On Porous Medium Equation: Gradient-flow perspective and Inverse Problems

Tuhin Ghosh

Harish-Chandra Research Institute, India.

A School on Gradient Flows, ICTS Bengaluru

Porous medium equation

Let $\rho(t,x) \geq 0$ be a density at time t and position x for a gas in a porous medium occupying the domain $\Omega \subseteq \mathbb{R}^N$. The porous medium equation is

$$\partial_t \rho = \Delta(\rho^m), \qquad m > 0.$$

- $m > 1$: slow diffusion, degeneracy at $\rho = 0$, finite-speed propagation.
- $m = 1$: heat equation.
- $0 < m < 1$: fast diffusion, singular behaviour at $\rho = 0$.

Mass is formally preserved:

$$\frac{d}{dt} \int_{\mathbb{R}^N} \rho(t,x)\, dx = 0.$$

So the natural state space is a space of probability densities.

Derivation

Equivalent divergence form:

$$\partial_t \rho = \operatorname{div}\left(m\rho^{m-1}\nabla\rho\right).$$

Diffusion speed depends on density.

$$\text{Continuity equation:} \qquad \partial_t \rho + \operatorname{div}(\rho v) = 0,$$

$$\text{Darcy law:} \qquad v = -\frac{k}{\mu}\nabla p.$$

where $k > 0$ is the permeability (the porosity) of the medium and $\mu > 0$ is the fluid viscosity.

- Assume a pressure law of the form

$$p(\rho) = \frac{m}{m-1}\rho^{m-1}$$

$m \neq 1$ depends on the property of the gas.

- Then (assuming $k = \mu = 1$)

$$\partial_t \rho = \operatorname{div}\left(\rho\nabla p(\rho)\right) = \operatorname{div}\left(m\rho^{m-1}\nabla\rho\right) = \Delta(\rho^m).$$

PME

- Similarities to the heat equation: has the maximum principle, comparison principle, ...
- Differences: no UCP, solutions not real analytic in time, ...

  Barenblatt (self-similar profile) solutions:

$$t^{-\alpha}(C - k|x|^2 t^{-2\beta})_+^{\frac{1}{m-1}}; \quad m > 1.$$

(α, β, k depend on m, N, C is arbitrary).

PME

- Similarities to the heat equation: has the maximum principle, comparison principle, ...
- Differences: no UCP, solutions not real analytic in time, ...

  Barenblatt (self-similar profile) solutions:

$$t^{-\alpha}(C - k|x|^2 t^{-2\beta})_+^{\frac{1}{m-1}}; \quad m > 1.$$

  (α, β, k depend on m, N, C is arbitrary).

If we do not ignore all the constants

$$\epsilon(x)\partial_t u(t,x) - \nabla \cdot (\gamma(x)\nabla u^m(t,x)) = 0, \quad (t,x) \in (0,\infty) \times \Omega$$

where the parameters

ϵ - the porosity of the medium

μ - related to the viscosity of the fluid.

The state space

We set

$$\mathcal{M} := \left\{ \rho \geq 0 : \int_{\mathbb{R}^N} \rho(x)\, dx = 1 \right\}.$$

A tangent vector at ρ is a signed density variation s with zero total mass:

$$T_\rho \mathcal{M} \simeq \left\{ s : \int_{\mathbb{R}^N} s(x)\, dx = 0 \right\}.$$

The central question is:

Question

Can we find a metric tensor g_ρ and an energy $\mathcal{E}$ such that

$$\partial_t \rho = -\operatorname{grad}_g \mathcal{E}(\rho)?$$

Abstract gradient flow

On a Riemannian manifold $(\mathcal{M}, g)$, the gradient of $\mathcal{E} : \mathcal{M} \to \mathbb{R}$ is defined by

$$g_\rho(\operatorname{grad} \mathcal{E}(\rho), s) = D\mathcal{E}(\rho)[s] \qquad \forall s \in T_\rho \mathcal{M}.$$

The gradient-flow equation is

$$\frac{d\rho}{dt} = -\operatorname{grad} \mathcal{E}(\rho).$$

Then

$$\frac{d}{dt}\mathcal{E}(\rho(t)) = -g_\rho(\dot\rho, \dot\rho) = -|\dot\rho|^2_{g_\rho} \le 0.$$

Energy decreases at exactly the metric speed squared.

