Hydrodynamics of Fractional Quantum Hall states

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Where is a place of the theory of FQHE?

For a long time the theory of FQHE survived without a Hamiltonian

- Adiabatic transport (the standard avenue of the theory): the ground state (say Laughlin wf) and the adiabatic properties (a gapped spectrum) suffices.

$$\Psi \sim \prod_{i>j}^{N} (z_i - z_j)^{\beta} e^{-\frac{B}{4}\sum_i |z_i|^2}, \quad \nu = \beta^{-1} - \text{ filling fraction}$$

- E.g., e may consider a system in multiply-connected geometry. Then the ground state is a bundle whose base is moduli space.

- A Hamiltonian is not necessary;

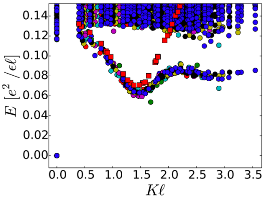
- For anything else, like optical properties of the bulk, and even dynamics on the edge one needs to know excited states.

  A Hamiltonian is necessary!

SPECTRUM POSSESSES UNIVERSAL FEATURES

- A popular approach is a few-particles ($\sim$ 10 particle the max) *ab initio* Hamiltonian. That does not extend beyond limited numerics.

- Still, numerics indicates that the spectrum possesses universal features



Magneto-roton mode:

Numerical spectrum (on a sphere) features a branch called a magneto-roton. It is isolated from continuum part and gapped from the ground state and features a minimum at some (not so small) momentum.

- Energy of the excitation vs momentum decreases: tendency to a crystallization which eventually had not been realized!
- Conjecture: The spectrum is geometrically determined. It is related to P. Sarnak extremal surfaces

Hydrodynamics

- The Hamiltonian could be obtained from the basic physical fact:

Electronic states in the Quantum Hall regime is *incompressible fluid*

- Incompressibility follows from the main property of the Lowest Landau Level: all states there are holomorphic!

- This fact alone allows to establish dynamics of the QH states beyond just the ground state!

- Incompressible flows is a geometrically governed dynamics requires minimal knowledge of microscopic

A search for a Hamiltonian is based on the basic facts

Fractional quantum Hall (interacting states on the Lowest Landau Level) form:

- **Liquid**
- **Incompressible** (*viz.* all states are *Holomorphic*)
- **Dissipation-free** liquid (inviscid), and non-resistive (at small T)
- **Ultra quantum**
- Flows are **chiral**

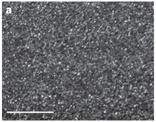
Quantum hydrodynamics is a natural approach.

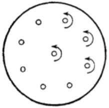
A minor obstacle on the way is that the Hydrodynamics had never been successfully/systematically quantized

Hydrodynamics of FQHE

The main thesis of this talk is the proposal tested against all available data about FQHE

Hydrodynamics of FQH states $\equiv$ Hydrodynamics of Fast Rotating Superfluid





Rotating superfluid is a dense array of quantum vortices (vortex matter)

In this correspondence vortices are identified with electrons (with attached magnetic flux)

Vortices $\longleftrightarrow$ Electrons

Historical landscape

- 1986: Girvin, MacDonald, Platzman were first who pointed out a similarity between superfluid and FQH liquid. They correctly used the Poisson structure of the rotating fluid but for unclear reason employed Feymann theory of non-rotating compressible superfluid with no vortices.
- The closest analog is rotating He^4: If rotation is fast, helium is almost incompressible and almost 2D;
- However, contrary to FQHE, He^4 is semiclassical, while QHE is ultra-quantum;
- Semiclassical theory of rotating superfluid is dated to 1962-75 (Khalatnikov, Fetter, ...)

