Renormalized solution and weak (variational) solution

If f is more regular, that is $f \in L^2(\Omega)$ then we recover that any renormalized solution belongs to $H_0^1(\Omega)$. More precisely

Proposition 3

Assume that $f \in L^2(\Omega)$ *and* $A(x,s) \in (L^\infty(\Omega \times \mathbb{R}))^{N^2}$*. Then* u *is a renormalized solution of* (1) *iff* $u \in H_0^1(\Omega)$ *is a weak solution of* (1).

Proposition 3 is a consequence of the following

Proposition 4

For any $k > 0$ *we have*

$$\lambda \int_\Omega u T_k(u) + \int_\Omega A(x,u)\nabla u \cdot \nabla T_k(u) = \int_\Omega f T_k(u)$$

Formally $T_k(u)$ is an admissible test function for equation (1) and if $\lambda > 0$ then $u \in L^1(\Omega)$.

Proof of Proposition 4.

We use in the renormalized formulation h_n in place of h and $T_k(u)$ as test function in (8) :

$$\lambda \int_\Omega u h_n(u) T_k(u) + \int_\Omega h_n(u) A(x,u)\nabla u \cdot \nabla T_k(u) + \int_\Omega h_n'(u) T_k(u) A(x,u) \nabla u \cdot \nabla u = \int_\Omega f h_n(u) T_k(u)$$

We now derive a priori estimates independent of n to pass to the limit as $n \to +\infty$.

□

Proof of Proposition 3.

With the previous proposition, for $k > 0$, since f belongs to L^2, we have using the ellipticity of the matrix A, Cauchy-Schwarz and Poincaré inequalities

$$\alpha \int_\Omega |\nabla T_k(u)|^2 \leq \int_\Omega A(x,u)\nabla u \cdot \nabla T_k(u) \leq C \|f\|^2_{L^2(\Omega)}.$$

□

It follows that $T_k(u)$ is bounded (uniformly with respect to k) in $H_0^1(\Omega)$. Then we obtain that $u \in H_0^1(\Omega)$.

To conclude that u is a weak solution, it is sufficient to take $h_n(u)\varphi$ with $\overset{\varphi}{\cancel{\xi}} \in \mathcal{C}_0^\infty(\Omega)$ and to pass to the limit à n goes to infinity.

Stability

As explained at the beginning, it is possible to derive a stability result.

Theorem 5

Let $(f_\varepsilon)_{\varepsilon>0}$ a sequence of L^1 functions and $(A_\varepsilon)_{\varepsilon>0}$ a sequence of matrix fields such that

- $A_\varepsilon(x,r)\xi\cdot\xi \geq \alpha|\xi|^2$, *a.e.* $x\in\Omega, \forall r\in\mathbb{R}, \forall\xi\in\mathbb{R}^N$,
- *for any* $k>0$, $A_\varepsilon(x,r)\in L^\infty(\Omega\times(-k,k))^{N\times N}$.

Assume that $f_\varepsilon \to f$ *strongly in* L^1 *and*

$$\begin{cases} A_\varepsilon(x,r_\varepsilon) \longrightarrow A(x,r) \\ \textit{for every sequence } r_\varepsilon\in\mathbb{R} \textit{ such that } r_\varepsilon \longrightarrow r. \end{cases}$$

If u_ε *denotes a renormalized solution of* $\lambda u_\varepsilon - \operatorname{div}\big(A_\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon)\big) = f_\varepsilon$ *in* Ω, *then* u_ε *converges a.e. to* u *where* u *is a renormalized solution of* $\lambda u - \operatorname{div}\big(A(x,u)\nabla u)\big) = f$ *in* Ω *(with Dirichlet boundary conditions). We have also* $T_k(u_\varepsilon)\to T_k(u)$ *strongly in* $H_0^1(\Omega)$, *for any* $k>0$.

Extensions

As far as the existence question is concerned, a wide class of problems/generalization is possible

- our model problem with data in $H^{-1}(\Omega) + L^1(\Omega)$
- adding the term div$(\Phi(u))$ where Φ is a continuous function with value in $\mathbb{R}^N$ without any growth condition (Dirichlet b.c.)
- adding a $g(s)|\nabla u|^2$ (with a strong control of $g(s)$)
- replace $A(x, u)\nabla u$ by general Leray-Lions operators $\mathbf{a}(x, u, \nabla u)$ with p growth and equation with $L^1 + W^{-1,p'}$ data
- noncoercive problem

Remark 6

Nonlinear operators like $\mathbf{a}(x, u, \nabla u)$ give additional difficulties and require the Minty trick for the identification of weak limits.