Traditional metric

One classical interpretation uses

$$-\Delta p = s, \qquad g_\rho^{\mathrm{tr}}(s,s) = \int_{\mathbb{R}^N} |\nabla p|^2\,dx.$$

This metric does *not* depend on ρ.
The energy is

$$\mathcal{E}_{\mathrm{tr}}(\rho) = \frac{1}{m+1}\int_{\mathbb{R}^N} \rho^{m+1}\,dx.$$

Then

$$\partial_t \rho = \Delta(\rho^m)$$

is recovered as an H^{-1}-gradient flow.

Traditional metric

One classical interpretation uses

$$-\Delta p = s, \qquad g_\rho^{\mathrm{tr}}(s,s) = \int_{\mathbb{R}^N} |\nabla p|^2 \, dx.$$

This metric does *not* depend on ρ.
The energy is

$$\mathcal{E}_{\mathrm{tr}}(\rho) = \frac{1}{m+1} \int_{\mathbb{R}^N} \rho^{m+1} \, dx.$$

Then

$$\partial_t \rho = \Delta(\rho^m)$$

is recovered as an H^{-1}-gradient flow.

- This is mathematically valid. Now we want to see the transport nature.

Otto's transport metric

The new metric identifies tangent vectors by

$$s = -\operatorname{div}(\rho \nabla p).$$

Then

$$g_\rho(s, s) = \int_{\mathbb{R}^N} \rho |\nabla p|^2 \, dx.$$

More generally, for

$$s_i = -\operatorname{div}(\rho \nabla \phi_i), \qquad i = 1, 2,$$

set

$$g_\rho(s_1, s_2) = \int_{\mathbb{R}^N} \rho \nabla \phi_1 \cdot \nabla \phi_2 \, dx.$$

Wasserstein distance: Monge formulation

Given two densities $\rho_0, \rho_1 \in \mathcal{P}_2(\mathbb{R}^N)$

$$\mathcal{P}_2(\mathbb{R}^N) = \left\{ \rho \geq 0 : \int_{\mathbb{R}^N} \rho \, dx = 1, \int_{\mathbb{R}^N} |x|^2 \rho(x) \, dx < \infty \right\}$$

, a transport map T sends ρ_0 to ρ_1:

$$T_{\#}\rho_0 = \rho_1.$$

The quadratic Monge cost is

$$\int_{\mathbb{R}^N} |x - T(x)|^2 \rho_0(x) \, dx.$$

If an optimal map exists,

$$\mathrm{W}_2^2(\rho_0, \rho_1) = \inf_{T_{\#}\rho_0 = \rho_1} \int_{\mathbb{R}^N} |x - T(x)|^2 \rho_0(x) \, dx.$$

This is the minimal squared displacement needed to rearrange one density into another.

Wasserstein distance: Kantorovich relaxation

A transport plan π is a probability measure on $\mathbb{R}^N \times \mathbb{R}^N$ with marginals ρ_0 and ρ_1:

$$\pi \in \Pi(\rho_0, \rho_1).$$

The relaxed formulation is

$$\mathrm{W}_2^2(\rho_0, \rho_1) = \inf_{\pi \in \Pi(\rho_0, \rho_1)} \int_{\mathbb{R}^N \times \mathbb{R}^N} |x - y|^2 \, d\pi(x, y).$$

- If ρ_0 is absolutely continuous, the optimal plan is induced by a map.
- Brenier's theorem: $T = \nabla \psi$ for a convex ψ.

Tangent vectors to probability densities

A tangent vector $s \in T_\rho \mathcal{P}$ must have zero total mass:

$$\int_{\mathbb{R}^N} s(x)\, dx = 0.$$

The transport interpretation writes s through a velocity field v:

$$s + \operatorname{div}(\rho v) = 0.$$

The minimal kinetic cost for producing s is

$$\inf_v \left\{ \int_{\mathbb{R}^N} \rho |v|^2 dx : s + \operatorname{div}(\rho v) = 0 \right\}.$$

The minimizer is a gradient field $v = \nabla \phi$.