The task is to lift the semiclassical treatment of vortex matter to ultra quantum

IDENTIFICATION

Quantum Vortex matter is characterized by two integers N_{vortex} and N_{Atom}

FQHE also is characterized by two integers $N_{\text{electrons}}$ and N_{Fluxes} of magnetic fluxes

electrons $\longleftrightarrow$ vortices,	$N_{\text{Electrons}} = N_{\text{Vortices}}$
magnetic flux quanta $\longleftrightarrow$ fluids atoms	$N_{\text{Fluxes}} = N_{\text{Atom}}$
filling fraction	$\nu = N_{\text{vortex}}/N_{\text{Atom}}$
spectral gap(Larmor energy)$\longleftrightarrow$ Frequency of rotation	$\Delta \sim \hbar\Omega = \hbar(eB)/2m_e$

Euler Equation for incompressible rotating fluid

$$\dot{\mathrm{u}} + (\mathrm{u}\cdot\nabla)\mathrm{u} + \nabla p = \Omega \times \mathrm{u}, \qquad \nabla\cdot\mathrm{u} = 0,$$

$$\frac{1}{V}\int \omega dV = 2\Omega, \qquad \omega = \nabla \times u$$

2D incompressible flows are completely characterized by positions of vortices.
Their states are holomorphic functions of positions.

The stationary flow happens to be the Laughlin's wave function (PW 2012)

$$\Psi \sim \prod_{i>j}^{N} (z_i - z_j)^{2\beta} e^{-\frac{B}{4}\sum_i |z_i|^2}$$

Kirchhoff Equations and Laughlin's state

$$\mathrm{u}(z,t) = u_x - \mathrm{i}u_y = -\mathrm{i}\Omega\bar{z} + \mathrm{i}\sum_{i=1}^{N} \frac{\Gamma}{z - z_i(t)}$$

Kirchhoff equations: $\mathrm{i}\dot{\bar{z}}_i = \Omega\bar{z}_i - \sum_{i\neq j}^{N} \frac{\Gamma}{z_i(t) - z_j(t)}$

Onsager quantization: $\Gamma = \frac{h}{m_{\text{Atom}}}$

- Quantization $\{z_i, \bar{z}_j\}_{P.B.} \to [z_i, \bar{z}_j] = 2\ell^2\delta_{ij}$, $\quad \bar{z}_i = 2\ell^2\partial_{z_i}$
- Stationary quantum flow: $-\hbar\partial_{z_i}|\Psi\rangle = \left(\Omega\bar{z}_i - x\sum_{j\neq i}\frac{\Gamma}{z_i - z_j}\right)|\Psi\rangle$
- Solution is Laughlin's w.f. $\Psi \sim \prod_{i>j}^{N}(z_i - z_j)^{2\beta}e^{-\frac{B}{4}\sum_i |z_i|^2}$, $\quad \beta = \frac{\Gamma}{\hbar\Omega}$

Quantum Hydro of vortex matter=rotating superfluid=FQHE

- Vortex flux: $\rho \mathrm{v} = \mathrm{i}\sum_i \dot{\bar{z}}_i \delta(z - z_i)$
- Symplectic structure: $\mathrm{v} = \nabla \times \frac{\delta \mathscr{H}}{\delta \rho}$
- The Hamiltonian could be expressed through the density of electrons=vorticity,

$$\rho = \frac{\nu}{2\pi}(\nabla \times \mathrm{v}) = \sum_i \delta(z - z_i)$$

A caveat:

This method restores the Hamiltonian up to a class function insensitive to variations.

Next slide: The Hamiltonian is expressed in units of $m_A \rho_A = (gap)^{-1}$ and quantum unit of circulation $\Gamma = h/m_A$

THE HAMILTONIAN

$$\mathscr{H} = \mathscr{H}_{\text{semi}} + \mathscr{H}_{\text{quantum}} + [\text{Class}(\rho)]$$

Semiclassical part (Khalatnikov, 1964), quantum part (PW, 2015).

The origin of the quantum part is the measure of integration over continuum flow in the Euler specification:

$$\text{Lagrangian measure} = dz_1 \dots dz_N \Rightarrow \text{Flat measure in Clebsch variables} = \mathscr{D}\lambda^1(z)\mathscr{D}\lambda^2(z),$$

$$\text{Eulerian measure} = [\text{Jacobian}] \times \mathscr{D}[\text{Flat measure in stream function}]$$