Uniqueness

For the uniqueness of the solution of

$$\begin{cases} \lambda u - \operatorname{div}\big(A(x,u)\nabla u\big) = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{13}$$

we need to have additional assumption on $A(x,s)$ with respect to s and to distinguish (whatever the regularity of f is)

- $\lambda > 0$: the uniqueness comes from the term λu
- $\lambda = 0$: the uniqueness question is more intricate

The case $\lambda > 0$

We need to assume a local Lipschitz condition on $A(x,s)$ in s.

Theorem 7

Under the previous assumptions giving the existence. Moreover if $\lambda > 0$ and if A verifies

$$\forall K > 0, \exists L_K > 0, \quad |A(x,s) - A(x,r)| \leq L_K |s - r|, \quad \forall s, r \in [-K, K], \ a.e.,$$

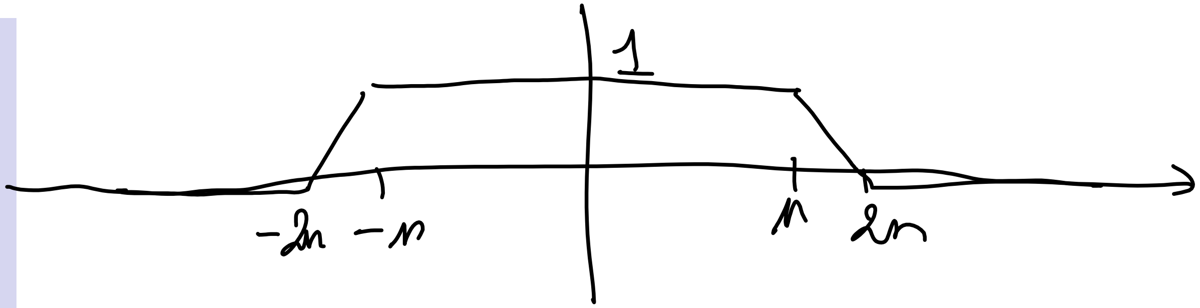
local Lipschitz condition.

then the renormalized solution is unique.

Proof.

Let u_1 and u_2 be two renormalized solutions. □

$$\begin{cases} \lambda u - \mathrm{div}(A(x,u)\nabla u) = f & \Omega \\ u = 0 & \partial\Omega \end{cases}$$



The strategy:

- use $h_n(u_1)\dfrac{T_k(u_1-u_2)}{k}$ in the equation in u_1,
  $h_n(u_2)\dfrac{T_k(u_1-u_2)}{k}$ in the equation in u_2
- compute the difference
- pass to the limit first as k goes to zero
- pass to the limit as n goes to infinity

Formaly we obtain

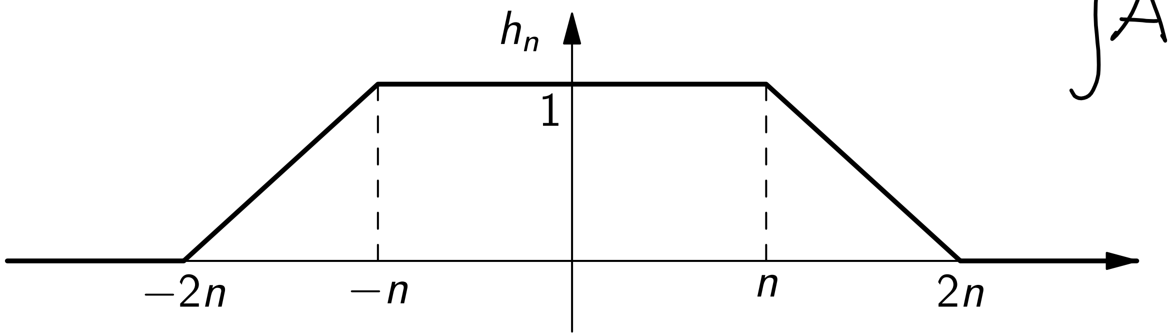
$$\lambda\int_\Omega |u_1-u_2| \le 0.$$

$T_k(u_1) \in H^1_0$
$T_k(u_2) \in H^1_0$

? $T_k(u_1-u_2) \in H^1_0(\Omega)$ (true $A(x,s)=A(x)$).

$$\int A(x)\nabla T_k(u_1-u_2)\cdot\nabla T_k(u_1-u_2) = 0$$

$\Rightarrow u_1=u_2$.



