Exponential Mixing Via Additive Combinatorics

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Speed of Decay of Correlations

M = Connected, closed, cpt Riemannian Manifold of negative curvature.

g_t = geodesic flow on T^1M.

μ = measure of maximal entropy.

Bowen–Ruelle Conjecture:

μ is exponentially mixing, i.e.

$\exists\, \sigma > 0 : \quad \forall \varphi, \psi \in C^1(T^1 M),$

$$\int \varphi \circ g_t \cdot \psi \, d\mu = \int \varphi \, d\mu \int \psi \, d\mu + O_{\varphi,\psi}(e^{-\sigma |t|})$$

Speed of Decay of Correlations

Thm (Dolgopyat '98)

$\dim M = 2 \Rrightarrow$ all equillibrium measures are exp. mixing.

* **Chernoff**: stretched exp. mixing.

Speed of Decay of Correlations

Thm (Liverani '04)

g_t is exp. mixing w.r.t. Liouville measure.

* Significance of Liouville is conditional measures along unstable foliation are Lebesgue

* Dolgopyat showed this under pinching assumptions

Speed of Decay of Correlations

Thm (Giulietti-Liverani-Pollicott '12)

MME is exp. mixing under strong pinching assumptions

Speed of Decay of Correlations

Thm (Giulietti - Liverani - Pollicott '12)

MME is exp. mixing under strong pinching assumptions

* Spirit of pinching here:

MME is "close" to Liouville.

What makes Liouville "easier"?

* Liverani's approach rests on an oscillatory integral estimate: $\exists \delta > 0$

$$\left| \int_B e^{i\lambda\tau(x)} \, d\mu^u(x) \right| \lesssim |\lambda|^{-\delta}, \quad \forall \lambda \neq 0.$$

* $\tau(x)$ "=" [stable, unstable]

Advantage of Liverani's Method

* More intrinsic:
  smoothness of $g_t \Longrightarrow$ more precise info on spectral gap

* Stability under perturbation

* . - - -

Towards Non-smooth Measures

We'll discuss an approach based on additive combinatorics to extend Liverani's approach beyond SRB measures.

Setting of Main Results

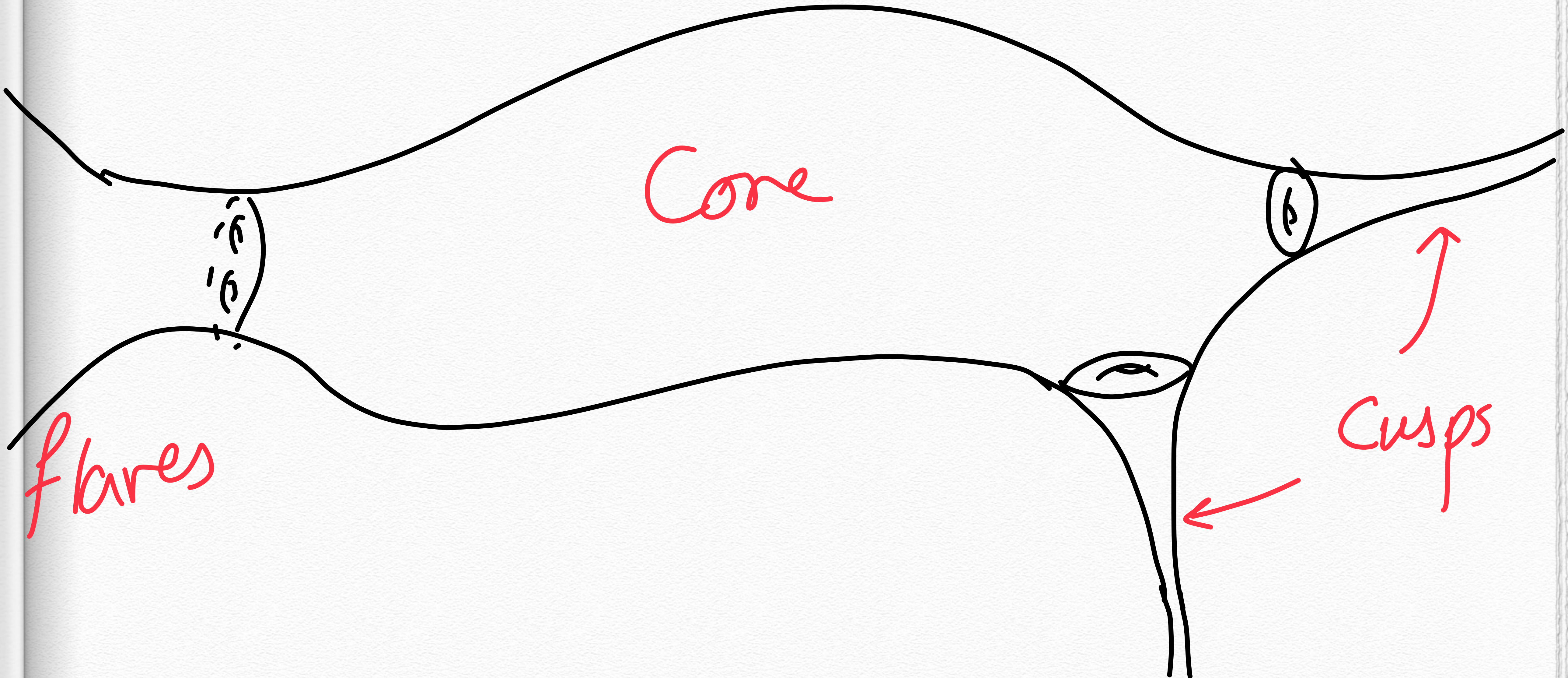
$\mathbb{H}^d$ = real, complex, quaternionic, or Octonionic hyperbolic space, dim = d

