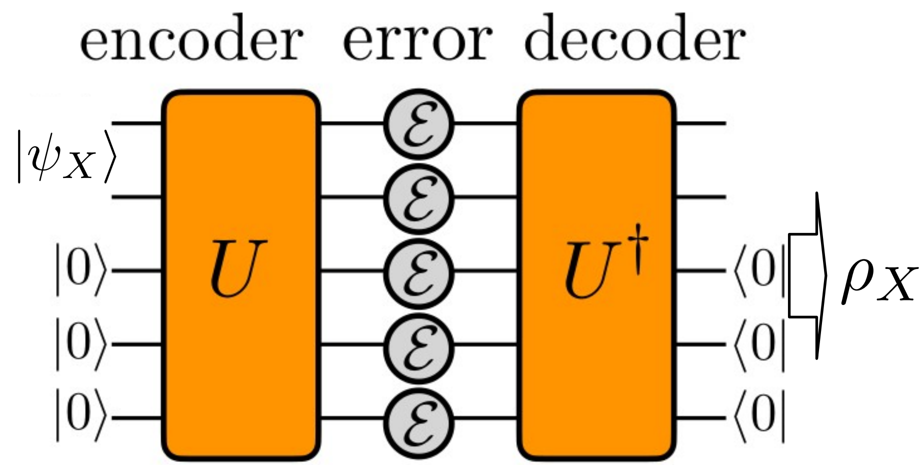
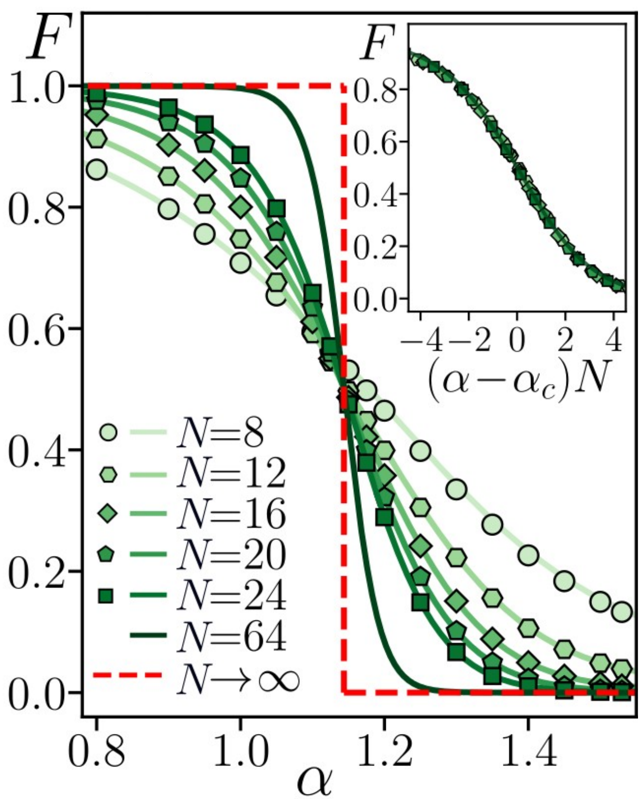
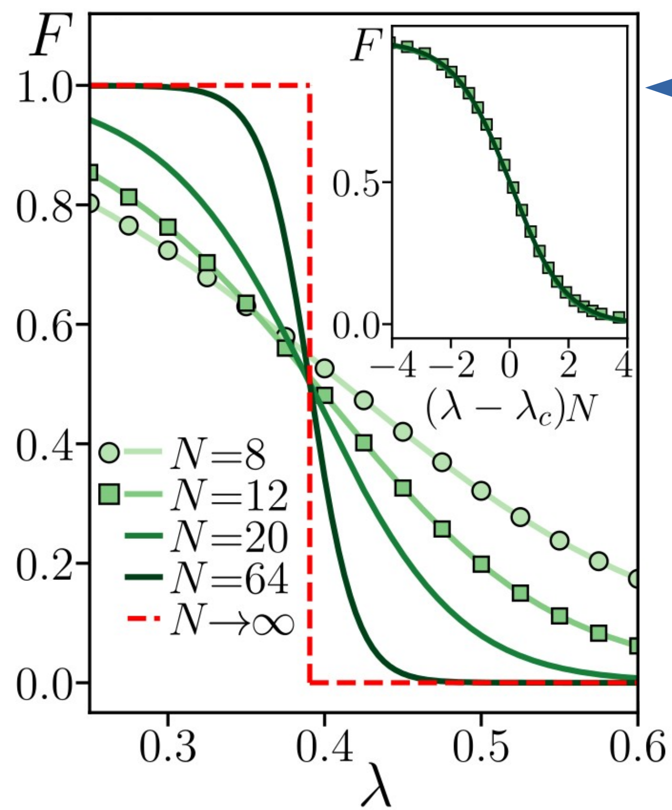



Coherent Errors (local rotations)

Markers: numerics for $\bar{F}$

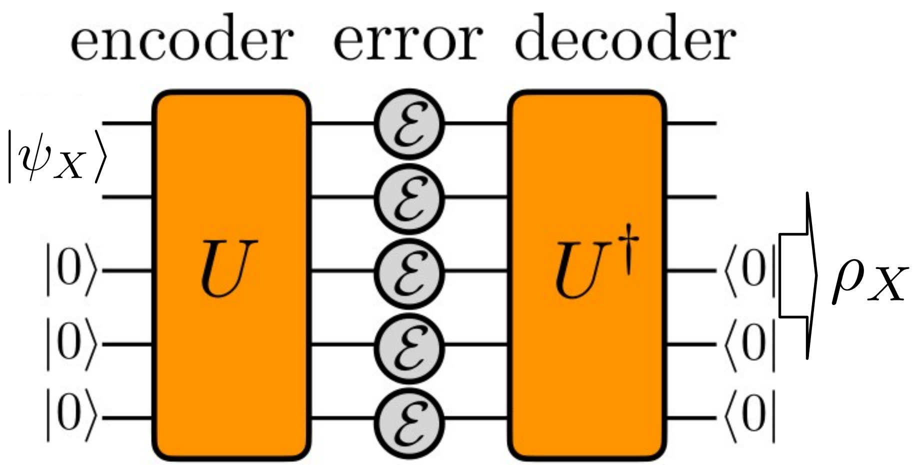
Lines – quenched averages $\tilde{F}$




Incoherent Errors (local depolarizing noise)

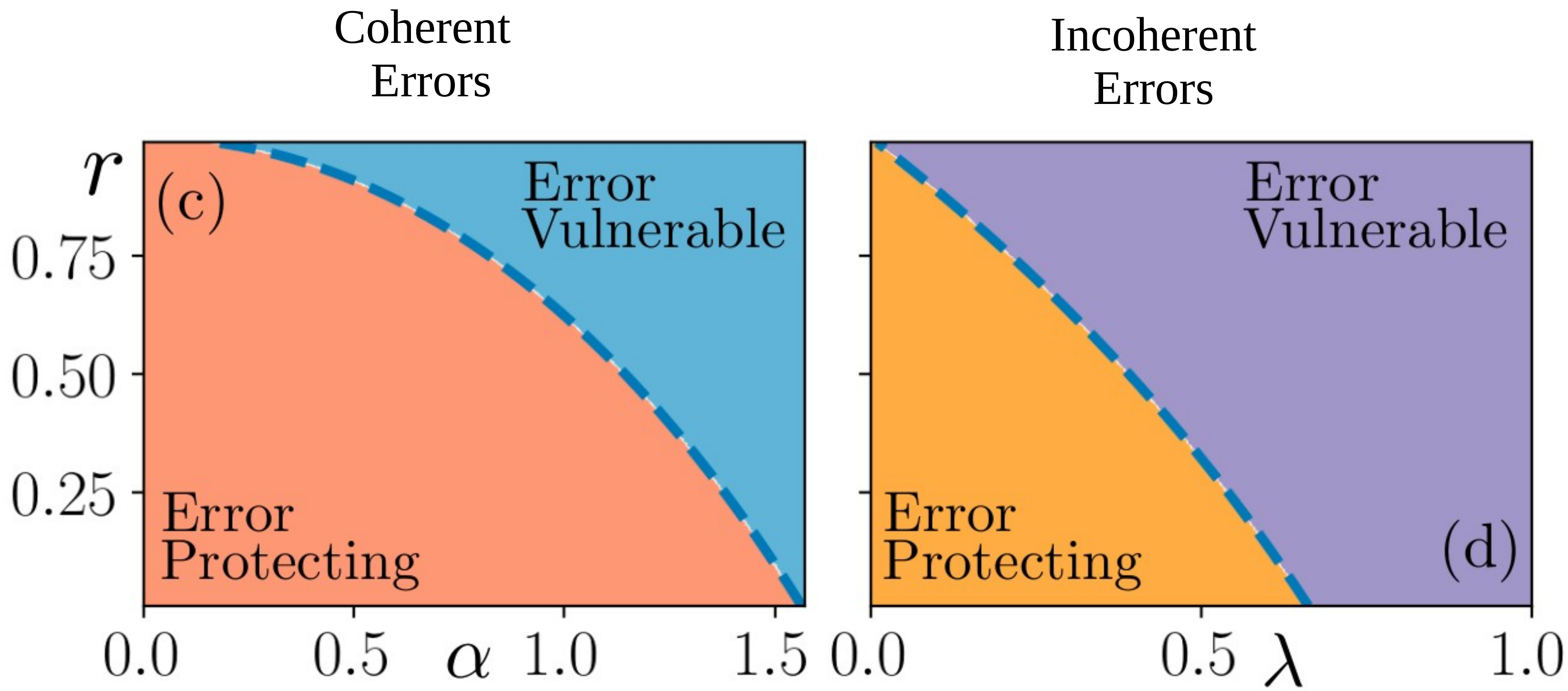
# Error-resilience phase transition




- For uniform error strength, the critical exponent:

$$\nu = 1$$

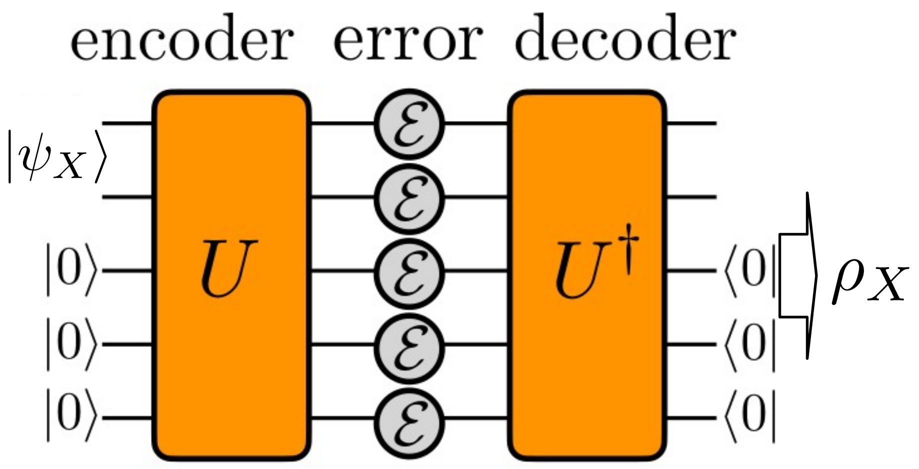
- Phase diagram, $r \equiv k/N$



# Disorder in error strength

- Uniform error strength $\nu = 1$
- For non-uniform errors of strength $\alpha_j$ or ( $\lambda_j$ ) trivial generalization:

$$\tilde{F} = \frac{(2^N - 1)\left(2^N \prod_{i=1}^N \cos^2(\alpha_i/2) + 1\right)}{2^N (2^N - 2^k) \prod_{i=1}^N \cos^2(\alpha_i/2) + 2^{k+N} - 1}$$



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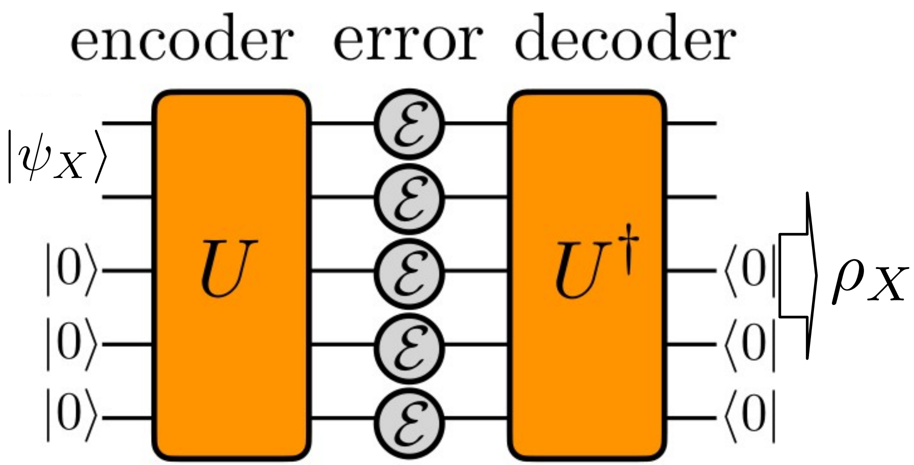


- Self-averaging: $\overline{\Phi} = \tilde{\Phi} + \mathcal{O}(2^{-\eta N})$

at *each fixed* realization of $\alpha_j$ or ( $\lambda_j$ )