Since $\mathrm{supp}(h_n) = [-2n; 2n]$ we have

$k<1$

$$h_n(u_1)T_k(u_1-u_2) = h_n(u_1)T_k(T_{2n+1}(u_1) - T_{2n+1}(u_2)) \quad \text{a.e.}$$

$\in L^\infty(\Omega)\cap H^1_0(\Omega)$

which is then an admissible test function.

$$\lambda\int_\Omega (u_1h_n(u_1) - u_2h_n(u_2))T_k(u_1-u_2)$$
$$+\int_\Omega h_n'(u_1)A(x,u_1)\nabla u_1\nabla u_1 T_k(u_1-u_2) - \int_\Omega h_n'(u_2)A(x,u_2)\nabla u_2\nabla u_2 T_k(u_1-u_2)$$
$$+\int_\Omega h_n(u_1)A(x,u_1)\nabla u_1\nabla T_k(u_1-u_2) - \int_\Omega h_n(u_2)A(x,u_2)\nabla u_2\nabla T_k(u_1-u_2)$$
$$= \int_\Omega f(h_n(u_1) - h_n(u_2))T_k(u_1-u_2)$$

We divide by k and let us study the behavior of each term as $k \to 0$.

- $$\lambda \int_\Omega \left(u_1 h_n(u_1) - u_2 h_n(u_2)\right) \frac{T_k(u_1-u_2)}{k} \xrightarrow[k\to 0]{} \lambda \int \left(u_1 h_n(u_1) - u_2 h_n(u_2)\right) sg(u_1-u_2)$$

$$\left|\frac{T_k(u_1-u_2)}{k}\right| \leq 1 \qquad \frac{T_k(u_1-u_2)}{k} \longrightarrow sg(u_1-u_2)$$

u_1, u_2 finite a.e. $\quad h_n(u_1) \xrightarrow[n\to+\infty]{} 1 \quad h_n(u_2) \longrightarrow$

$$\lambda \int \left(u_1 h_n(u_1) - u_2 h_n(u_2)\right) sg(u_1-u_2) \longrightarrow \lambda \int (u_1-u_2)\, sg(u_1-u_2) = \lambda \int |u_1-u_2|$$

- Similarly

$$\int f\left(h_n(u_1) - h_n(u_2)\right) \frac{T_k(u_1-u_2)}{k} \xrightarrow[k\to 0]{} \int f\left(\underbrace{h_n(u_1) - h_n(u_2)}_{\xrightarrow[n\to+\infty]{} 0}\right) sg(u_1-u_2) \xrightarrow[n\to+\infty]{} 0$$

$f \in L^1$

- Decay of the truncated energy

$$\left|\int_\Omega h_n'(u_1) A(x,u_1) \nabla u_1 \nabla u_1 \frac{T_k(u_1-u_2)}{k}\right| \leq \frac{1}{n} \int_{|u_1|<2n} A(x,u_1) \nabla u_1 \cdot \nabla u_1 \xrightarrow[n\to+\infty]{} 0$$

$$\int_\Omega h_n(u_1)A(x,u_1)\nabla u_1\cdot\nabla T_k(u_1-u_2) - \int_\Omega h_n(u_2)A(x,u_2)\nabla u_2\cdot\nabla T_k(u_1-u_2)$$

$$= \int_\Omega h_n(u_1)A(x,u_1)(\nabla u_1-\nabla u_2)\cdot\nabla T_k(u_1-u_2) + \int_\Omega\big(h_n(u_1)A(x,u_1)-h_n(u_2)A(x,u_2)\big)\nabla u_2\cdot\nabla T_k(u_1-u_2)$$

$$= \underbrace{\int_\Omega h_n(u_1)A(x,u_1)\nabla T_k(u_1-u_2)\cdot\nabla T_k(u_1-u_2)}_{\geq 0} + \underbrace{\int_{\substack{0<|u_1-u_2|<k\\ |u_1|<2n+1\\ |u_2|<2n+1}}\big(h_n(u_1)A(x,u_1)-h_n(u_2)A(x,u_2)\big)\nabla u_2\cdot\nabla T_k(u_1-u_2)}_{B_{k,n}}$$

