ORIGAMI CONSTRUCTIONS

1. Basic Origami Operations

It is easy to classify what operations are possible under straight edge and compass because we know exactly what our tools can do. It was proved that if we only allow one fold at a time, and assume all our creases are straight lines, then the only folding operations possible are:

- O1 Given two points P_1 and P_2, we can fold a crease line connecting them.
- O2 Given two lines, we can locate their point of intersection, if it exists.
- O3 Given two points P_1 and P_2, we can fold the point P_1 onto P_2 (perpendicular bisector).
- O4 Given two lines L_1 and L_2 , we can fold the line L_1 onto the line L_2 (angle bisector).
- O5 Given a point P and a line L, we can make a fold line perpendicular to L passing through the point P (perpendicular through a point).
- O6 Given two points P_1 and P_2 and a line L, we can, whenever possible, make a fold that places P_1 onto the line L and passes through the point P_2.
- O7 Given two points P_1 and P_2 and two lines L_1 and L_2 , we can, whenever possible, make a fold that places P_1 onto L_1 and also places P_2 onto L_2 .
- O8 Given a point P and two nonparallel lines L_1 and L_2, we can make a fold perpendicular to L_2 that places P onto L_1.

2. Constructible numbers in Origami-

We would call a number constructible, if we can construct a line segment of the specified length using BOO(Basic Origami Operations listed above). Notice that, in measurement, the value of 1 is a matter of convention - you may think of it as a unit. And, this unit would help us define all other numbers as lengths. When we talk about constructibility of numbers, we are actually talking about constructibility of lengths.

Theorem 1. *If x and y can be constructed, then we can also construct $x+y$, $x-y$, xy and x/y.*

Proof. Exercise. (**Hint:** Similar triangles) □

Example 2.1. *Show that all rational numbers can be constructed using basic origami operations.*

Example 2.2. *Show that $\sqrt{2}$ can be constructed using basic origami operations. More generally, show that $\sqrt{n}$ can be constructed using basic origami operations. Finally, show that $\sqrt{\frac{p}{q}}$ can be constructed using basic origami operations.*

Theorem 2. *If x is constructible then $\sqrt{x}$ is also constructible.*

Proof. Exercise. (**Hint:** Take a line l. Mark an arbitrary point O on l. Further mark points A and B on l such that $|OA| = 1$, $|OB| = 1 + a$ and $|AB| = a$. Draw circle of radius $\frac{1+a}{2}$ centred at the midpoint of OB.) $\square$

Theorem 3. *Let A be the set of numbers that can be constructed using basic origami operations and let B be the set of numbers that can be constructed using compass and straight-edge. Then $B \subset A$.*

There are two approaches to proving this theorem and both approaches have their merits.

Approach 1: Observe that we cannot draw circles using basic origami operations. However, given two points p_1 and p_2, we can find the intersection of circles $C(p_i, r_i)$ (circle with centre p_i and radius r_i) using basic origami operations. Also, we can find the intersection of a line we can draw (using basic origmai operations) and the circle $C(p_1, r_1)$.

Approach 2: Recall that the numbers that can be constructed using compass and straight-edge had a simple description.

Exercise 1. *Prove Theorem 3 using both the approaches.*

Question. *We showed that $B \subset A$. Is $A \subset B$?*

Recall that an angle θ can be constructed (using compass and straight edge) iff we can construct $\cos(\theta)$. Recall that we cannot construct $20°$ using compass and straight-edge and therefore we cannot construct $\cos(20°)$. So, the most natural question is whether we can construct $20°$ using origami.

Exercise 2. *Construct an equilateral triangle using origami - notice that in the process you have constructed* $60°$

Theorem 4. *If we can construct an angle θ using origami, we can also construct $\theta/3$.*

Question. *While using Approach 1 we mapped every compass scale construction to an origami construction, did you use all the BOO listed above?*

Question. *Notice that we can trisect any length using compass and straight-edge. Why doesn't this imply that we can trisect any angle?*