$\Gamma < \mathrm{Isom}(\mathbb{H}^d)$: discrete, geometrically finite, non-elementary

Λ_Γ = limit set $\overline{\Gamma . o} \cap \partial \mathbb{H}^d$

$\delta_\Gamma = \dim \Lambda_\Gamma$ = top. entropy of g_t

Geometrically Finite Manifolds



• Geom. finite "$\Longleftrightarrow$" (thin part = cusps)

Exponential Mixing

Theorem (K. '22+)

The measure of maximal entropy for the geodesic flow on $T^1(\mathbb{H}^d/\Gamma)$ is exponentially mixing.

Meromorphic Continuation

* **Laplace transform:**

$$\hat{\rho}_{\varphi,\psi}(z) = \int_0^\infty e^{-zt} \left(\int \varphi \circ g_t \cdot \psi \, d\mu \right) dt$$

holomorphic for $\mathrm{Re}(z) > 0$.

* $$\beta = \begin{cases} \infty, & \text{if } X \text{ has no cusps,} \\ \frac{1}{2} \min\{\delta_\Gamma, k_{min}, 2\delta_\Gamma - k_{max}\}, & \text{else.} \end{cases}$$

Meromorphic Continuation

Theorem (K. '22+)

$\forall \varphi \in C_c^\infty(X)$, $\hat{e}_{\varphi,\psi}$ extends meromorphically to $\mathrm{Re}(z) > -\beta_0$

* Smoothness is essential here.