$k<1$

local Lipschitz

$$\frac{|B_{k,n}|}{k} \leq C\int_{0<|u_1-u_2|<k} |\nabla T_{2n+1}(u_2)|^2 + |\nabla T_{2n+1}(u_1)|^2$$

Lebesgue Dominated convergence $\quad \mathbb{1}_{\{0<|u_1-u_2|<k\}} \xrightarrow[k\to 0]{} 0$ a.e. in Ω

$$\lim_{k\to 0}\left|\frac{B_{k,n}}{k}\right| = 0$$

<u>Conclusion</u> $\quad \lambda\int|u_1-u_2| + \boxed{\geq 0} \leq 0 \qquad u_1-u_2=0$

$u_1=u_2$ a.e. in Ω

uniqueness

$\lambda > 0$: dependence with respect to f

Theorem 8

Under the previous assumptions giving the existence. Moreover assume that $\lambda > 0$ *and that* A *verifies*

$$\forall K > 0, \exists L_K > 0, \quad |A(x,s) - A(x,r)| \leq L_K |s-r|, \quad \forall s, r \in [-K, K], \ a.e.$$

Let f_1 *and* f_2 *two elements of* $L^1(\Omega)$*. Let* u_1 *(resp.* u_2*) the renormalized solution of* (1) *with* f_1 *in place of* f *(resp.* f_2 *in place of* f*). Then*

$$\lambda \|u_1 - u_2\|_{L^1(\Omega)} \leq \|f_1 - f_2\|_{L^1(\Omega)}.$$

The variational case and $\lambda = 0$

$$\begin{cases} -\operatorname{div}(A(x,u)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with

- $A(x,r)$ bounded elliptic matrix, global Lipschitz in the second variable,
- $f \in H^{-1}(\Omega)$

If u_1 and u_2 are two weak solutions, let us use $T_k(u_1 - u_2)$ as a test function in the difference of the equations.

But dividing by k and letting $k \to 0$ gives

$$\lim_{k\to 0} \frac{1}{k} \int_\Omega |\nabla T_k(u_1 - u_2)|^2 = 0$$

but not the uniqueness.

$$\frac{1}{k} \int |T_k(u_1 - u_2)|^2 \xrightarrow[k \to 0]{} 0$$

$$\int \left(A(x,u_1)\nabla u_1 - A(x,u_2)\nabla u_2\right)\cdot \nabla T_k(u_1-u_2) = 0$$

The idea of Artola (86): (dropping the x dependence of $A(x,r)$)

$$\int_\Omega A(u_1)(\nabla u_1 - \nabla u_2)\cdot \nabla T_k(u_1-u_2) \le \left| \int_\Omega (A(u_1)-A(u_2))\nabla u_2 \cdot \nabla T_k(u_1-u_2) \right|$$

$$\alpha \int_\Omega |\nabla T_k(u_1-u_2)|^2 \le \left| \int_\Omega (A(u_1)-A(u_2))\nabla u_2 \cdot \nabla T_k(u_1-u_2) \right|$$

$$\le Ck\Big(\int_\Omega |\nabla T_k(u_1-u_2)|^2\Big)^{1/2}\Big(\int_{\{0<|u_1-u_2|<k\}} |\nabla u_2|^2\Big)^{1/2}$$

So that

$\underline{\forall k>0}$:

$$\alpha \int_\Omega |\nabla T_k(u_1-u_2)|^2 \le Ck^2\Big(\int_{\{0<|u_1-u_2|<k\}} |\nabla u_2|^2\Big)$$

Divide by k^2, $k\to 0$, $|\nabla u_2|^2 \in L^1(\Omega)$, Poincaré inequality and Lebesgue theorem

Lebesgue Th, $\mathbb{1}_{\{0<|u_1-u_2|<k\}} \xrightarrow[k\to 0]{} 0$, $\int_{\{0<|u_1-u_2|<k\}} |\nabla u_2|^2 \xrightarrow[k\to 0]{} 0$

