Step 1 : construction of the approximate problem

Since the matrix A is not supposed to be bounded, for $\varepsilon > 0$ let us define

$$A_\varepsilon(x, s) = A(x, T_{1/\varepsilon}(s))$$

and let $f_\varepsilon \in L^2(\Omega)$ such that

$$f_\varepsilon \to f \text{ strongly in } L^1(\Omega).$$

We now consider $u_\varepsilon \in H_0^1(\Omega)$ a weak solution of the approximated problem :
$\forall v \in H_0^1(\Omega)$

$$\lambda \int_\Omega u_\varepsilon v + \int_\Omega A_\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon \cdot \nabla v = \int_\Omega f_\varepsilon v \tag{12}$$

Step 2 : a priori estimates

$T_k(u_\varepsilon)$ is bounded in $H_0^1(\Omega)$ uniformly with respect to ε

$$\lim_{M\to+\infty} \limsup_{\varepsilon\to 0} \text{meas}\{|u_\varepsilon| \geq M\} = 0$$

Using $T_k(u_\varepsilon)$ as a test function

$$\underbrace{\lambda \int u_\varepsilon T_k(u_\varepsilon)}_{\geq 0} + \int_\Omega A_\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon \cdot \nabla T_k(u_\varepsilon) = \underbrace{\int_\Omega f_\varepsilon T_k(u_\varepsilon)}$$

Since $f_\varepsilon \to f$ in $L^1(\Omega)$ $\quad \|f_\varepsilon\|_{L^1(\Omega)} \leq M$; $\quad |\cdot| \leq k M$

The ellipticity condition on A_ε implies

$$\alpha \int_\Omega |\nabla T_k(u_\varepsilon)|^2 \leq k M \quad + \text{ Poincaré Inequality}$$

$\Longrightarrow$ $T_k(u_\varepsilon)$ bounded in $H_0^1(\Omega)$.

With Poincaré inequality $\quad k^2 \,\text{meas}\{|u_\varepsilon| \geq k\} \leq \int_\Omega (T_k(u_\varepsilon))^2 \leq k M'$

$\Longrightarrow$ $\text{meas}\{|u_\varepsilon| \geq k\} \leq M'/k$

Step 3 : extraction of subsequences

There exists a measurable function u, finite a.e. such that, up to a subsequence

$$u_\varepsilon \to u \text{ a.e. in } \Omega$$

$$\forall k > 0, \quad T_k(u_\varepsilon) \rightharpoonup T_k(u) \text{ weakly in } H_0^1(\Omega)$$

$k \in \mathbb{N}$, Rellich-Kondrasov Theorem, $\forall k \in \mathbb{N}$ let $u_k \in H_0^1(\Omega)$ /

$$\begin{cases} T_k(u_\varepsilon) \to u_k \text{ a.e, } L^2\text{-strongly} \\ T_k(u_\varepsilon) \rightharpoonup u_k \quad H_0^1(\Omega)\text{-weakly} \end{cases}$$

Diagonal process, up to a subsequence.

$\{u_\varepsilon\}_{\varepsilon>0}$ is a Cauchy sequence in measure.

Notation: $\{|u_\varepsilon| > k\} = \{x \in \Omega;\ |u_\varepsilon(x)| > k\}$

$$\text{meas}\{|u_\varepsilon - u_{\varepsilon'}| > \eta\} \leq \text{meas}\{|T_k(u_\varepsilon) - T_k(u_{\varepsilon'})| > \eta\} + \text{meas}\{|u_\varepsilon| > k\} + \text{meas}\{|u_{\varepsilon'}| > k\}$$

Let $\delta > 0$. Let $k \in \mathbb{N}$, large / $\forall \varepsilon > 0$ $\quad \text{meas}\{|u_\varepsilon| > k\} < \delta$.