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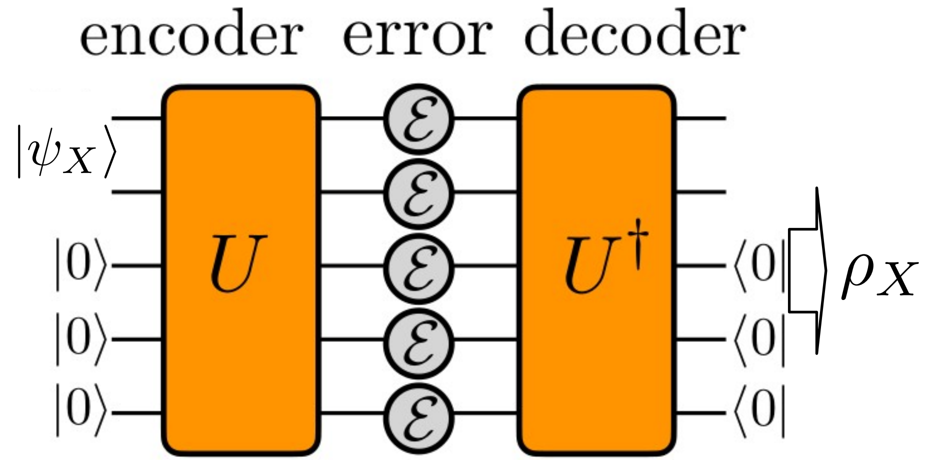
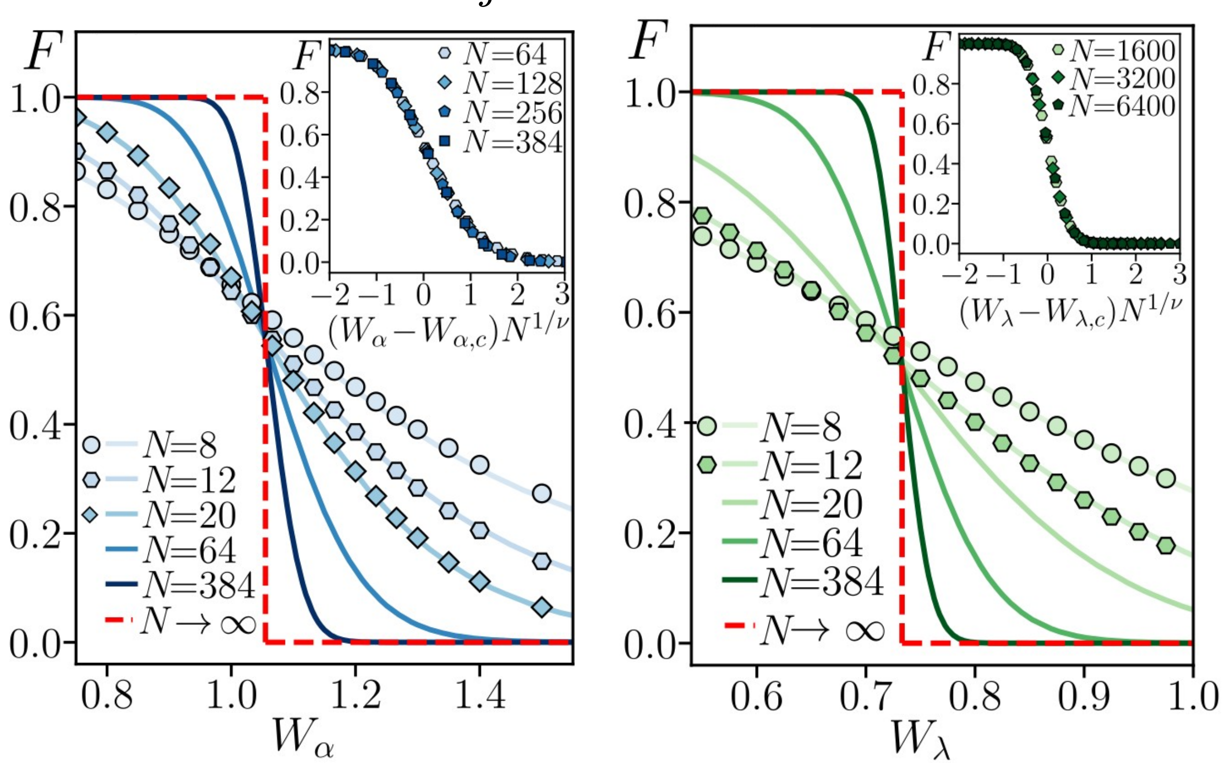



- Self-averaging: $\overline{\Phi} = \tilde{\Phi} + \mathcal{O}(2^{-\eta N})$

at *each fixed* realization of $\alpha_j$ or ( $\lambda_j$ )

$\alpha_j = N(0, W_\alpha)$

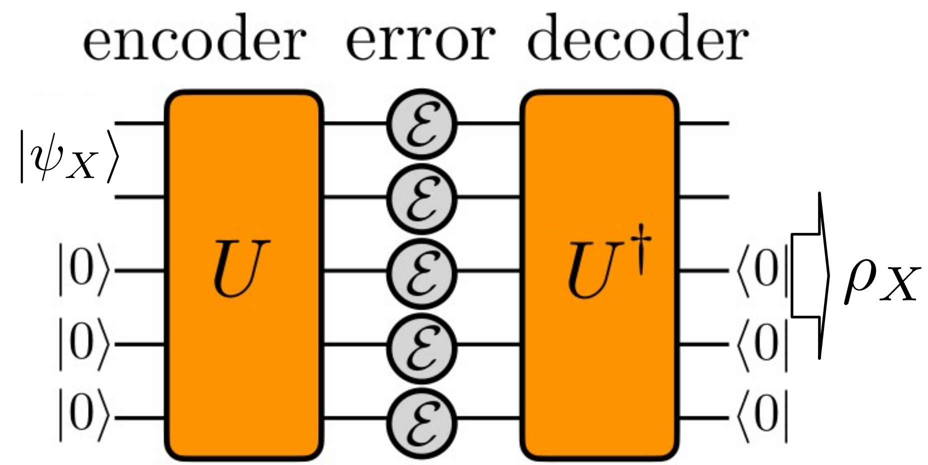



$\lambda_i = [0, W_\lambda]$

- For disordered error strength:

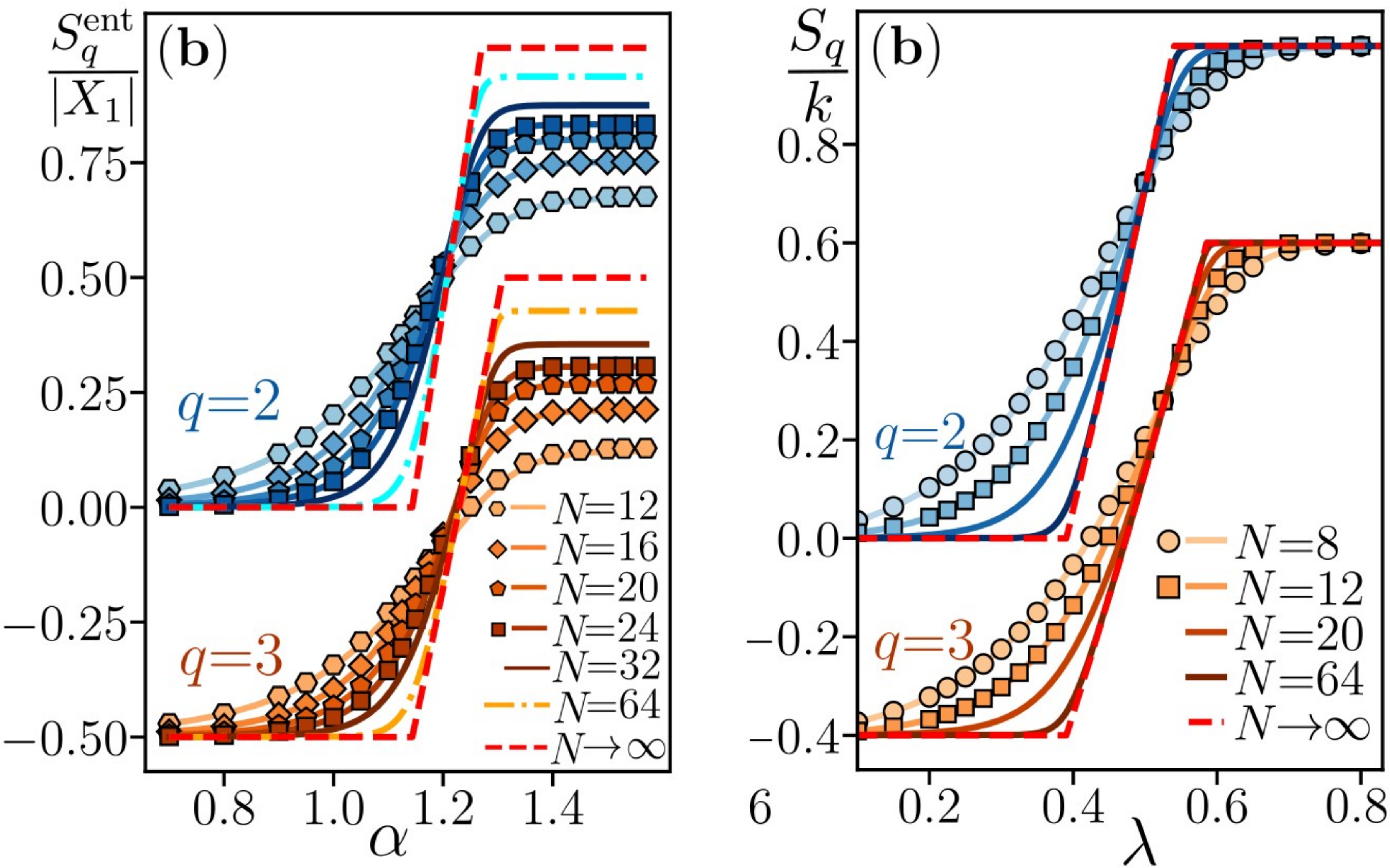
$\nu = 2$

# Conclusion 2

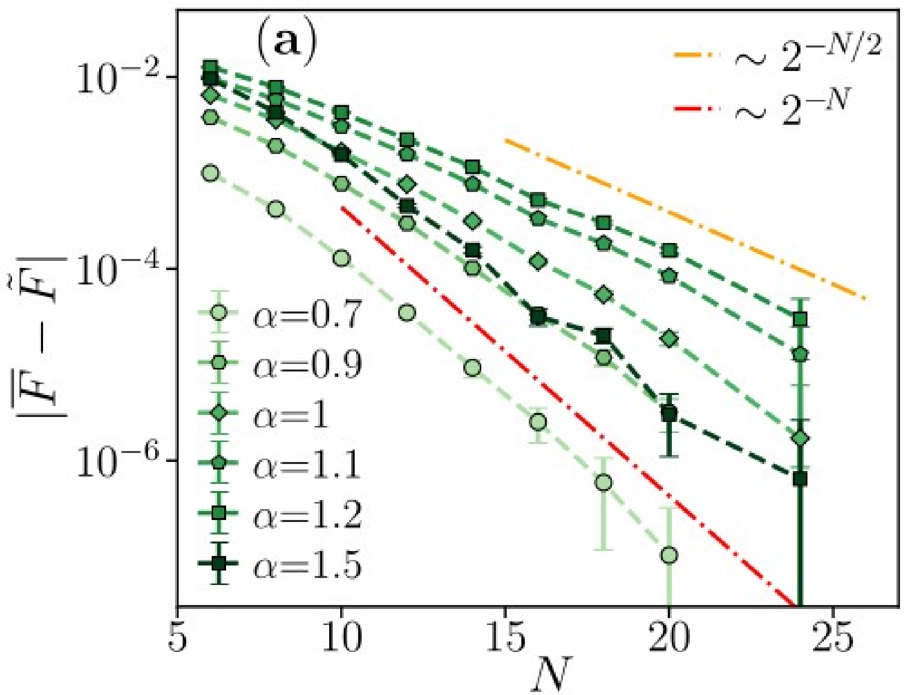


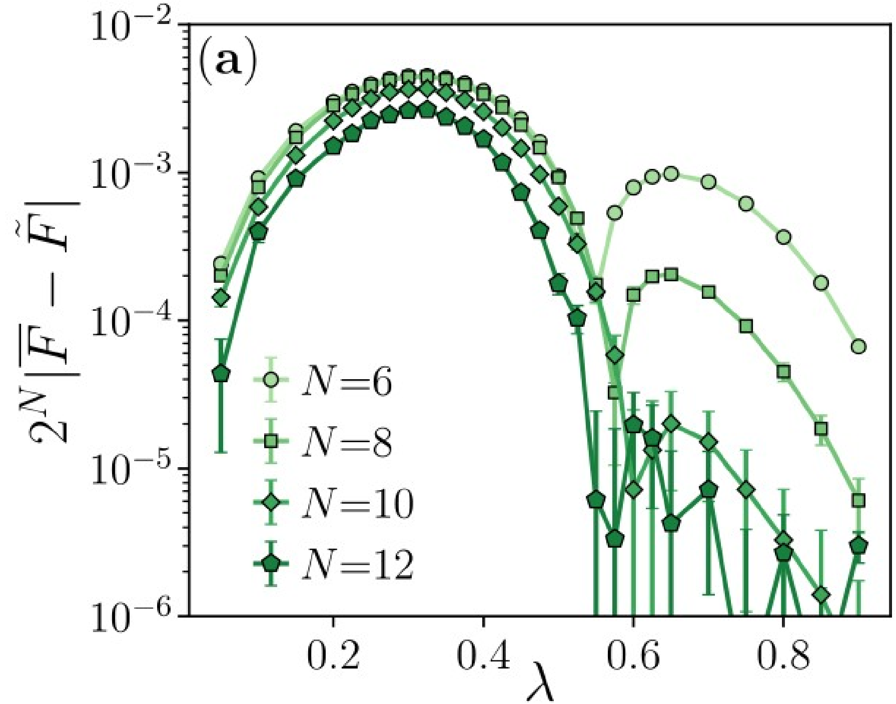

- Exact analytical solution for encoding-decoding circuits
- Works for both coherent and incoherent errors, even non-uniform
- Features an Error-Resilience Phase Transition
- Higher number $(2q)$ of replicas => Entropies (better characterization of the Error-vulnerable phase)
- Possible generalizations: geometry, stabilizers, error-models
- Why does it work?