Poincaré

$$\int \Big(\frac{T_k(u_1-u_2)}{k}\Big)^2 \le C' \int \Big|\frac{\nabla T_k(u_1-u_2)}{k}\Big|^2 \xrightarrow[k\to 0]{} 0$$

$$\xrightarrow[k\to 0]{} \int \mathbb{1}_{\{|u_1-u_2|\neq 0\}} = 0 \Rightarrow u_1 = u_2$$

The variational case and $\lambda = 0$

Boccardo-Murat-Gallouët (1994), Chipot-Michaille (1989), Carrillo-Chipot (1985):

- $\mathbf{a}(x, s, \xi)$ nonlinear operators (strong monotonicity, Hölder continuous in ξ, global Lipschitz in s or with a strong control of the Lipschiz coefficient) with p growth $\underline{1 < p \leq 2}$
- add div$(\Phi(u))$ term

but since the p-Laplace is degenerated in 0 when $p > 2$

- counter example when $p > 2$. In the particular case of non negative right-hand side Murat-Casado Diaz-Porretta (2007) proved some uniqueness results

L^1 data and $\lambda = 0$

The method of Artola, which is

$$\int_\Omega |\nabla T_k(u_1 - u_2)|^2 \leq k^2 \Big(\int_{\{0<|u_1-u_2|<k\}} |\nabla u_2|^2 \Big)$$

requires $|\nabla u_2|^2$ in $L^1(\Omega)$ which is not the case (in general) for L^1 data.

With respect to the case $\lambda > 0$ the main difference is that (to my knowledge) with test function in $h_n(u)T_k(u_1 - u_2)/k^2$ we have first to let $n \to +\infty$ and then k goes to zero.

Idea, $f \in L^1$, global assumption

$$A(x,u)\nabla u = \frac{A(x,u)}{\varphi'(u)}\nabla\varphi(u)$$

Assume, for the sake of simplicity, that

- $A(x,r)$ bounded and global Lipschitz in r
- $f \geq 0$: u_1 and u_2 are two non negative solutions

Denote $\varphi(r) = (1+r)^3 - 1$ and let us use **formally** the test function $W_k = T_k(\varphi(u_1) - \varphi(u_2))$ to $-\mathrm{div}(A(x,u_1)\nabla u_1 - A(x,u_2)\nabla u_2) = 0$

$$0 = \int_\Omega (A(u_1)\nabla u_1 - A(u_2)\nabla u_2)\cdot\nabla W_k = \overbrace{\int_\Omega \frac{A(u_1)}{\varphi'(u_1)}\nabla W_k\cdot\nabla W_k}^{\geq 0}$$
$$+\int_\Omega\left(\frac{A(u_1)}{\varphi'(u_1)} - \frac{A(u_2)}{\varphi'(u_2)}\right)\varphi'(u_2)\nabla u_2\cdot\nabla W_k$$

$A(x,r)$ is "more" than Lipschitz with respect to $\varphi(r)$: for k small enough

$$\frac{A(x,r)}{\varphi'(r)} - \frac{A(x,s)}{\varphi'(s)} \qquad \mathbb{1}_{\{|\varphi(r)-\varphi(s)|<k\}} \qquad \underline{k<k_0}$$

$$\mathbb{1}_{\{0<|\varphi(u_1)-\varphi(u_2)|<k\}}\left|\frac{A(u_1)}{\varphi'(u_1)}-\frac{A(u_2)}{\varphi'(u_2)}\right| \leq \frac{Ck}{(1+u_1+u_2)^4}\mathbb{1}_{\{0<|\varphi(u_1)-\varphi(u_2)|<k\}}$$

Then we have

$$\boxed{\int_\Omega \frac{1}{\varphi'(u_1)}|\nabla W_k|^2} \leq Ck^2 \int_{\{0<|\varphi(u_1)-\varphi(u_2)|<k\}} \frac{|\nabla u_2|^2}{(1+u_1+u_2)^2}.$$

If $f \in L^1(\Omega)$, see Boccardo-Gallouët (1989) (since $\int_0^{u_2} \frac{ds}{(1+|s|)^2}$ is a bounded test function)

$$\frac{|\nabla u_2|^2}{(1+u_2)^2} \in L^1(\Omega)$$

It follows that

$$\frac{1}{k^2}\int_\Omega \frac{1}{\varphi'(u_1)}|\nabla W_k|^2 \to 0 \quad \text{as } k \to 0.$$

How to conclude that $u = v$?