Equivalently,

$$g_\rho(s, s) = \inf_u \left\{ \int_{\mathbb{R}^N} \rho |u|^2 dx : s + \operatorname{div}(\rho u) = 0 \right\}.$$

This is the infinitesimal form of optimal transport: the cost is kinetic energy of moving mass.

Dynamic formulation

The same distance admits the dynamic characterization

$$\mathrm{W}_2^2(\rho_0, \rho_1) = \inf_{(\rho, v)} \int_0^1 \int_{\mathbb{R}^N} \rho |v|^2 \, dxdt,$$

subject to

$$\partial_t \rho + \mathrm{div}(\rho v) = 0, \qquad \rho_{t=0} = \rho_0, \qquad \rho_{t=1} = \rho_1.$$

• The action density $\int \rho |v|^2$ is precisely the infinitesimal Otto metric. Thus W_2 is the genuine distance induced by this Riemannian structure.

The entropy functional

The physically natural energy is

$$\mathcal{E}(\rho) = \begin{cases} \dfrac{1}{m-1} \displaystyle\int_{\mathbb{R}^N} \rho^m \, dx, & m \neq 1, \\ \displaystyle\int_{\mathbb{R}^N} \rho \log \rho \, dx, & m = 1. \end{cases}$$

Its first variation is

$$\frac{\delta \mathcal{E}}{\delta \rho} = e'(\rho) = \begin{cases} \dfrac{m}{m-1} \rho^{m-1}, & m \neq 1, \\ \log \rho + 1, & m = 1. \end{cases}$$

The pressure law satisfies

$$P(z) = z e'(z) - e(z) = z^m.$$

Recovering the PDE from the metric

Let $\dot\rho = -\operatorname{grad}\mathcal{E}(\rho)$. For every $s = -\operatorname{div}(\rho\nabla p)$,

$$g_\rho(\dot\rho, s) + D\mathcal{E}(\rho)[s] = 0.$$

Since

$$g_\rho(\dot\rho, s) = \int \dot\rho\, p, \qquad D\mathcal{E}(\rho)[s] = \int e'(\rho) s,$$

we get

$$\int \dot\rho p - \int e'(\rho)\operatorname{div}(\rho\nabla p) = 0.$$

After integration by parts,

$$\int (\dot\rho - \operatorname{div}(\rho\nabla e'(\rho)))\, p = 0.$$

Hence

$$\partial_t\rho = \operatorname{div}(\rho\nabla e'(\rho)) = \Delta(\rho^m).$$

Energy dissipation identity

Along smooth solutions,

$$\frac{d}{dt}\mathcal{E}(\rho) = -\int_{\mathbb{R}^N} \rho \left| \nabla \frac{\delta \mathcal{E}}{\delta \rho}(\rho) \right|^2 dx.$$

Equivalently, if

$$v = -\nabla \frac{\delta \mathcal{E}}{\delta \rho}(\rho),$$

then

$$\partial_t \rho + \mathrm{div}(\rho v) = 0 \qquad \text{and} \qquad \frac{d}{dt}\mathcal{E}(\rho) = -\int_{\mathbb{R}^N} \rho |v|^2 dx.$$

So the rate of energy loss equals the rate of dissipation induced by the metric.

Why the L^2 gradient is not the right one

If one incorrectly used the flat L^2-metric, then the gradient flow of $\mathcal{E}$ would be

$$\partial_t \rho = -\frac{\delta \mathcal{E}}{\delta \rho} = -\frac{m}{m-1}\rho^{m-1},$$

which is not a diffusion equation at all. The missing ingredient is the continuity equation constraint:

$$\partial_t \rho + \mathrm{div}(\rho v) = 0.$$

This is exactly what the Wasserstein geometry builds into the notion of gradient.