The Hamiltonian

Density of electrons=vorticity $\rho = \frac{\nu}{2\pi}(\nabla \times \mathrm{u})$

$$\mathscr{H}_{\text{semi}} = \int \left(\tfrac{1}{2}u^2 - \overbrace{\tfrac{\Gamma^2}{8\pi}\rho \log \rho}^{\text{odd viscosity}}\right) = \int \rho \int \left(\log \tfrac{1}{|r-r'|}\rho(r') + \tfrac{1}{2}\log \rho\right)$$

$$\mathscr{H}_{\text{quantum}} = -\nu \int \rho \log \rho + \underbrace{\tfrac{1}{2}\log \mathrm{Det}(-\Delta_\rho)}_{\text{gravitational anomaly}}.$$

$\Delta_\rho = \rho\,\partial_z \partial_{\bar{z}}$ – Laplace-Beltrami operator with metric $ds^2 = \rho\, dz d\bar{z}$

Polyakov formula: $\log \mathrm{Det}(-\Delta_\rho) = -\frac{1}{12\pi}\int (\nabla \log \rho)^2 + [\mathrm{Class}(\rho)]$

Geometric interpretation/representation of vortex dynamics

- Density of vortices can be thought as a metric of an auxiliary surface

$$\boxed{ds^2 = \rho\, dz d\bar{z}, \quad \rho = \nabla \times u > 0}$$

- Measure over flows is identical to the measure in 2D quantum gravity

Hamiltonian

$$\mathscr{H} = \iint \rho(r) \log \tfrac{1}{|r-r'|} \rho(r') + \int \left((\tfrac{1}{2} - \nu) \log \rho + \frac{1}{12\pi} (\nabla \log \rho)^2 \right) + [\mathrm{Class}(\rho)]$$

The structure function $\overline{S}(k) = \langle \rho_k \rho_{-k} \rangle$ follows from the Hamiltonian $\rho = \bar{\rho} + \sum_k e^{ikr} \rho_k$

$$\overline{S}(k) = \langle \rho_k \rho_{-k} \rangle = \tfrac{1-\nu}{8\nu} k^4 \left(1 - \tfrac{10\nu-3}{12\nu} (\ell k)^2 + \dots \right)$$

Now one can compute optical properties, like inelastic light scattering as a linear response to a smooth variation of the density, etc.

Central charge and remarkable 1/3

The energy computed on the solution of the Euler-Lagrange equation reads

$$E[\rho]-E_0=[\mathrm{Class}(\rho)]-\frac{c}{96\pi}\int(\nabla\log\rho)^2+\text{Higher gradients},$$

$$\boxed{c=1-6\left(\sqrt{2\nu}-\frac{1}{\sqrt{2\nu}}\right)^2}$$

Miraculously

$$\boxed{c=0,\ \text{at}\ \nu=\frac{1}{3}}$$

In this case $\nu=1/3$ the energy is the class-function determined only by the class (singularities). It does not depend on a density profile- solution of the Euler-Lagrange equation minimizing the energy

$$\boxed{\nu=\frac{1}{3}:\quad E[\rho]-E_0=[\mathrm{Class}(\rho)]}$$

Specifically, for the magneto-roton mode we have to find the minimal class and within the class determine the minimum.

Spectrum as Extremal surfaces

Geometric interpretation of density of electrons (or vorticity- density of vortices)

$$ds^2 = \rho dz d\bar{z} = (\nabla \times u) dz d\bar{z}$$

Could be seen as a metric of an auxiliary surface

Spectrum corresponds to *Extremal surfaces*: surface which minimize class-functions (A problem posted by P. Sarnak in mid-90th)

Topology of a sphere (genus-0):

Magneto-roton mode corresponds to a surface of revolution. Then there are two classes:

- Punctured sphere with two antipodal punctures threaded by magnetic flux
- Antipodal conical singularities

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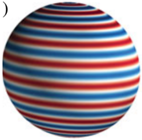

The spectrum is the class-function on a sphere as the function of the degree of antipodal singularities

Class function of a spindle: Spreafico and Zerbini

$$\log \operatorname{Det}(-\Delta) = -4\zeta_2(0,\alpha,1,1) + \frac{\alpha}{2} - 2\log\alpha - \left(\frac{\alpha}{6} + \frac{1}{6\alpha} - 1\right)\log\frac{\alpha}{2}$$

the Barnes double zeta function

$$\zeta_2(s,\alpha,1,1) = \sum_{m,n=0} \frac{1}{(\alpha m + n + 1)^s}$$