Since $T_k(u_\varepsilon)$ converges a.e, let $\varepsilon_0 > 0$ / $\forall\, 0 < \varepsilon, \varepsilon' < \varepsilon_0$

$$\text{meas}\{|T_k(u_\varepsilon) - T_k(u_{\varepsilon'})| > \eta\} < \delta$$

$$\Rightarrow \forall\, 0 < \varepsilon, \varepsilon' < \varepsilon_0 \qquad \text{meas}\{|u_\varepsilon - u_{\varepsilon'}| > \eta\} < 3\delta.$$

Up a subsequence (still denoted by $\varepsilon \to 0$) $\quad u_\varepsilon \longrightarrow u$ a.e. in Ω.

There exists a measurable function u

? $T_k(u) \in H^1_0(\Omega)$? $\quad$? u finite a.e. in Ω ?

We know $\forall k \in \mathbb{N}$ $\quad T_k(u_\varepsilon) \longrightarrow u_k$ a.e. $\quad \Big| \; T_k(u_\varepsilon) \longrightarrow T_k(u)$ a.e.

$\qquad\qquad\qquad\qquad \rightharpoonup u_k$ $\quad H^1_0(\Omega)$-weak.

. By identification we deduce $\quad u_k = T_k(u) \qquad T_k(u) \in H^1_0(\Omega)$.

. $\displaystyle \lim_{M \to +\infty} \lim_{\varepsilon \to 0} \operatorname{meas}(\{|u_\varepsilon| > M\}) = 0 \quad \Rightarrow \quad u$ finite a.e. in Ω.

<u>Step 3</u>. The constructed function u is a candidate (to be a renormalized sol^o of (1))

Step 4 : uniform control of the decay of the truncated energy

$$\lim_{n\to+\infty} \limsup_{\varepsilon\to 0} \frac{1}{n}\int_{\{|u_\varepsilon|<n\}} A_\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\cdot\nabla u_\varepsilon = 0$$

A very important condition in Definition : $\lim_{n\to+\infty} \frac{1}{n}\int_{\{|u|<n\}} A(x,u)\nabla u \nabla u = 0$.

$T_n(u_\varepsilon)\in H_0^1(\Omega)$ as a test function in the approximate problem.

$$\lambda\int_\Omega u_\varepsilon T_n(u_\varepsilon) + \int_\Omega A_\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\cdot\nabla T_n(u_\varepsilon) = \int_\Omega f_\varepsilon T_n(u_\varepsilon)$$

$\nabla T_n(u_\varepsilon) = \mathbb{1}_{\{|u_\varepsilon|<n\}}\nabla u_\varepsilon$, $u_\varepsilon T_n(u_\varepsilon)\geq 0$

$$\int_{\{|u_\varepsilon|<n\}} A_\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\cdot\nabla u_\varepsilon \leq \int_\Omega |f_\varepsilon|\,|T_n(u_\varepsilon)|$$? pass to the limit as $\varepsilon\to 0$?

$f_\varepsilon\to f$ strongly in L^1
$T_n(u_\varepsilon)\to T_n(u)$ a.e. + bounded L^∞ weak $*$
$\Rightarrow \int_\Omega |f_\varepsilon|\,|T_n(u_\varepsilon)| \xrightarrow[\varepsilon\to 0]{} \int |f|\,|T_n(u)|$

$$\overline{\lim_{\varepsilon\to 0}}\ \frac{1}{n}\int_{\{|u_\varepsilon|<n\}} A_\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\cdot\nabla u_\varepsilon \leq \frac{1}{n}\int |f|\,|T_n(u)|$$

$$\overline{\lim_{\varepsilon\to 0}} \frac{1}{n}\int_{\{|u_\varepsilon|<n\}} A_\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\cdot\nabla u_\varepsilon \le \frac{1}{n}\int_\Omega |f|\,|T_n(u)|$$

? $n\to+\infty$?

$$0 \le |f| \times \frac{|T_n(u)|}{n} \le |f| \in L^1(\Omega)$$

. u finite a.e. $\quad \dfrac{|T_n(u)|}{n} \xrightarrow[n\to+\infty]{} 0$ a.e. in Ω $\quad \dfrac{|f|\,|T_n(u)|}{n} \xrightarrow[n\to+\infty]{} 0$ a.e. in Ω.