$W_k = T_k(\varphi(u_1) - \varphi(u_2))$

$$\int_{\{|u_1|\leq n\}} |\nabla W_k|^2 \leq \max_{[-n,n]} (\varphi'(s)) \int \frac{1}{\varphi'(u_1)} |\nabla W_k|^2$$

In particular, for any $n > 0$

$$\frac{1}{k^2}\int_{\{|u_1|<n\}} |\nabla W_k|^2 \to 0 \quad \text{as } k \to 0.$$

We use again the function h_n

Poincaré with “two steps” : $h_n(u_1)\dfrac{W_k}{k} \in H_0^1(\Omega) \cap L^\infty(\Omega)$

$$\int_\Omega h_n(u_1)\left(\frac{W_k}{k}\right| \qquad \nabla(\ ) = h'_n(u_1)\frac{W_k}{k}\nabla u_1 + h_n(u_1)\frac{\nabla W_k}{k}$$

- first as $k \to 0$
- then as $n \to +\infty$.
- the decay of the truncated energy

$$k \to 0 \quad 0$$

$$n \to 0 \quad \to \quad 0$$

$$\int |sg(\varphi(u_1)) - \varphi(u_2))| = 0$$

$$= \lim_{n\to+\infty} \int h_n(u_1)\, sg(\varphi(u_1) - \varphi(u_2))|$$

Conclusion : $|\operatorname{sign}(\varphi(u_1) - \varphi(u_2))| = 0 \Rightarrow \varphi(u_1) = \varphi(u_2) \Rightarrow u_1 = u_2$ a.e. in Ω.

Remark 9

It remains to justify the **formal** computations. Using the functions h_n it is true !

A not easy to read uniqueness result

$$(14) \qquad A(x,s) \text{ uniformly coercive,}$$

$\exists K_0, C_1$ and $\delta > 1/2$ and $\varphi \in \mathcal{C}^1(\mathbb{R})$ with $\varphi' \geq 1$, such that if $|\varphi(s) - \varphi(t)| < K < K_0$ then

$$(15) \qquad \left| \frac{A(x,s)}{\varphi'(s)} - \frac{A(x,t)}{\varphi'(t)} \right| \leq \frac{C_1 K}{\varphi'(s)^{1/2}\varphi'(t)^{1/2}\big(1 + |\varphi(s)| + |\varphi(t)|\big)^{\delta}}$$

$\delta > \frac{1}{2}$

$2\delta > 1$

a.e. in Ω.

Theorem 10 (Blanchard-Désir-G (2005))

For any $f \in L^1(\Omega)$ the renormalized solution of $-\mathrm{div}(A(x,u)\nabla u) = f$ in Ω and $u = 0$ on $\partial\Omega$ is unique.

See also Porretta (2004) for uniqueness results with L^1 data and modulus of continuity of $A(x,\cdot)$ with exponential growth.

As in the previous example, the method is formally to use $W_K = T_K(\varphi(u) - \varphi(v))$ as a test function

$$0 = \int_\Omega (A(u)\nabla u - A(v)\nabla v) \cdot \nabla W_K = \int_\Omega \frac{A(u)}{\varphi'(u)} \nabla W_K \cdot \nabla W_K$$
$$+ \int_\Omega \left(\frac{A(u)}{\varphi'(u)} - \frac{A(v)}{\varphi'(v)} \right) \varphi'(v) \nabla v \cdot \nabla W_K$$

Play with the (technical) structure condition on A with respect to φ

$$\int_\Omega \frac{1}{\varphi'(u)} |\nabla W_K|^2 \leq CK^2 \int_{0<|W_K|<K} \frac{\varphi'(v)|\nabla v|^2}{(1+\varphi(v))^{2\delta}}.$$

$$2\delta > 1 \Rightarrow \frac{\varphi'(v)|\nabla v|^2}{(1+\varphi(v))^{2\delta}} \in L^1(\Omega).$$

$$\lim_{K\to 0} \frac{1}{K^2} \int_\Omega \frac{1}{\varphi'(u)} |\nabla W_K|^2 \to 0 \quad + 2 \text{ steps Poincaré} \Rightarrow u = v.$$

Almost readable uniqueness result

Theorem 11 (differential inequality condition)