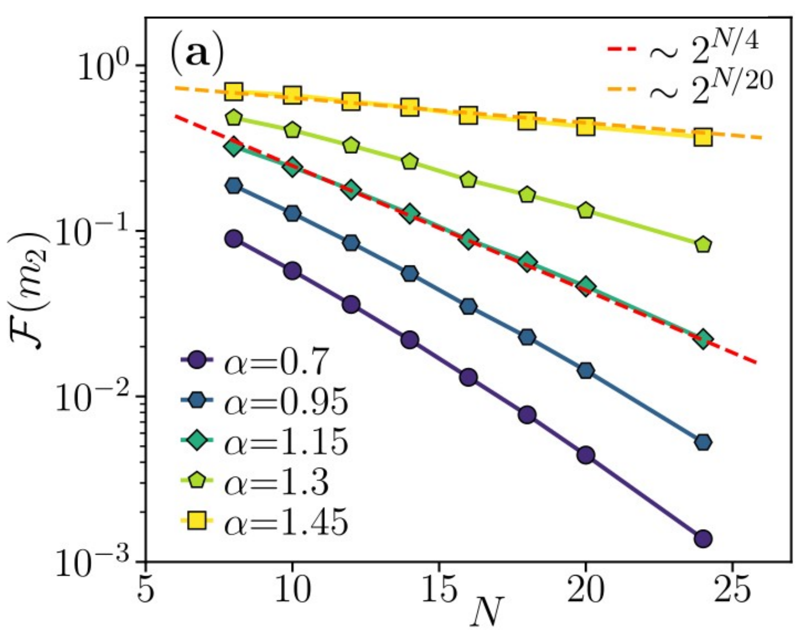
# Entropies: Error-Vulnerable Phase

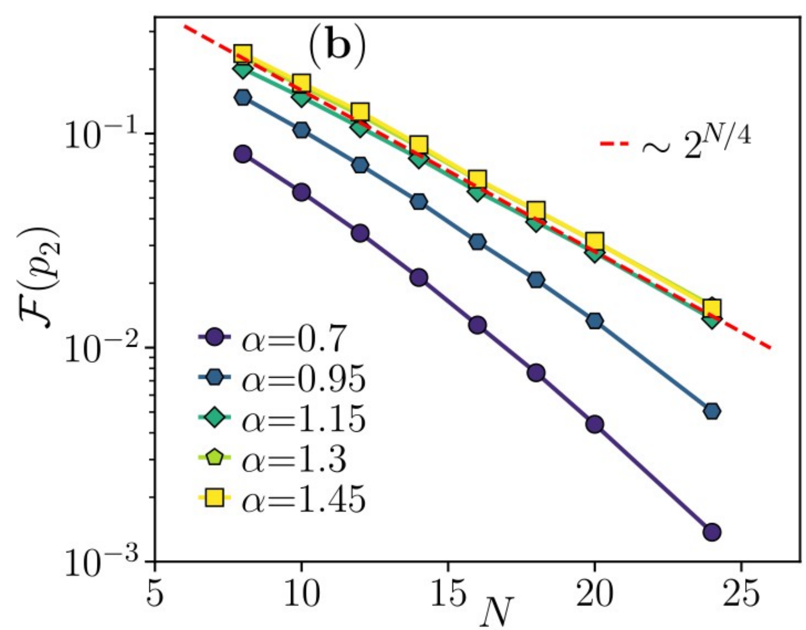


# Self-averaging



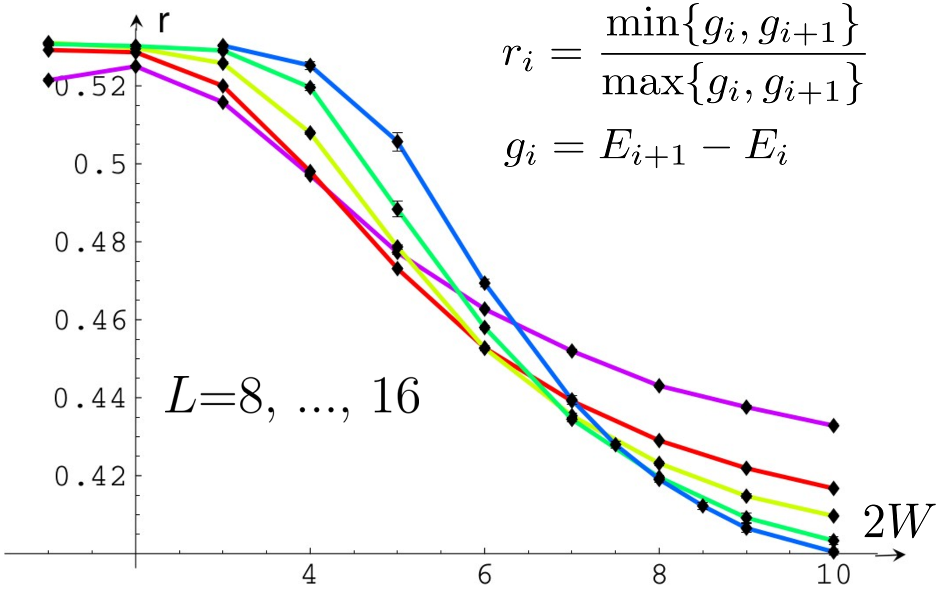



# Random field Heisenberg spin chain

$$H_{XXZ} = \sum_{i=1}^{L} \left( S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right) + \sum_{i=1}^{L} h_i S_i^z, \quad h_i \in [-W, W] \ \ (i.i.d.)$$

- V. Oganesyan, D. Huse, Phys. Rev. B 75, 155111 (2007)

  "(...), we find that the crossings of the r(W ) curves for adjacent L's take place at points that, as L is increased, **"drift"** progressively towards larger W and smaller (more insulating) r; see Fig. 2. As this drift precludes the straightforward quantitative analysis of our data in terms of one-parameter scaling theory, we have exerted considerable effort to attempt to eliminate it (...). While this drift of the crossings can be reduced (…), it appears that **it is intrinsic to this model's spectral statistics** and none of the many things we have tried eliminated or reversed it. Accepting this, there are two very distinct possible implications ..."




$$r_i = \frac{\min\{g_i, g_{i+1}\}}{\max\{g_i, g_{i+1}\}}$$

$$g_i = E_{i+1} - E_i$$

- Transition to MBL phase at $W_C = 3.7$

  D. Luitz, N. Laflorencie, F. Alet, Phys. Rev. B **91**, 081103(R) (2015)

- The system remains ergodic at any $W$ in the $L \to \infty$ limit

  J. Šuntajs, J. Bonča, T. Prosen, L. Vidmar, Phys. Rev. E **102**, 062144 (2020)

"Does MBL exist?"

# Lack of MBL in constrained spin chains

- The Rydberg blockade regime (V>>1)

$$\hat{H} = \sum_{i=1}^{L} P_i^\alpha S_i^x P_{i+1+\alpha}^\alpha + \sum_{i=1}^{L} h_i S_i^z$$

where $P_i^\alpha = \prod_{j=i-\alpha}^{i-1} (1/2 + S_j^z)$

$\alpha = 1$: PXP model

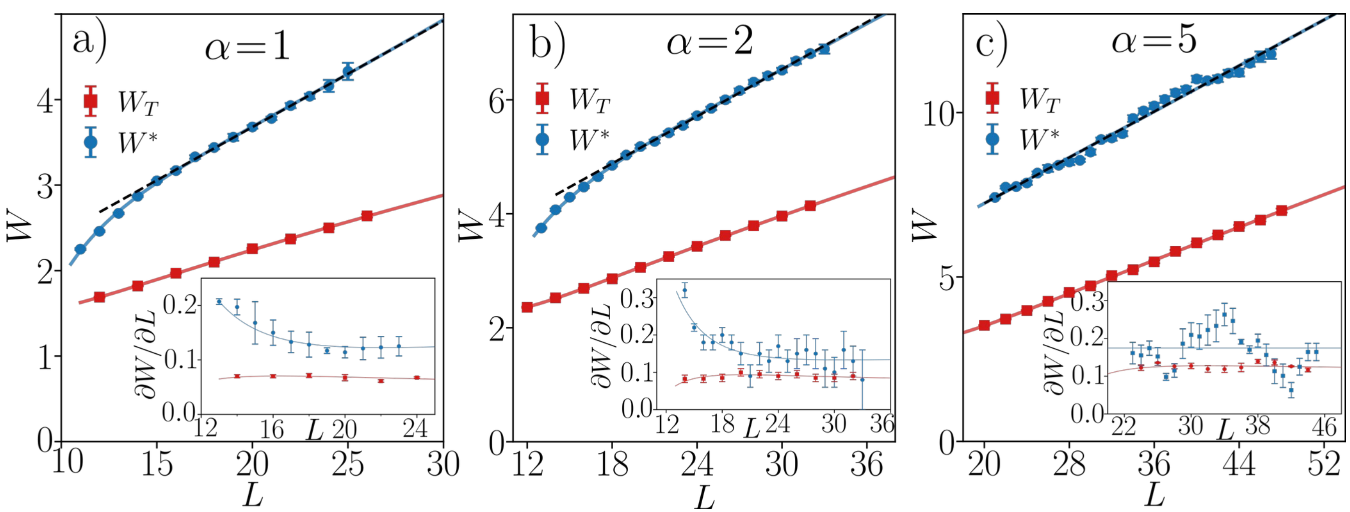
C. Turner et al., Nature Physics **14**, 745–749 (2018)

The Hilbert space dimension:

$\mathcal{N}_\alpha = (\Phi_\alpha)^L$ where

$$\Phi_{\alpha=1,2,5} \approx 1.6180, 1.4656, 1.2852$$

- The crossover between ergodic and MBL regimes observed when *W* is increased




Slow delocalization:

$$W_T(L) \sim L$$
$$W^*(L) \sim L$$

PS, E. Lazo, M. Dalmonte, A. Scardicchio, J. Zakrzewski, Phys. Rev. Lett. **127**, 126603 (2021)

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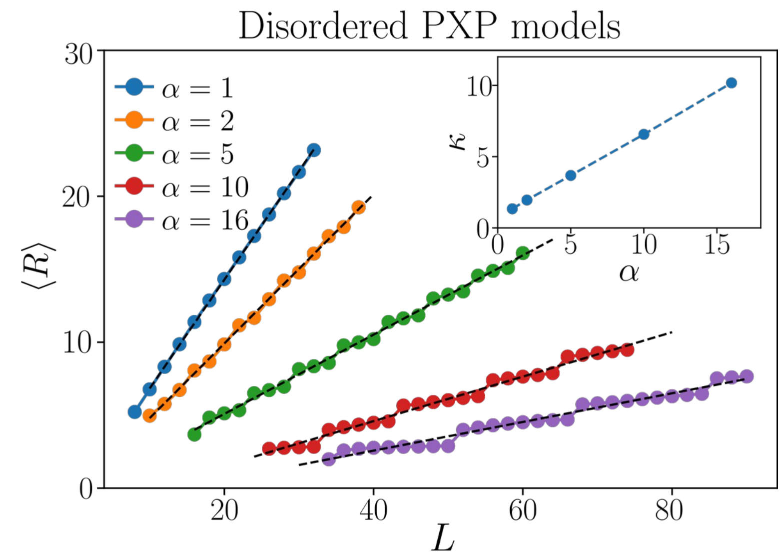
- Hilbert space graph radius R

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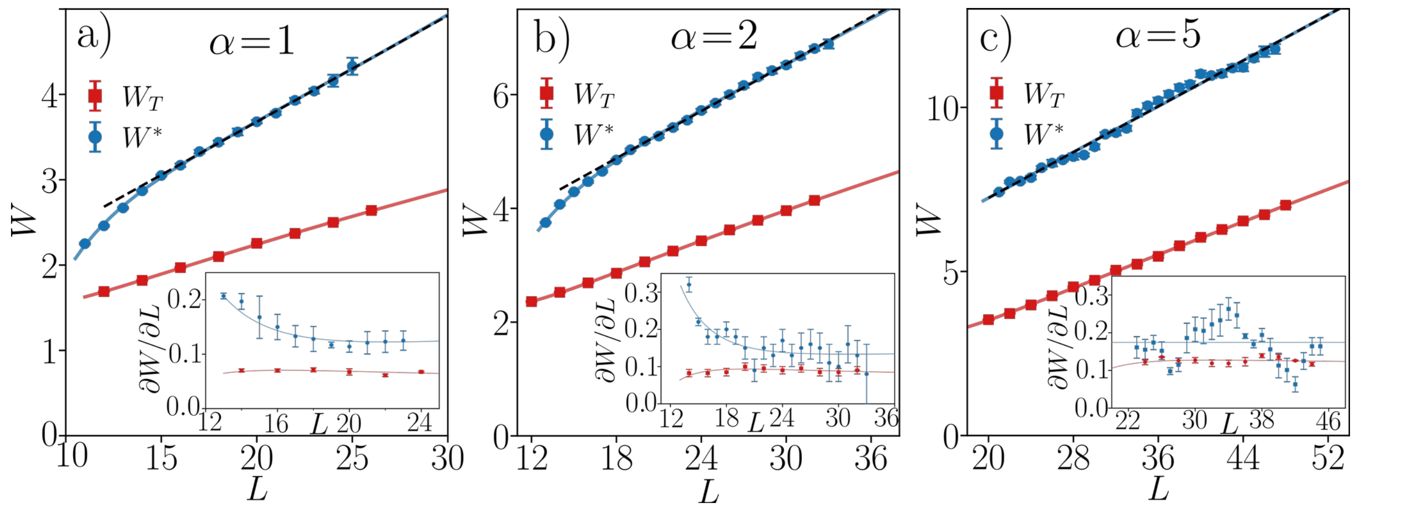