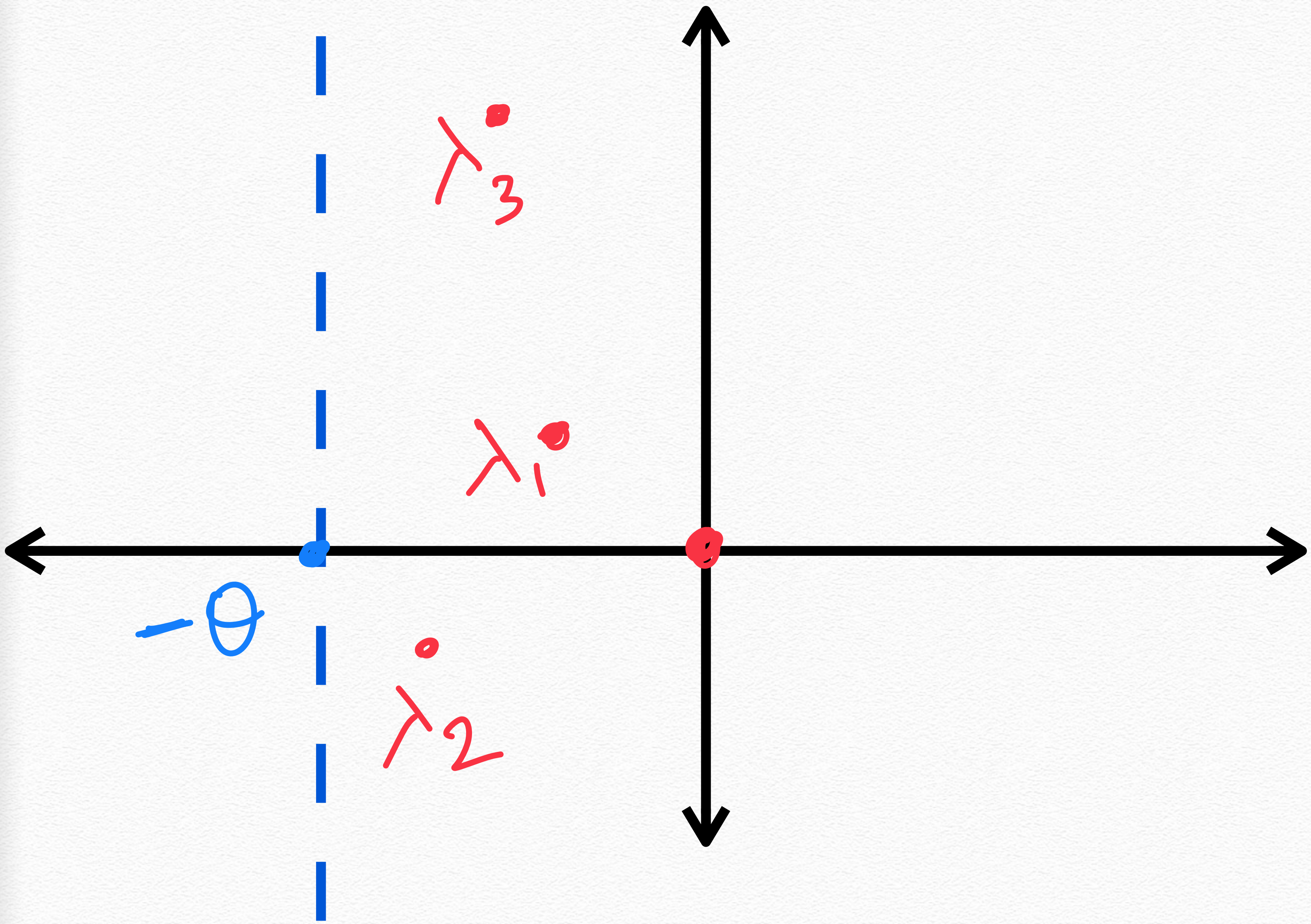
Pollicott-Ruelle Resonances

Theorem (K. '22+)

$\exists\, \theta > 0$, depending only on δ_Γ & ranks of cusps of Γ if any & $\exists\, \lambda_1, \dots, \lambda_n \in \mathbb{C}$ with $-\theta < \mathrm{Re}(\lambda_i) < 0$:

$$\int \varphi \circ g_t \, \psi \, d\mu = \int \varphi \int \psi + \sum_{i=1}^{n} e^{\lambda_i t} c_i(\varphi, \psi) + O_{\varphi,\psi}(e^{-\theta t}).$$

Pollicott-Ruelle Resonances



* E.g.: θ doesn't change on finite covers.