If there exists $w \in \mathcal{C}^1(\mathbb{R})$, $w > 0$ *such that*

$$|w'| < C_2 w^{1+\eta} \quad \text{with } \eta > 0 \text{ and } C_2 > 0, \tag{16}$$

$$\big|A(x,s) - A(x,t)\big| \leq \left| \int_s^t w(z)dz \right| \tag{17}$$

$\forall s, t \in \mathbb{R}$, *a.e. in* Ω, *then we can construct a function* ε *such that* A *verifies assumption* (15) *in the above theorem and uniqueness holds.*

Arguments

Step 1. For $\mu > 0$ by taking n large enough in

$$\rho_n(t) = \left[\int_0^t (1 + |w'(z)| + |w'(-z)|) dz + w(0) + 1 \right]^n .$$

so that there exists $\psi = \rho_n \in \mathcal{C}^1(\mathbb{R}^+)$ verifying

$$\left|A(x,s) - A(x,t)\right| \le \left| \int_s^t \psi(|z|) dz \right|$$

and

$$(18) \qquad \exists M > 0,\ \forall t \ge 0, \quad \begin{cases} 1 \le \psi'(t) \le M(\psi(t))^{1+\mu}, \\ 1 \le \psi(t) \le (\psi'(t))^{1+\mu}, \end{cases}$$

Arguments

Step 2. Let $0 < \mu < 1$ and let $\psi \in \mathcal{C}^1(R^+)$ (as in Step 1)

$$\varphi(t) = \left((1 + \widetilde{\psi}(|t|))^3 - 1\right) \operatorname{sign}(t) \qquad \widetilde{\psi}(t) = \int_0^t \psi(z)dz.$$

We have for $t > 0$

$$1 \leq \varphi''(t) \leq M_1 \big(\varphi'(t)\big)^{1+\mu}, \qquad \psi(t)^{1-\mu} \leq M_2 \big(\widetilde{\psi}(t) + 1\big),$$
$$\widetilde{\psi}(t) \leq M_3 \big(\psi(t)\big)^{1+\mu+\mu^2}.$$

Step 3. For μ small enough the function φ verifies the “not easy to read condition” : $\delta > 1/2$ (depends on μ), $|\varphi(s) - \varphi(t)| < K < K_0$ implies

$$\left| \frac{A(x,s)}{\varphi'(s)} - \frac{A(x,t)}{\varphi'(t)} \right| \leq \frac{C_1 K}{\varphi'(s)^{1/2} \varphi'(t)^{1/2} \big(1 + |\varphi(s)| + |\varphi(t)|\big)^{\delta}}$$

Example 12

If $\mathbf{B} \in L^\infty(\Omega)^{N\times N}$ coercive and $b \in L^\infty(\Omega)$, $b \geq 0$ then

$$A(x,s) = \big(1 + b(x)\exp(s)\sin^2(\exp(s^2))\big)\mathbf{B}(x) \tag{19}$$

verifies (16)–(17). We can have highly oscillating and/or increasing coefficients of A. Here we have only $A(x,r)\xi\cdot\xi \geq |\xi|^2$.

Remark 13

The result is new also in the variational case. The Lipschitz condition is global but fairly general.

Readable uniqueness result

The previous condition is very general but not usual. A natural question is : “If $A(x,r)$ is local Lipschitz in r” is it possible to construct w such that the “differential inequality condition” holds?

Theorem 14 (Feo-G 2017)

Assume that $A(x,r)$ is local Lipschitz in r, that is

$$\forall K>0, \exists L_K>0 \quad |A(x,s)-A(x,r)| \leq L_K|s-r|, \quad \forall s,r \in [-K,K], \ a.e. \ in\, \Omega.$$

Then the renormalized solution is unique.

Proof.

It is sufficient to construct a function φ verifying the “differential inequality condition”. In fact it is sufficient to use Hermite interpolation and the family of functions $r \mapsto 1/(n-r)$ which blows up in n and verify some differential inequality. $\square$

Remark 15

- the uniqueness results are new even in the variational case
- possible generalization to nonlinear operators with p growth ($1 < p \leq 2$), structure condition, local Lipschitz conditions
- this techniques allow one to give some generalization to the results of Casado Diaz-Murat-Porretta for $p > 2$, non negative right-hand side and very local condition on the operator

Thank you for your attention