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# MBL in Kicked Ising model

- Kicked Ising model

$h_j \in [0, 2\pi]$ uniformly distributed

$$U_F = e^{-ig\sum_j X_j} e^{-i\sum_j (gZ_jZ_{j+1} + h_jZ_j)}$$

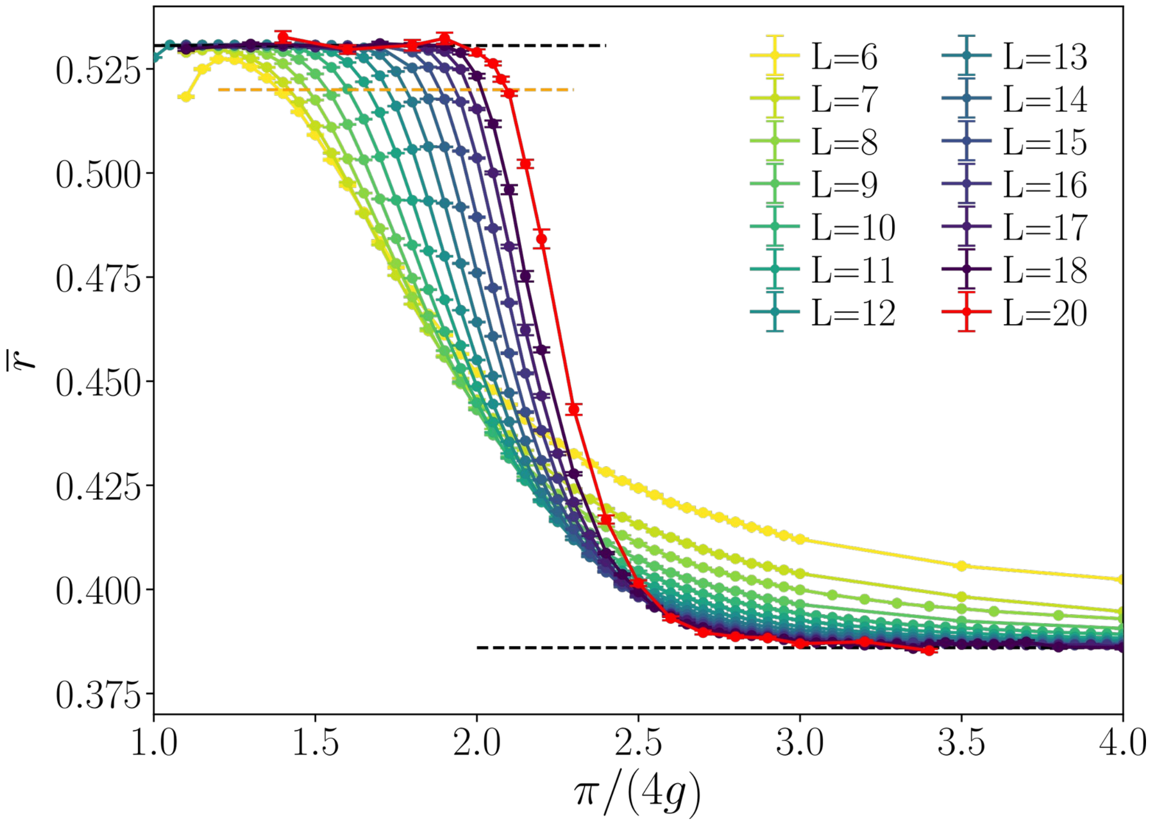
$W = \pi/(4g)$ disorder strength

L. Zhang, V. Khemani, D. Huse, Phys. Rev. B **94**, 224202 (2016)

T. Lezama, S. Bera, J. Bardarson, Phys. Rev. B **99**, 161106(R) (2019)

- Eigenstates with POLFED (up to $L \leq 20$): ETH-MBL crossover

PS, M. Lewenstein, A. Scardicchio, J. Zakrzewski, Phys. Rev. B **107**, 115132 (2023)



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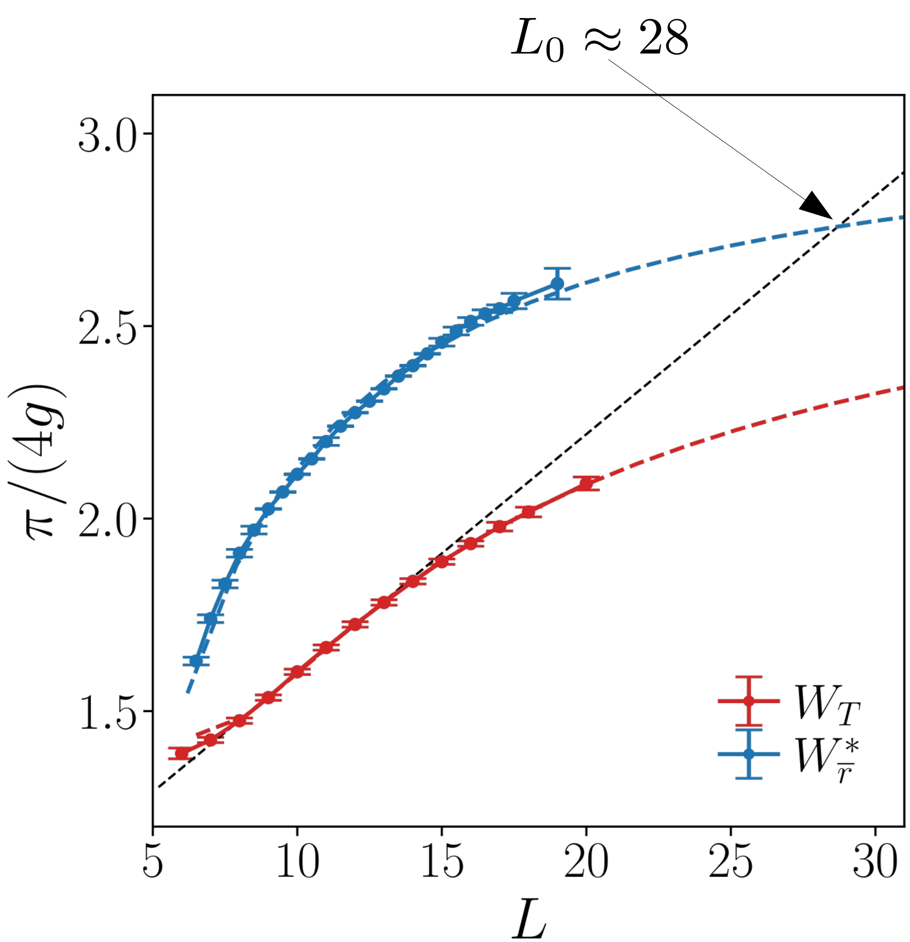


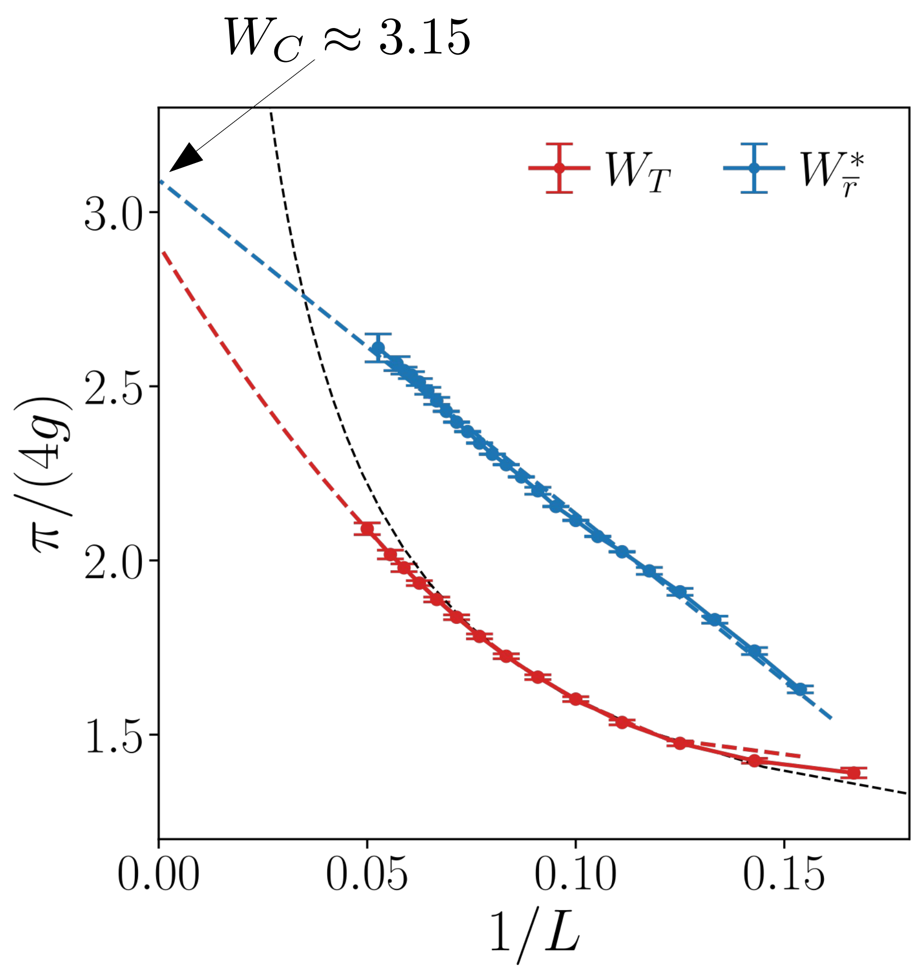

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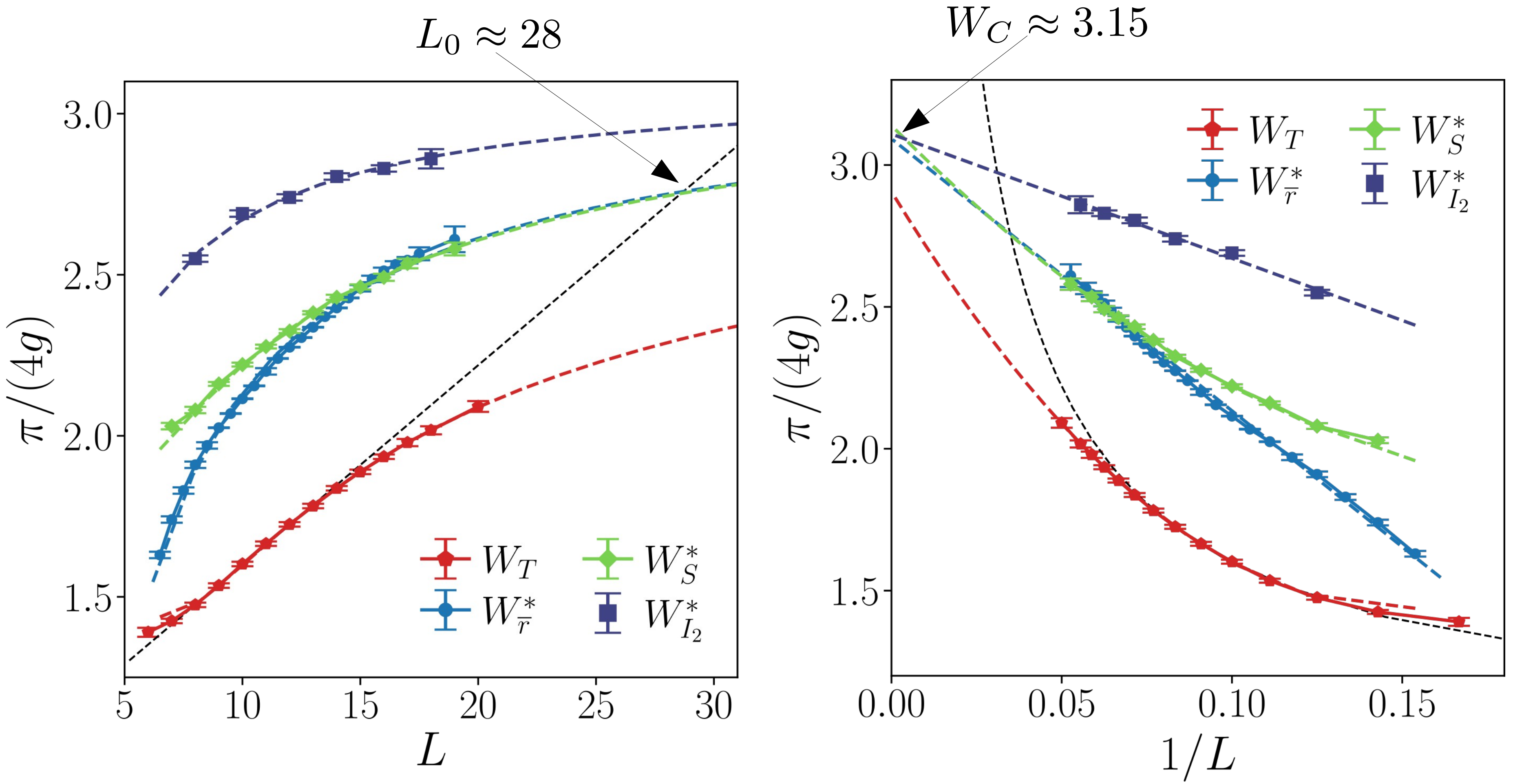
PS, M. Lewenstein, A. Scardicchio, J. Zakrzewski, Phys. Rev. B **107**, 115132 (2023)