Self-similarity and rescaling

The PME admits self-similar spreading solutions (Barenblatt solutions). After a suitable self-similar change of variables,

$$\rho(t,x) = t^{-\alpha} u(\tau, y), \qquad y = t^{-\beta} x, \qquad \tau = \log t,$$

the equation becomes a confined diffusion

$$\partial_\tau u = \Delta(u^m) + \beta \,\mathrm{div}(yu).$$

This is the gradient flow of the new energy

$$\mathcal{F}(u) = \frac{1}{m-1} \int u^m \, dy + \frac{\beta}{2} \int |y|^2 u(y) \, dy.$$

Barenblatt profile as the equilibrium of the rescaled flow

At equilibrium, the first variation of $\mathcal{F}$ is constant on the support of u:

$$\frac{m}{m-1}u^{m-1} + \frac{\beta}{2}|y|^2 = C.$$

Hence the stationary profile is of the form

$$u_*(y) = \left(C - k|y|^2\right)_+^{\frac{1}{m-1}}.$$

This is the Barenblatt profile in self-similar variables. So the long-time behavior of PME can be interpreted geometrically as convergence of the confined gradient flow toward the unique minimizer of $\mathcal{F}$.

Weak solutions

A natural weak formulation is:

$$\int_0^T \int_{\mathbb{R}^N} \left(\rho\, \partial_t \varphi + \rho^m \Delta \varphi\right) dxdt + \int_{\mathbb{R}^N} \rho_0(x) \varphi(0,x)\, dx = 0$$

for smooth compactly supported test functions φ. A basic energy quantity is

$$\mathcal{E}(\rho(t)) = \frac{1}{m-1} \int_{\mathbb{R}^N} \rho(t,x)^m\, dx,$$

which decreases in time.

The JKO scheme for the porous medium equation

Fix a time step $h > 0$, and given $\rho^{k-1} \in \mathcal{P}_2(\mathbb{R}^N)$, define

$$\rho^k \in \underset{\rho \in \mathcal{P}_2(\mathbb{R}^N)}{\arg\min} \left\{ \frac{1}{2h} \mathrm{W}_2^2(\rho, \rho^{k-1}) + \mathcal{E}(\rho) \right\}.$$

That is,

$$\rho^k \in \underset{\rho \in \mathcal{P}_2(\mathbb{R}^N)}{\arg\min} \left\{ \frac{1}{2h} \mathrm{W}_2^2(\rho, \rho^{k-1}) + \frac{1}{m-1} \int \rho^m \, dx \right\}.$$

The JKO scheme for the porous medium equation

Fix a time step $h > 0$, and given $\rho^{k-1} \in \mathcal{P}_2(\mathbb{R}^N)$, define

$$\rho^k \in \underset{\rho \in \mathcal{P}_2(\mathbb{R}^N)}{\arg\min} \left\{ \frac{1}{2h} \mathrm{W}_2^2(\rho, \rho^{k-1}) + \mathcal{E}(\rho) \right\}.$$

That is,

$$\rho^k \in \underset{\rho \in \mathcal{P}_2(\mathbb{R}^N)}{\arg\min} \left\{ \frac{1}{2h} \mathrm{W}_2^2(\rho, \rho^{k-1}) + \frac{1}{m-1} \int \rho^m \, dx \right\}.$$

Existence

A minimizer exists at each step, thanks to the direct method of Calculus of Variation using the Prokhorov compactness theorem.

Euler–Lagrange condition for JKO

Take a smooth perturbation by transport maps

$$T_\tau = \mathrm{Id} + \tau\xi, \qquad \rho_\tau = (T_\tau)_\# \rho^k.$$

Differentiating the discrete functional at $\tau = 0$ gives formally

$$0 = \frac{d}{d\tau}\bigg|_{\tau=0} \left\{ \frac{1}{2h} \mathrm{W}_2^2(\rho_\tau, \rho^{k-1}) + \mathcal{E}(\rho_\tau) \right\}.$$

This yields the weak discrete equation

$$\frac{\rho^k - \rho^{k-1}}{h} \approx \Delta\big((\rho^k)^m\big).$$

So the JKO minimizer is the implicit Euler step for PME in Wasserstein geometry.