Prior Work

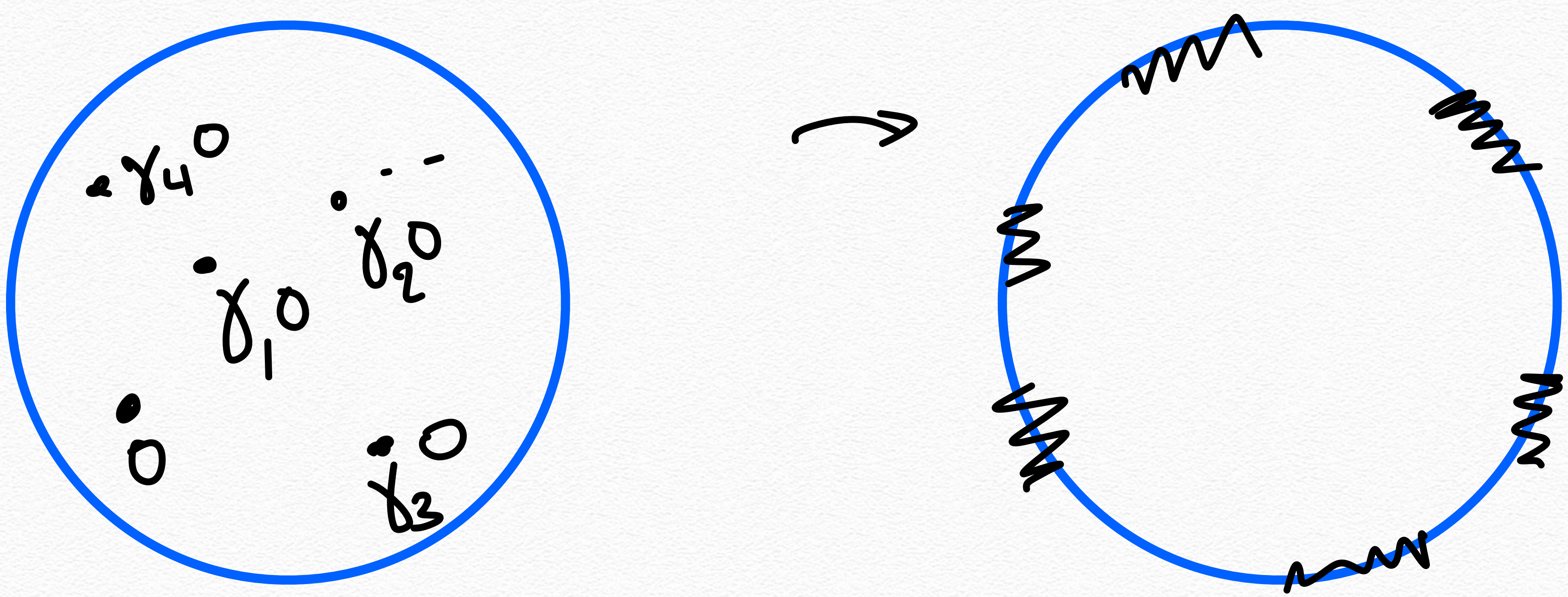
* Naud, Stoyanov, Li-Pan:
  Dolgopyat method, symbolic, no info on ess spectral gap, no resonances.
* Mohammadi-Oh, Edwards-Oh:
  representation theory, precise info., limited to $\delta_\Gamma > \frac{\text{Vol entropy}}{2}$.