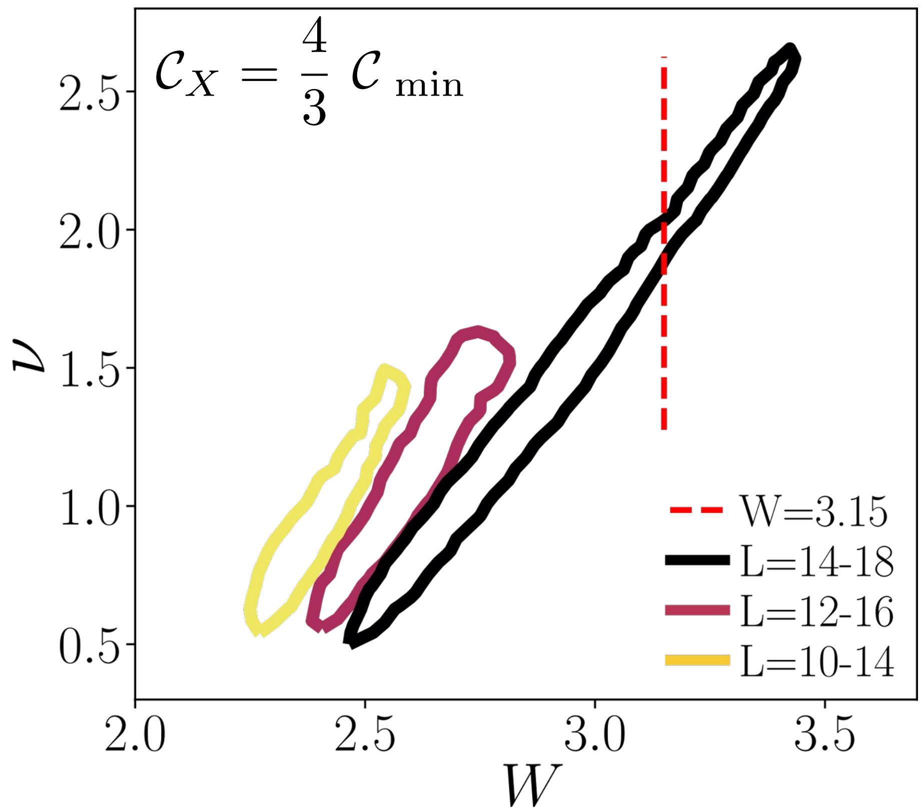
# MBL in Kicked Ising model

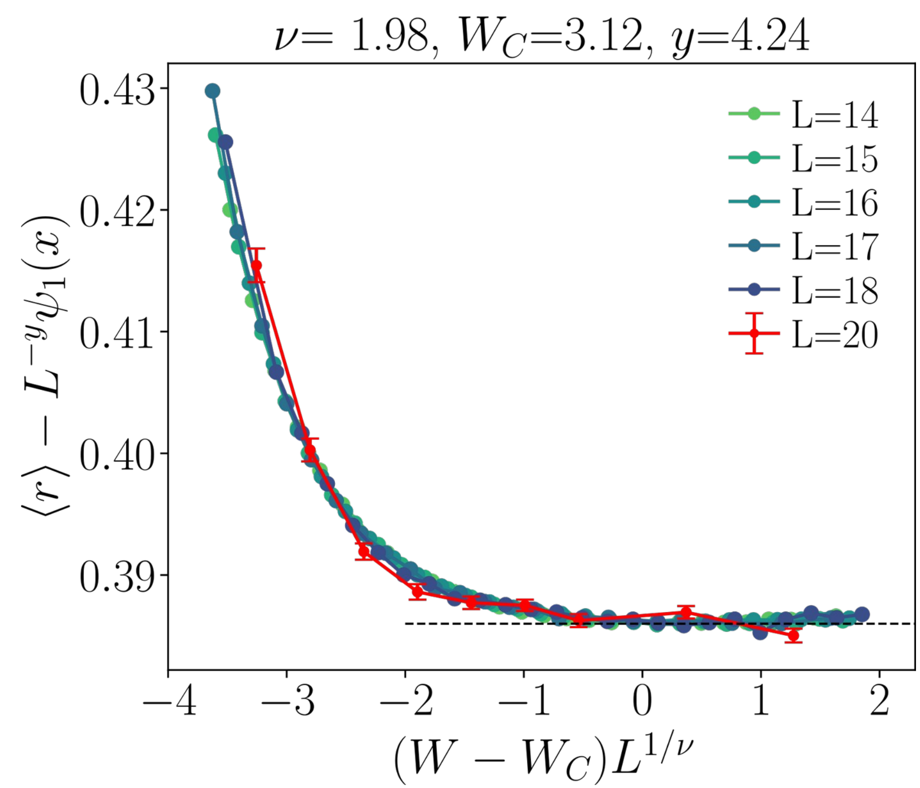
- $U_F = e^{-ig\sum_j X_j} e^{-i\sum_j (gZ_jZ_{j+1} + h_jZ_j)}$
- Finite size scaling: $\overline{r}(W,L) - L^{-y}\psi_1[(W-W_C)L^{1/\nu}] = f[(W-W_C)L^{1/\nu}]$

Sub-leading correction    Universal function

$$\mathcal{C}_X = \frac{\sum_{j=1}^{N-1} |X_{j+1} - X_j|}{\max\{X_j\} - \min\{X_j\}} - 1$$

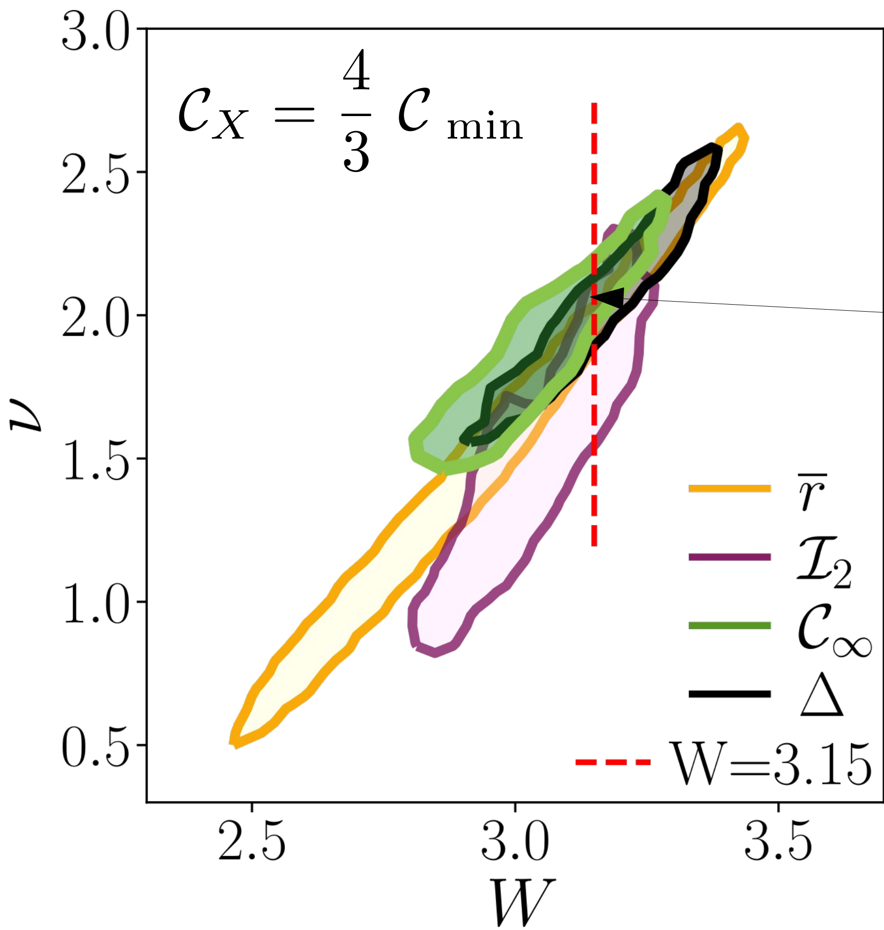
$$X_j = \overline{r}(W_j) - \psi_1[(W_j - W_C)L^{1/\nu}]$$



# MBL in Kicked Ising model

- $U_F = e^{-ig\sum_j X_j} e^{-i\sum_j (gZ_jZ_{j+1} + h_jZ_j)}$
- Finite size scaling: $\overline{r}(W,L) - L^{-y}\psi_1[(W-W_C)L^{1/\nu}] = f[(W-W_C)L^{1/\nu}]$
- Superimposing results for different quantities:




$\nu \approx 2$

- Consistent with Harris bound
  $\nu \geq 2/d$

A. Harris, J. Phys. C: Solid State Phys. **7** 1671 (1974)

# Sanity check 1: Anderson model in 3D

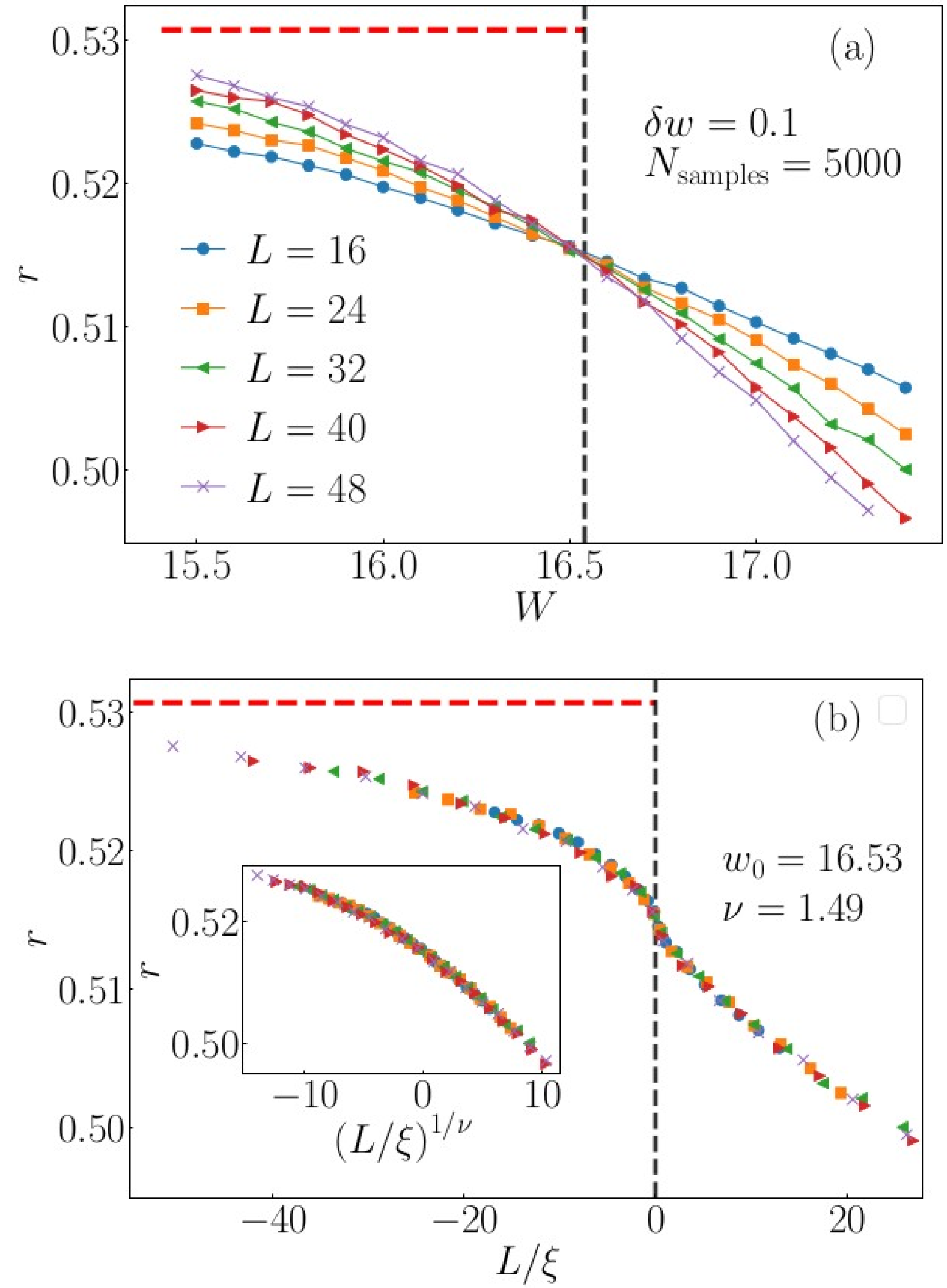
J. Šuntajs, T. Prosen, L. Vidmar,
Annals of Physics **435**, 168469 (2021)

- No shift of the crossing point:

- Finite size scaling reproduces the known critical properties

- 0-dim Quantum Sun model

J. Šuntajs, L. Vidmar, Phys. Rev. Lett, **129**, 060602 (2022)



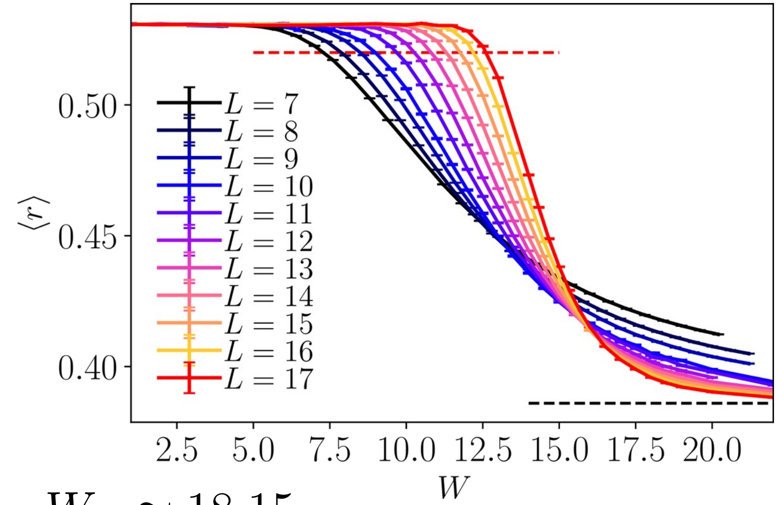
# Sanity check 2: Anderson model on RRG

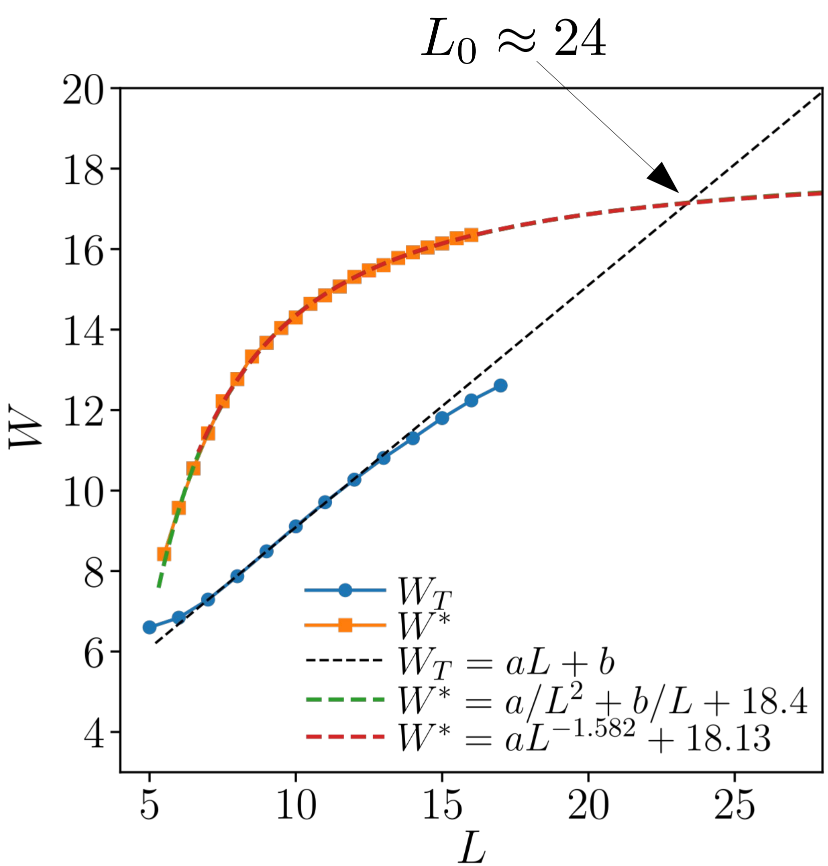
- Critical disorder strength known to be