Passing to the limit: existence of continuous flow

Define the piecewise constant interpolation

$$\rho_h(t) = \rho^k, \qquad t \in ((k-1)h, kh].$$

From the discrete estimates, one extracts a subsequence with

$$\rho_h \to \rho$$

in a suitable weak/metric sense as $h \downarrow 0$. The discrete Euler–Lagrange identity passes to the limit, giving a weak solution of

$$\partial_t \rho - \Delta(\rho^m) = 0.$$

This is the minimizing movement construction of the PME flow.

Transition to inverse problems

The forward gradient-flow story explains how the density evolves.

The inverse problem asks whether boundary measurements determine the medium.

On a bounded smooth domain $\Omega \subset \mathbb{R}^n$, consider

$$\epsilon(x)\partial_t u - \mathrm{div}(\gamma(x)\nabla u^m) = 0.$$

Here

- $\epsilon(x)$: porosity/storage coefficient;
- $\gamma(x)$: permeability-conductivity coefficient.

Forward problem for an inhomogeneous medium

Let $Q_T = (0, T) \times \Omega$, $S_T = (0, T) \times \partial\Omega$. For the forward theory one often includes a source:

$$\begin{cases} \epsilon(x)\partial_t u - \operatorname{div}(\gamma(x)\nabla u^m) = f, & \text{in } Q_T, \\ u(0, x) = 0, & x \in \Omega, \\ u = \phi, & \text{on } S_T, \\ u \geq 0. \end{cases}$$

For the inverse DN map, one sets $f = 0$.

Assume

$$\epsilon,\, \gamma \in C^\infty(\overline{\Omega}), \qquad \epsilon,\, \gamma > 0.$$

Weak solution for the inhomogeneous problem

A weak solution satisfies

$$u \in L^\infty(Q_T), \qquad \nabla(u^m) \in L^2(Q_T), \qquad u^m|_{S_T} = \phi^m,$$

and

$$\int_{Q_T} \gamma \nabla\varphi \cdot \nabla(u^m)\, dtdx - \int_{Q_T} \epsilon\, \partial_t \varphi\, u\, dtdx = \int_{Q_T} \varphi f\, dtdx$$

for admissible test functions φ vanishing on the lateral boundary and at $t = T$.

Forward well-posedness

For $0 < T < \infty$, if

$$\phi^m \in C(S_T) \cap C^{0,1}(S_T), \qquad \phi \geq 0, \qquad \phi(0,x) = 0,$$

and $f \in L^\infty(Q_T)$, $f \geq 0$, then there is a unique weak solution.
It satisfies an estimate of the form

$$\|u^m\|_{L^2((0,T);H^1(\Omega))} \leq C_T\left(\|\phi^m\|_{C^{0,1}(S_T)} + \|\phi\|_{L^\infty(S_T)} + \|f\|_{L^\infty(Q_T)}\right).$$

If $f = 0$, then

$$0 \leq \sup_{Q_T} u \leq \sup_{S_T} \phi.$$

Dirichlet-to-Neumann map

For $f = 0$, define

$$\Lambda^{\mathrm{PM}}_{\epsilon,\gamma}(\phi) = \gamma\partial_\nu(u^m)|_{S_T}.$$

Weakly,

$$\langle\gamma\partial_\nu(u^m), \psi|_{S_T}\rangle = \int_{Q_T} (\gamma\nabla\psi \cdot \nabla(u^m) - \epsilon\partial_t\psi\, u)\, dtdx.$$

The inverse problem is:

$$\Lambda^{\mathrm{PM}}_{\epsilon,\gamma} \quad \Longrightarrow \quad \epsilon \text{ and } \gamma?$$

Main inverse results

Typical uniqueness statements are:

Infinite-time full data

If $\Lambda^{\mathrm{PM}}_{\epsilon_1,\gamma_1} = \Lambda^{\mathrm{PM}}_{\epsilon_2,\gamma_2}$, then

$$\gamma_1 = \gamma_2, \qquad \epsilon_1 = \epsilon_2.$$

Finite-time full data

For any $T > 0$, full boundary measurements determine both coefficients.

Partial data, special case

If $\gamma = 1$, measurements on an open boundary part $\Sigma \subset \partial\Omega$ determine ϵ.