Proof Ideas

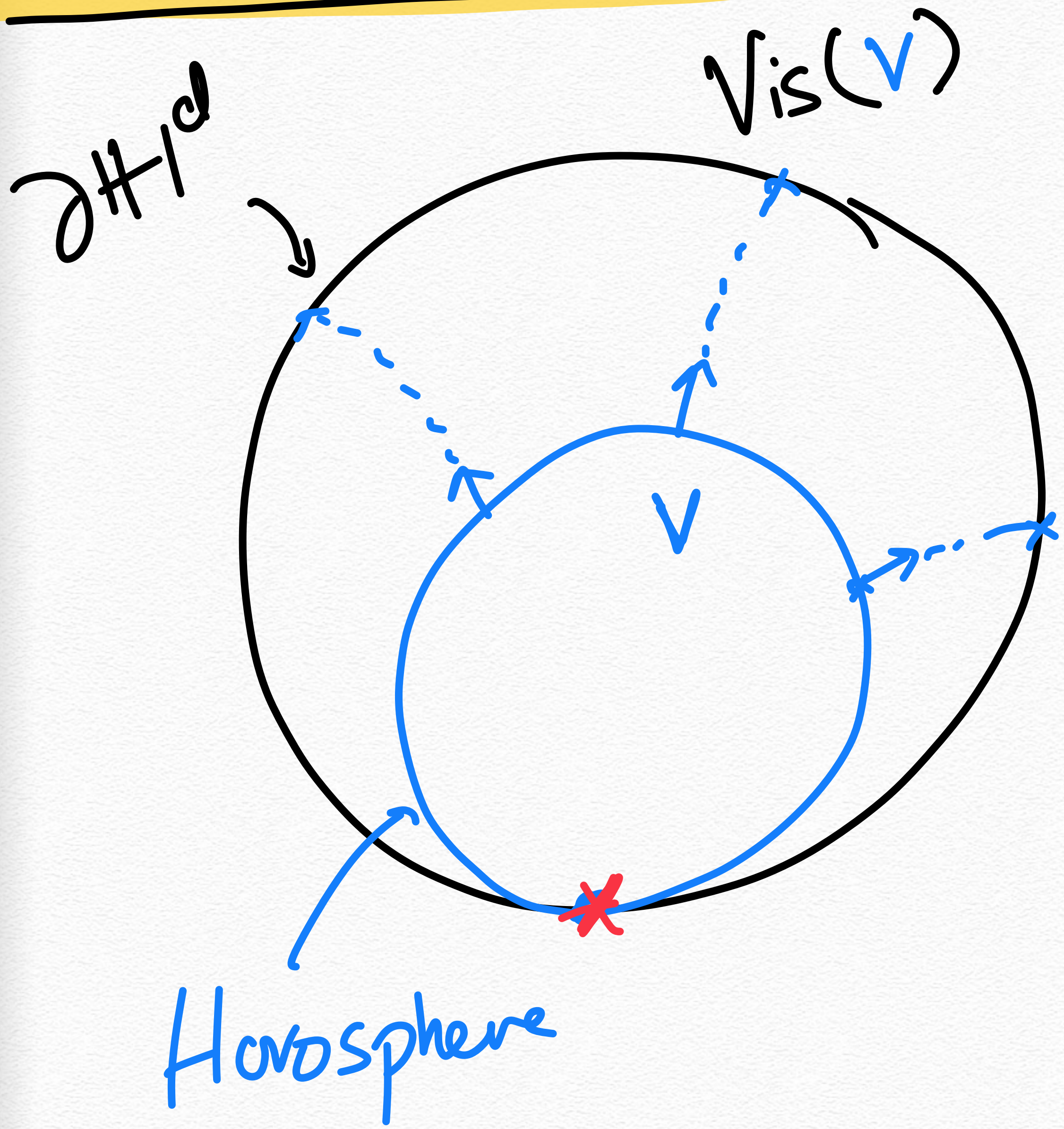
BMS Conditionals

* $\nu_o^{PS} := h_{top}$- Hausdorff measure on Λ_Γ

Patterson-Sullivan measure



BMS Conditionals



$\mathrm{Vis}: H(v) \to \partial \mathbb{H}^d \setminus \{*\}$

Visual map

$$\boxed{\mu_v \propto (\mathrm{Vis}^{-1})_* \nu_o^{PS}}$$

unstable conditional

Towards Oscillatory Integral Estimates

* A major difficulty is estimating integrals like:

$$\int_B \sum_\ell e^{i\lambda\langle v_\ell, x\rangle} \, d\mu_w(x) \quad , \quad |\lambda| \gg 1$$

* $\{v_\ell\}$ = discretized PS measure

* Key Idea: L^2-flattening of PS measures

L^2 - Flattening

L^2-dim of $\mu \in \mathrm{Prob}(\mathbb{R}^d)$:

$$\dim_2 \mu := d - \varlimsup_{R\to\infty} \frac{\log \int_{\|\xi\|<R} |\hat{\mu}(\xi)|^2 \, d\xi}{\log R}$$

L^2 Flattening of Unstable Conditionals

Theorem (K. '22+, Shmerkin)

Iterated convolution $\Longrightarrow$ smoothing of BMS conditionals:

$$\dim_2(\mu_w^{*n}) \xrightarrow{n \to \infty} \text{dimension of } \partial \mathbb{H}^d$$

L^2 Flattening - Ingredients

* Balog-Szemeredi-Gowers Lemma
* Hochman's Inverse Theorem for Entropy
* Non-concentration:
  - uniform doubling
  - friendliness

Uniform Doubling

Prop. (K. '22+) μ_w is uniformly doubling:

$\forall r > 0, \sigma > 1$:

$$\sigma^{\Delta} \lesssim \frac{\mu_w(B_{\sigma r})}{\mu_w(B_r)} \lesssim \sigma^{\Delta_+}$$

* Significant in the presence of cusps.

Friendliness of BMS Conditionals

Theorem (K. '22+)

$$\lim_{\varepsilon \downarrow 0} \sup_{w, \mathcal{L}} \frac{\mu_w(\mathcal{L}^{(\varepsilon)} \cap B_1)}{\mu_w(B_1)\, V(w)} = 0.$$

$\mathcal{L}$ = proper subspace of a horosphere $H(w)$.

B_1 = unit ball in $H(w)$.

$\mathcal{L}^{(\varepsilon)}$ = ε-nbhd of $\mathcal{L}$.

Friendliness of BMS Conditionals

* Main Input:

Thm (Connell-Muchnik)

PS measures on $\partial \mathbb{H}^d$ are stationary for a Zariski-dense random walk

Margulis Function

Ω = non-wandering set of g_t

= closure of periodic orbits.

Theorem (K. '22+): $\exists V : \Omega \to \mathbb{R}_{>0}$

a proper function; $\forall t \geq 0$,

$$\int_{B_1} V(g_t x) \, d\mu_w(x) \leq e^{-\Delta t} V(x) + B.$$

Margulis Function

* Orbits are biased away from cusps

* Well-understood in finite volume

* Fractal nature of μ_w requires new ideas in representation theory.

Happy Birthday Professor Dani!