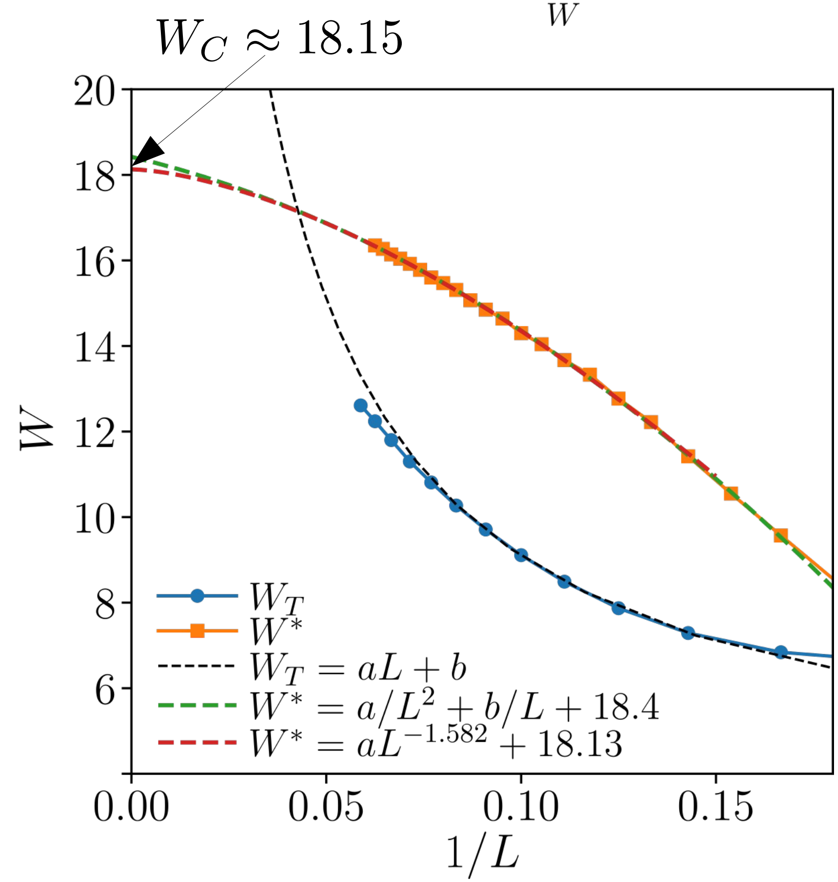
$$W_C = 18.15 \pm 0.05 \quad \text{for} \quad K = 2$$

G. Parisi et al, J. Phys. A: Math. Theor. **53** 014003 (2020)
K. Tikhonov, A. Mirlin, Phys. Rev. B **99**, 214202 (2019)
PS, M. Lewenstein, A. Scardicchio, arXiv:2205.14614

Number of vertices: $\mathcal{N} = 2^L$



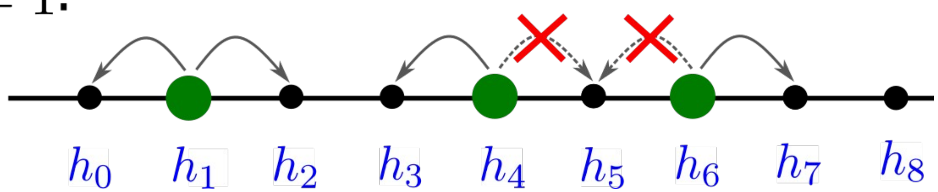
# Constrained spin chains

$$H = \sum_{i=1}^{L} P_i^{\alpha} \left( c_i^{\dagger} c_{i+1} + c_{i+1}^{\dagger} c_i \right) P_{i+2+\alpha}^{\alpha} + \sum_{i=1}^{L} h_i n_i$$

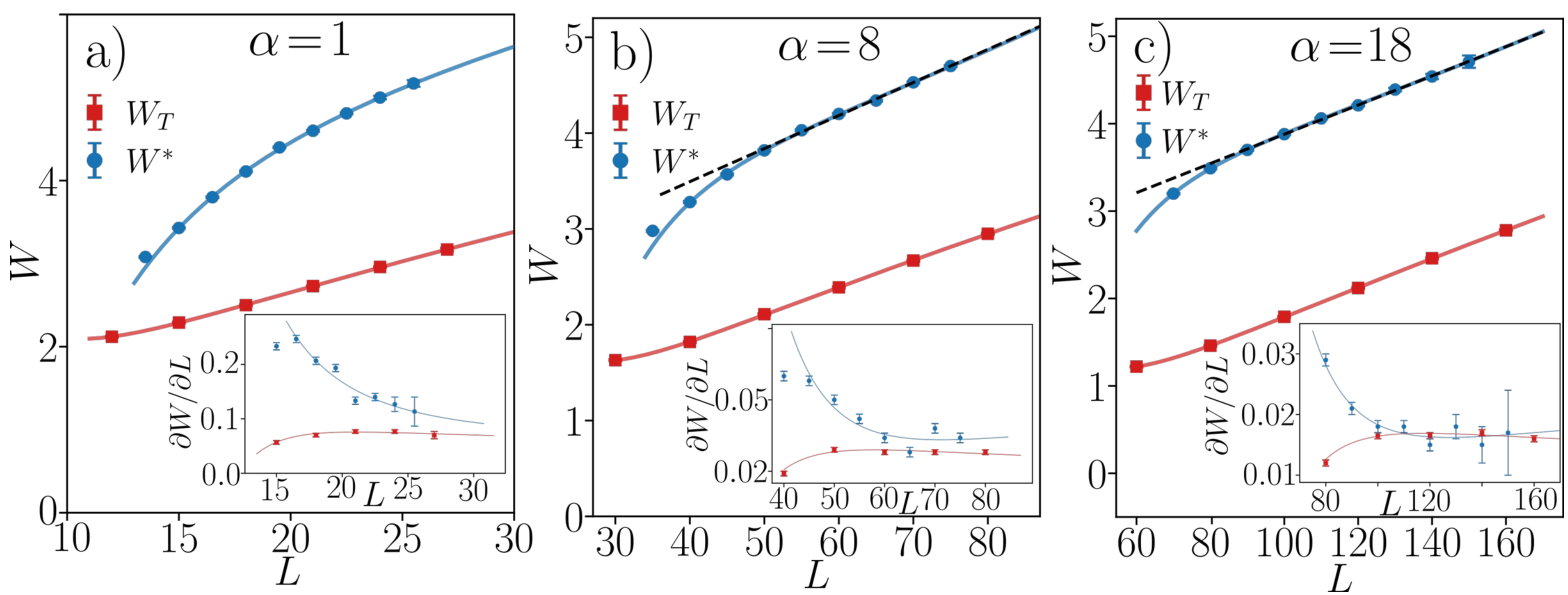
- U(1) symmetry:

$$\nu = \frac{N}{L} = \frac{1}{\alpha + 2}$$

$\alpha = 1$:




- Rydberg dressing



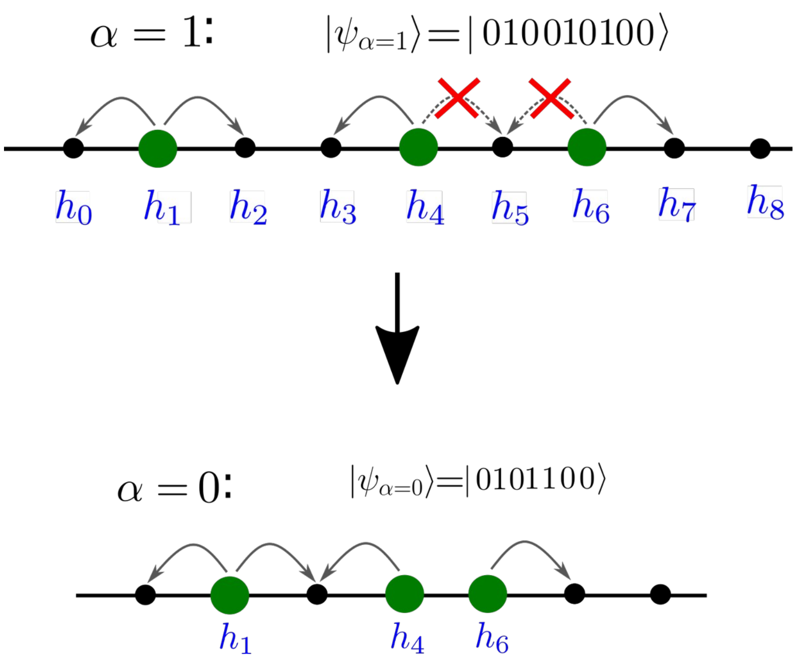

- Also delocalize at large L:

$$W_T(L) \sim L$$
$$W^*(L) \sim L$$

PS, E. Lazo, M. Dalmonte, A. Scardicchio, J. Zakrzewski, Phys. Rev. Lett. **127**, 126603 (2021)

# Constraints induced delocalization

$$H = \sum P_i^\alpha \left( c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i \right) P_{i+2+\alpha}^\alpha + \sum h_i n_i$$




- Constrained model with $\alpha = 1$:

| Constrained model with $\alpha = 1$: | | Unconstrained model: |
|---|---|---|
| 010100010 | ➡ | 0110010 |
| 010100001 | | 0110001 |
| 100100001 | | 1010001 |

- N particles on L sites in a model with constraint radius $\alpha$ and OBC

  ⬍

  N particles on $L - \alpha(N-1)$ sites in unconstrained model

- Introduce disorder: $W > 0$

$$\begin{matrix} h_0 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} & h_1 & h_3 & & & h_7 & \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{matrix}$$

$h'_0 \; h'_1 \; h'_2 \; h'_3 \; h'_4 \; h'_5 \; h'_6 \; h'_7 \; h'_8$

$$\sum_i h_i n_i \Rightarrow \sum_i h_i n_i \prod_{j<i} (1-n_j) + \sum_i h_{i+\alpha} n_i \sum_k n_k \prod_{j<i, j \neq k} (1-n_j) + \sum_i h_{i+2\alpha} n_i \sum_{k_1,k_2} n_{k_1} n_{k_2} \prod_{j<i, j\neq k_1,k_2} (1-n_j) + \ldots$$

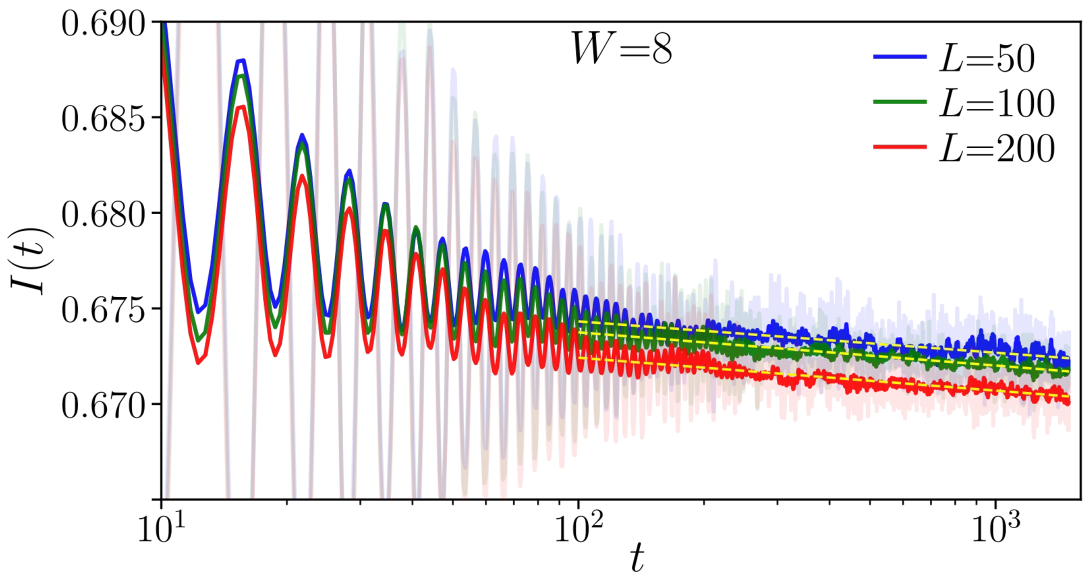
# Strong disorder and interactions (W=8)