First reduction: set $v = u^m$

Set

$$v = u^m.$$

Then $u = v^{1/m}$, and the equation becomes

$$\epsilon(x)\partial_t(v^{1/m}) - \mathrm{div}(\gamma(x)\nabla v) = 0.$$

The boundary value is

$$v|_{S_T} = \phi^m =: f.$$

The DN map becomes the conormal derivative of v:

$$\Lambda^v_{\epsilon,\gamma}(f) = \gamma\partial_\nu v|_{S_T}.$$

This puts the principal spatial part in linear divergence form.

Second reduction: Laplace transform

For infinite-time measurements, choose special boundary data

$$f(t,x) = tg(x).$$

Define

$$V(h,x) = \int_0^\infty e^{-t/h} v(t,x)\, dt, \qquad h > 0.$$

Then

$$\begin{cases} \operatorname{div}(\gamma(x)\nabla V(h,x)) = N(h,x), & x \in \Omega, \\ V(h,x)|_{\partial\Omega} = h^2 g(x), \end{cases}$$

where

$$N(h,x) = \frac{1}{h}\epsilon(x) \int_0^\infty e^{-t/h} v(t,x)^{1/m}\, dt.$$

Transformed DN map

The original PME DN map determines

$$\Lambda^h_{\epsilon,\gamma}(g) = h^{-2}\gamma\partial_\nu V(h,\cdot)|_{\partial\Omega}.$$

So one studies $\Lambda^h_{\epsilon,\gamma}$ as $h \to \infty$.

The large-h expansion separates the coefficients:

- leading term recovers γ;
- next term recovers ϵ.

First asymptotic ansatz: recovering γ

Write

$$V(h,x) = h^2 V_0(x) + R_1(h,x),$$

where

$$\begin{cases} \operatorname{div}(\gamma \nabla V_0) = 0, & x \in \Omega, \\ V_0|_{\partial\Omega} = g. \end{cases}$$

Then

$$\Lambda^h_{\epsilon,\gamma}(g) = \gamma \partial_\nu V_0|_{\partial\Omega} + o(1).$$

Therefore equality of PME DN maps implies equality of the classical Calderon DN maps for γ, hence

$$\gamma_1 = \gamma_2.$$

Refined ansatz: recovering ϵ

Once γ is known, use

$$V(h,x) = h^2 V_0(x) + h^{1/m} V_1(x) + R_2(h,x).$$

The correction V_1 solves

$$\begin{cases} \operatorname{div}(\gamma \nabla V_1) = \Gamma\left(1 + \frac{1}{m}\right) \epsilon(x) V_0(x)^{1/m}, \quad x \in \Omega, \\ V_1|_{\partial\Omega} = 0. \end{cases}$$

Thus the second asymptotic coefficient of $\Lambda^h_{\epsilon,\gamma}$ contains ϵ.

Integral identity for ϵ

Let W solve

$$\mathrm{div}(\gamma \nabla W) = 0 \quad \text{in } \Omega.$$

Green's identity gives

$$\langle \gamma \partial_\nu V_1, W|_{\partial\Omega} \rangle = \Gamma\left(1 + \frac{1}{m}\right) \int_\Omega \epsilon(x) V_0(x)^{1/m} W(x)\, dx.$$

For two media with the same DN map,

$$\int_\Omega (\epsilon_1 - \epsilon_2) V_0^{1/m} W\, dx = 0.$$

Taking $V_0 = 1 + sU$ and differentiating at $s = 0$ yields

$$\int_\Omega (\epsilon_1 - \epsilon_2) UW\, dx = 0,$$

which implies $\epsilon_1 = \epsilon_2$ by density of products of harmonic functions.

Selected references

- F. Otto, *The geometry of dissipative evolution equations: the porous medium equation*.
- R. Jordan, D. Kinderlehrer, F. Otto, *The variational formulation of the Fokker–Planck equation*.
- J. L. Vázquez, *The Porous Medium Equation: Mathematical Theory*.
- C. I. Cârstea, T. Ghosh, G. Nakamura, *An inverse boundary value problem for the inhomogeneous porous medium equation*.
- C. I. Cârstea, T. Ghosh, G. Uhlmann, inverse problems for finite-time and partial-data PME settings.