Random field XXZ spin-1/2 chain: $H_{XXZ} = \sum_{i=1}^{L} J \left( S_i^x S_{i+l}^x + S_i^y S_{i+l}^y + \Delta S_i^z S_{i+l}^z \right) + \sum_{i=1}^{L} h_i S_i^z$

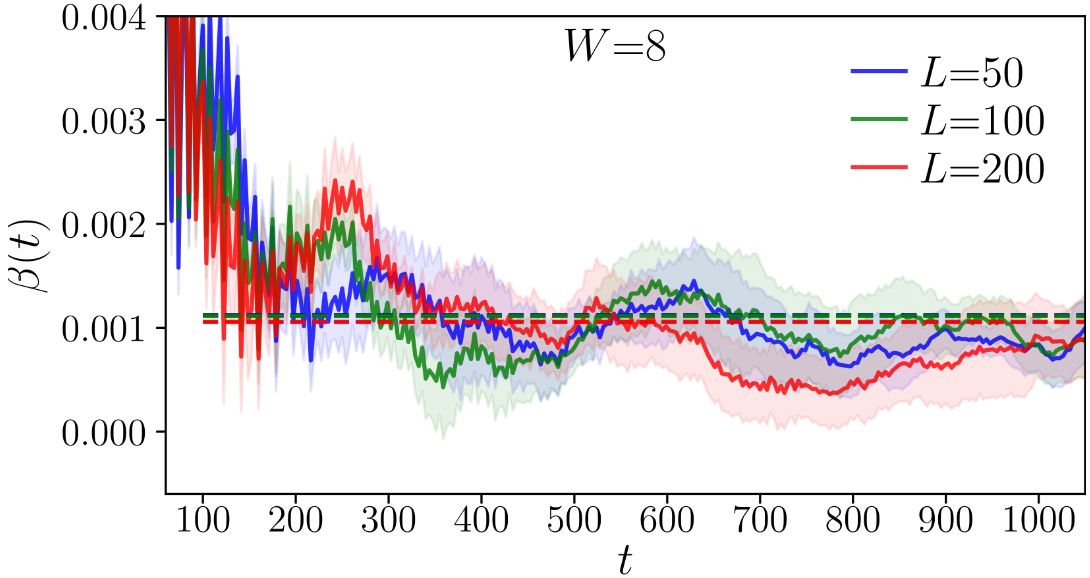
we set $\Delta = 1$

**(A)** : $\overline{\beta} \sim L^{-1}$

**(B)** : $\beta(t)$ decreases in $t$



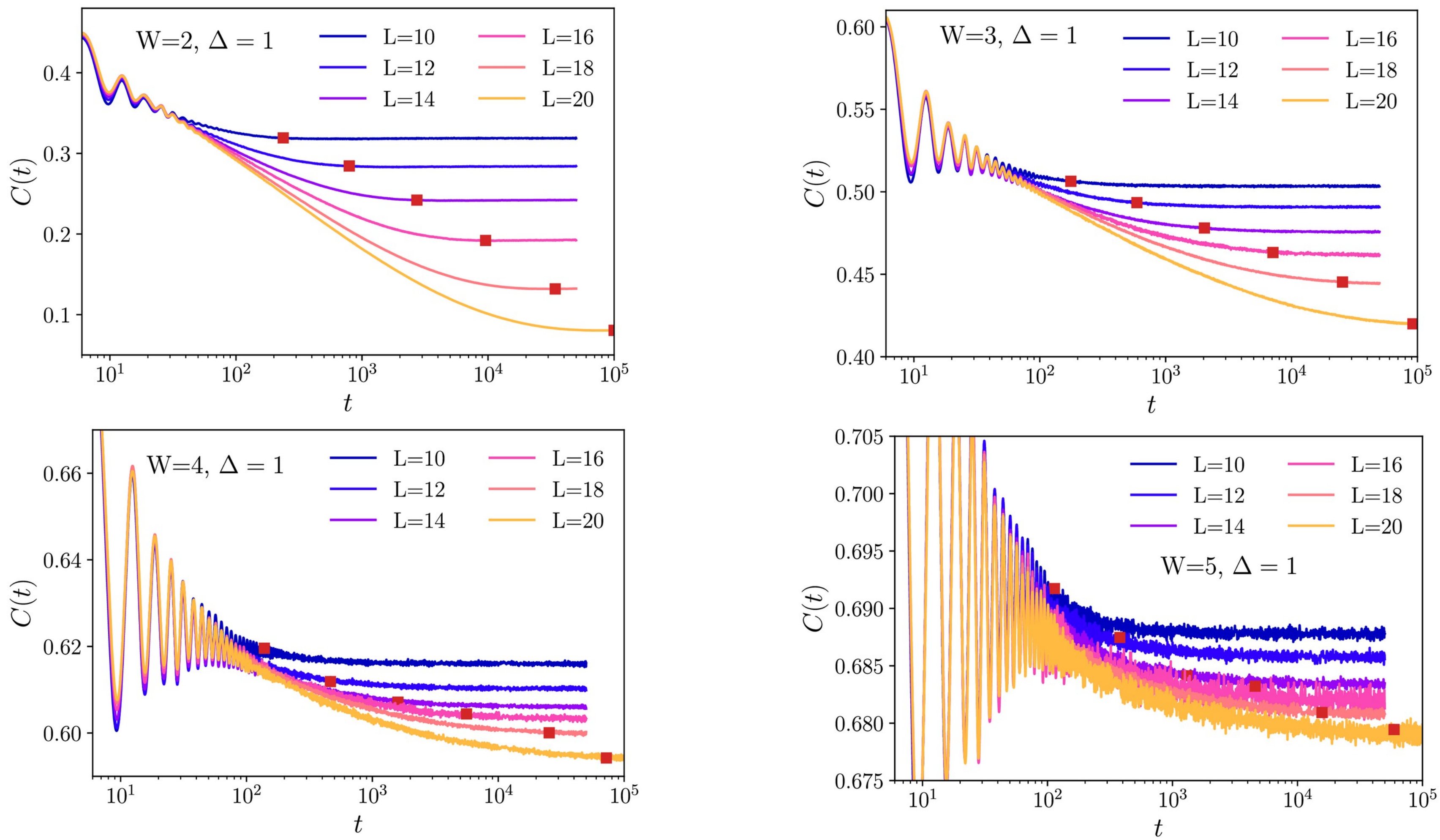

| | $t_{\max}$ | $\chi$ | $n_{\text{real}}$ | $\overline{\beta}$ |
|---|---|---|---|---|
| $L$=50 | 1500 | 128 | 4000 | $(10.03 \pm 1.23) \cdot 10^{-4}$ |
| $L$=100 | 1500 | 128 | 2000 | $(11.07 \pm 0.97) \cdot 10^{-4}$ |
| $L$=200 | 1500 | 160 | 1000 | $(11.03 \pm 0.81) \cdot 10^{-4}$ |



# Slow delocalization due to interactions

Random field XXZ spin-1/2 chain: $H_{XXZ} = \sum_{i=1}^{L} J\left(S_i^x S_{i+l}^x + S_i^y S_{i+l}^y + \Delta S_i^z S_{i+l}^z\right) + \sum_{i=1}^{L} h_i S_i^z$

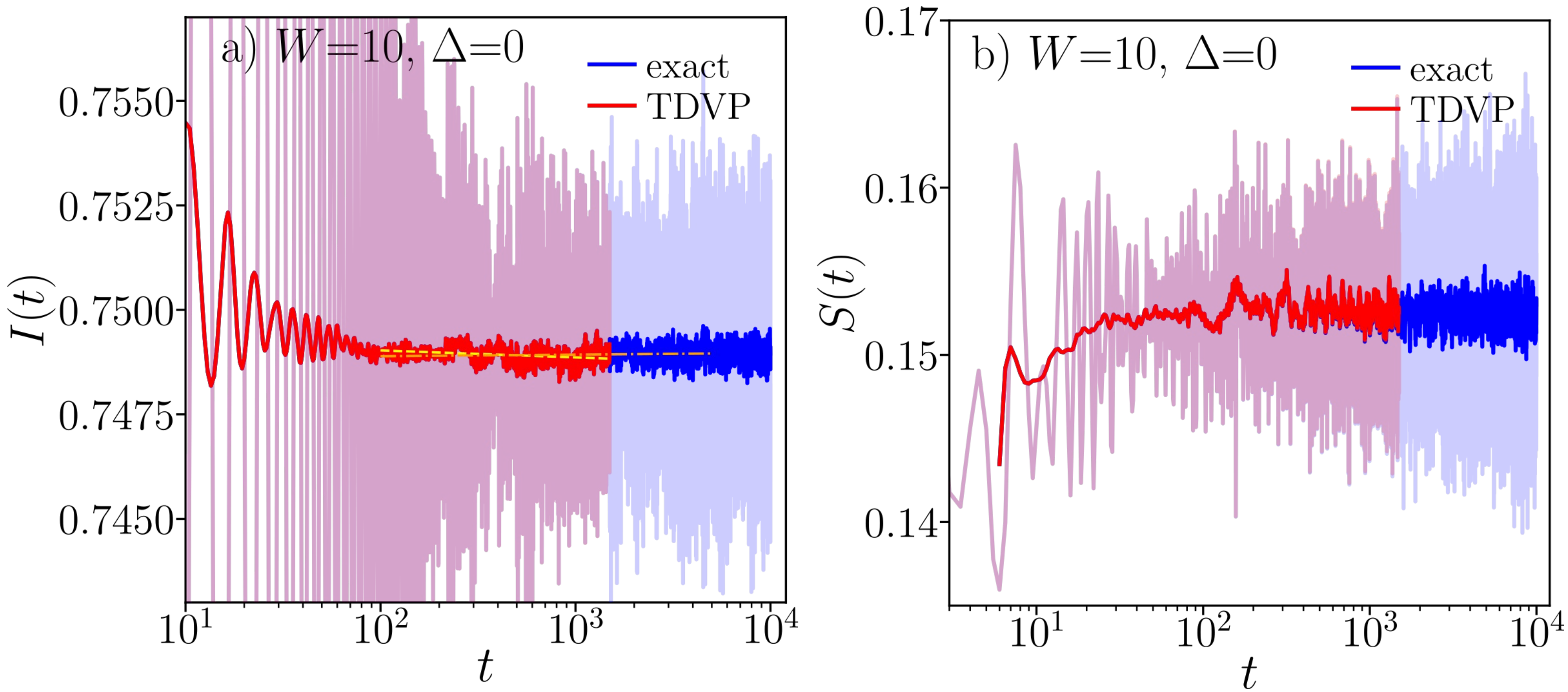
Density correlations: $C(t) = \frac{4}{L}\sum_{i=1}^{L}\langle S_i^z(t) S_i^z(0)\rangle$ $\quad J = 1, \epsilon_i \in [-W, W], W_C = 3.7$?



# Example: Anderson localization

Random field XXZ spin-1/2 chain: $H_{XXZ} = \sum_{i=1}^{L} J\left(S_i^x S_{i+l}^x + S_i^y S_{i+l}^y + \Delta S_i^z S_{i+l}^z\right) + \sum_{i=1}^{L} h_i S_i^z$

we set $\Delta = 0$




**(A)** : $\overline{\beta} \sim L^{-1}$

**(B)** : $\beta(t)$ decreases in $t$

| | $t_{\max}$ | $\chi$ | $n_{\text{real}}$ | $\overline{\beta}$ |
|---|---|---|---|---|
| $L$=50 | 1500 | 128 | 1000 | $(0.97 \pm 1.12) \cdot 10^{-4}$ |
| $L$=50 | 1500 | - | 1000 | $(0.96 \pm 1.12) \cdot 10^{-4}$ |
| $L$=50 | 5000 | - | 1000 | $(-0.23 \pm 0.43) \cdot 10^{-4}$ |

# Outlook

- MBL phase: $\lim_{L\to\infty} \lim_{t\to\infty}$ vs MBL regime: finite $t$, $L$

- Exact numerics yield unclear answers for interacting many-body systems

- Better understanding of the mechanism of the thermalization/resonances in strongly disordered systems is needed

  A. Morningstar, L. Colmenarez, V. Khemani, D. Luitz, D. Huse, Phys. Rev. B **105**, 174205 (2022)
  D. Sels, Phys. Rev. B **106**, L020202 (2022)

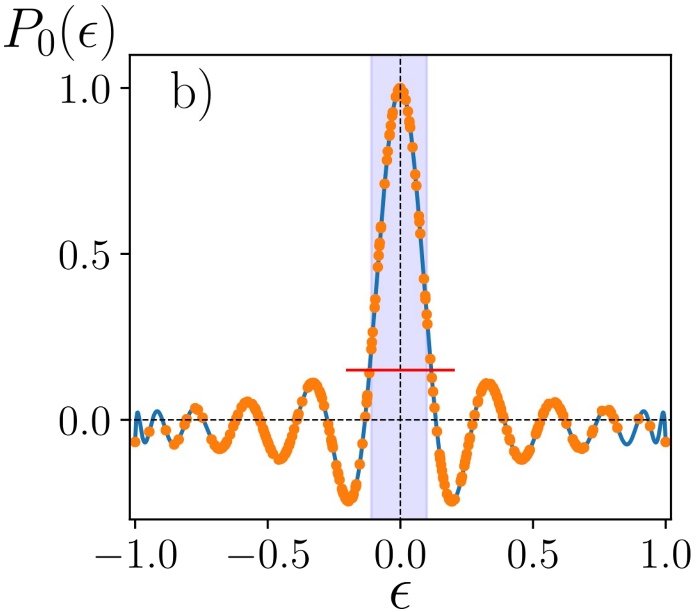
- Understanding of the regime of slow dynamics is as important:

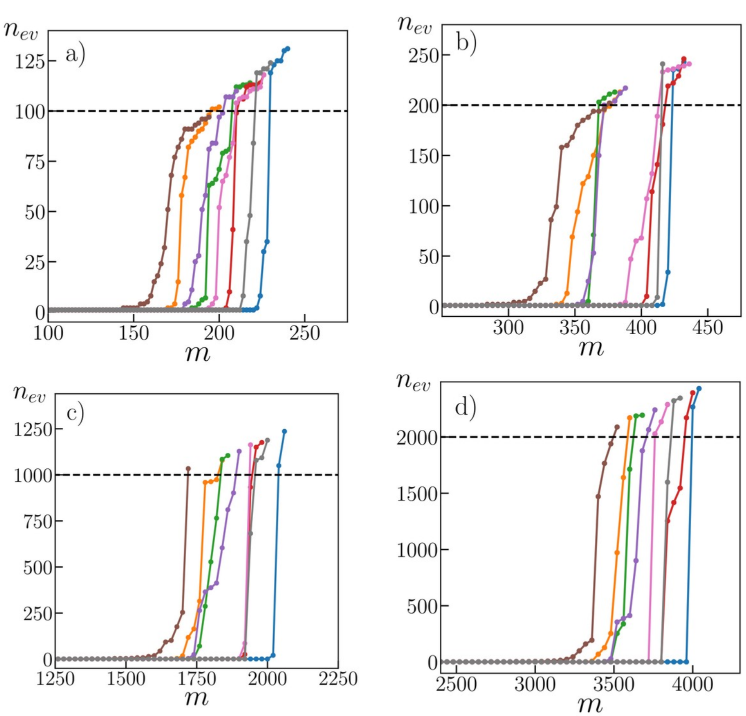
  F. Evers, S. Bera, arXiv:2302.11384

- Finding models with clearer numerical characteristics

  B. Krajewski, L. Vidmar, J. Bonča, M. Mierzejewski, Phys. Rev. Lett. **129**, 260601 (2022)

# POLFED algorithm




- Rescale the Hamiltonian:
  $[2H-(E_0+E_1)]/(E_1-E_0) \to H$

- Calculate the order *K* of the transformation using density of states of *H*

$$P_\sigma^K(H) = \frac{1}{D}\sum_{n=0}^{K} c_n^\sigma T_n(H)$$

- Choose block size *s*, initialize $Q_1 \in \mathbb{R}^{\mathcal{N}\times s}$

  And perform block Lanczos iteration, *j=0, 1,…, m*

$$U_j = P_\sigma^K(\tilde{H})Q_j - Q_{j-1}B_j^T, \quad A_j = Q_j^T U_j$$

$$R_{j+1} = U_j - Q_j A_j, \quad Q_{j+1}B_{j+1} = R_{j+1},$$

where $A_j, B_j \in \mathbb{R}^{s\times s}$, $Q_j, U_j, R_j \in \mathbb{R}^{\mathcal{N}\times s}$.

- Finally, with $\mathcal{Q}_m = [Q_1, \ldots, Q_m] \in \mathbb{R}^{\mathcal{N}\times ms}$

  One gets a block tridiagonal matrix:

$$T_m = \mathcal{Q}_m^T P_\sigma^K(\tilde{H})\mathcal{Q}_m$$

# Features of POLFED

- The order K of the transformation $P_\sigma^K(H) = \frac{1}{D}\sum_{n=0}^{K} c_n^\sigma T_n(H)$

  grows like $K \sim \mathcal{N}$ , so $P_\sigma^K(\tilde{H})Q_j$ dominates time consumption;
  – *two ways of parallelization*

- The matrix $\mathcal{Q}_m = [Q_1, \ldots, Q_m] \in \mathbb{R}^{\mathcal{N}\times ms}$ dominates memory consumption

  – *larger only by a factor of 2-3 than the memory to store calculated eigenvectors*

- Time consumption increases *only linearly with increasing number of non-zero elements*

- It can be used for Floquet systems: D. Luitz, arXiv:2102.05054

  Floquet operator: $U = U_1 U_2 = e^{-iH_1} e^{-iH_2}$

# Thouless time

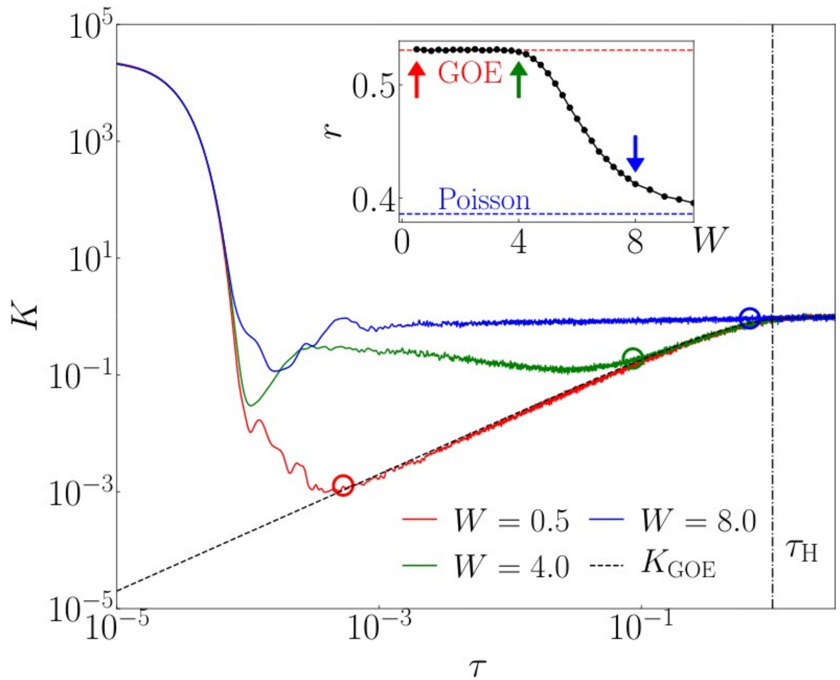
- Thouless time: time to reach boundary of the system D. Thouless, Physics Reports **13**, 93 (1974)

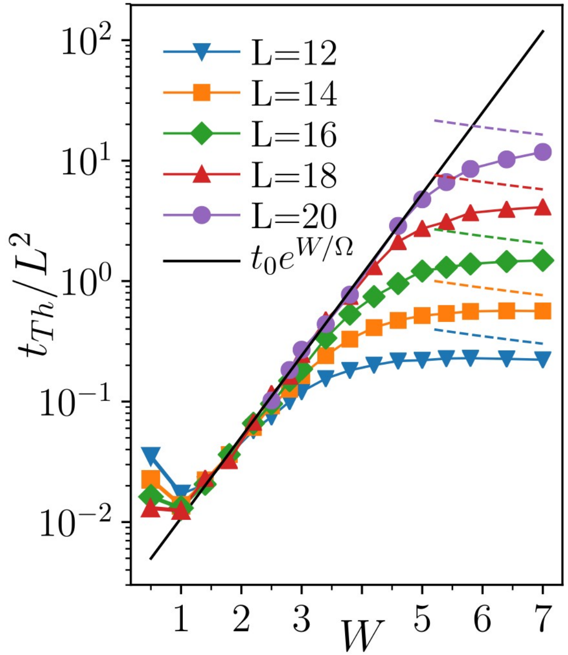
Diffusion: $\langle r^2(t)\rangle = Dt\,,$ so $t_{Th} = L^2/D$

- Analysis of spectral form factor J. Šuntajs, J. Bonča, T. Prosen, L. Vidmar, Phys. Rev. E **102**, 062144 (2020)

$$K(\tau) = \frac{1}{Z}\left\langle \left| \sum_{j=1}^{\mathcal{N}} g(\epsilon_j)\mathrm{e}^{-i\epsilon_j\tau} \right|^2 \right\rangle$$

$$t_{Th} = t_0 \mathrm{e}^{W/\Omega} L^2 ??$$




$$H_{J_1-J_2} = \sum_{i=1}^{L}\sum_{l=1}^{2}\left(S_i^x S_{i+l}^x + S_i^y S_{i+l}^y + \Delta_l S_i^z S_{i+l}^z\right) + \sum_{i=1}^{L} h_i S_i^z$$

# Thouless time at Anderson transition

PS, D. Delande, J. Zakrzewski, Phys. Rev. Lett. **124**, 186601 (2020)

- Anderson transition in 3D and 5D models:

$$W_C^{3D} = 16.54 \qquad W_C^{5D} = 57.3$$

- Subdiffusion at the transition:

$$\langle r^2(t)\rangle \sim t^{\alpha} \qquad \alpha_{3D} = 2/3 \qquad \alpha_{5D} = 2/5$$

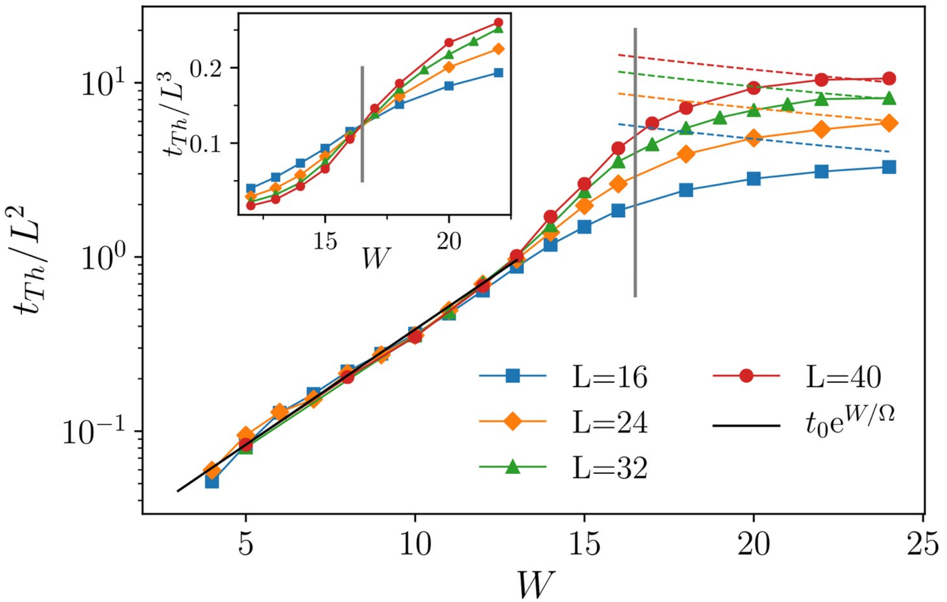
implying: $t_{Th} \sim L^{2/\alpha}$

- Diffusion at $W < W_C$

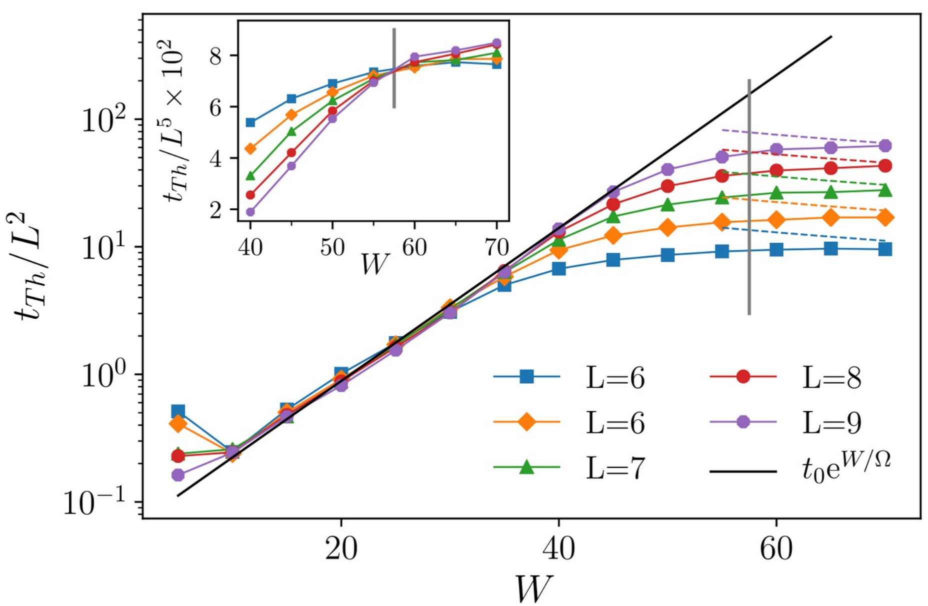
$t_{Th} = t_0 e^{W/\Omega} L^2$ works well at small W;

Diffusion constant $D = t_0^{-1} e^{-W/\Omega}$

$3D$ model :




$5D$ model :



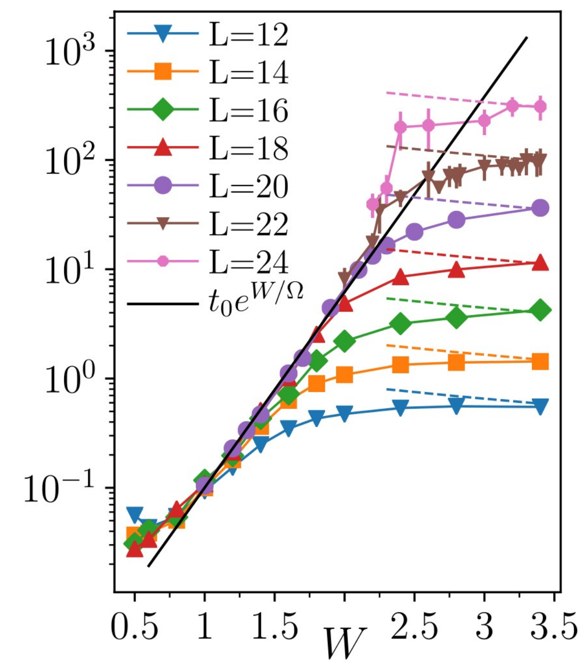
# Thouless time at MBL transition

$$H = \sum_{i=1}^{L}\sum_{l=1}^{2} J_l \left(S_i^x S_{i+l}^x + S_i^y S_{i+l}^y + \Delta_l S_i^z S_{i+l}^z\right) + \sum_{i=1}^{L} h_i S_i^z$$

- At small disorder W:

$$t_{Th} = t_0 e^{W/\Omega} L^2$$

- But this scaling is broken for largest system sizes considered, similarly to Anderson model




Is there MBL??