Then the Lebesgue dominated convergence theorem gives $\displaystyle\lim_{n\to+\infty}\frac{1}{n}\int |f|\,|T_n(u)| = 0$

<u>Conclusion</u> $\displaystyle\lim_{n\to+\infty}\overline{\lim_{\varepsilon\to 0}}\frac{1}{n}\int_{\{|u_\varepsilon|<n\}} A_\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\cdot\nabla u_\varepsilon = 0$ (the term is non negative).

As a consequence we have $u : \Omega \to \overline{\mathbb{R}}$ finite a.e.

$$\forall k > 0, \quad T_k(u) \in H^1_0(\Omega)$$

and the decay of the energy : . n large $A_\varepsilon(x, T_n(u_\varepsilon)) = A(x, T_n(u_\varepsilon))$
. A ellipticity . $u_\varepsilon \to u$ a.e . $\nabla T_n(u_\varepsilon) \to \nabla T_n(u)$ weakly-$(L^2)^N$
. Step 4!

$$\frac{1}{n}\int_{\{|u|<n\}} A(x,u)\nabla u \cdot \nabla u \longrightarrow 0 \text{ as } n \to +\infty.$$

It remains to prove (9) : $\forall h \in W^{1,\infty}(\mathbb{R})$ with h having compact support

$$\lambda u h(u) - \mathrm{div}(h(u)A(x,u)\nabla u) + h'(u)A(x,u)\nabla u \cdot \nabla u = f h(u)$$

in $\mathcal{D}'(\Omega)$. formally $\left(\lambda u - \mathrm{div}(A(x,u)\nabla u) = f\right) \times h(u)$ in $\mathcal{D}'(\Omega)$.

a test function $\in L^\infty(\Omega) \cap H^1_0(\Omega)$.

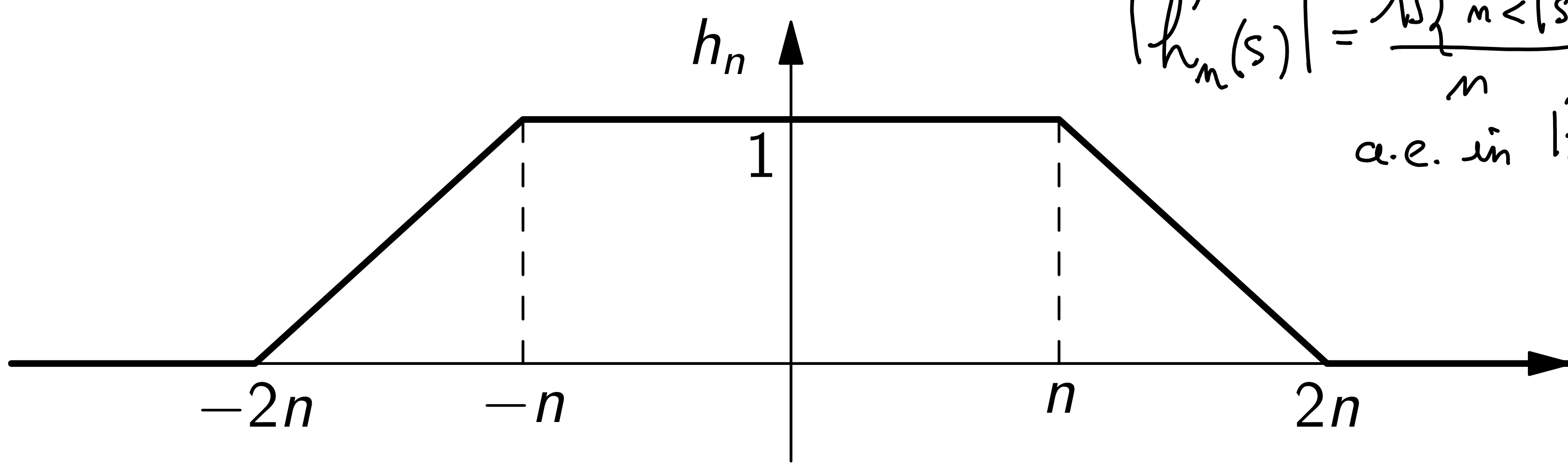
Remark. We have only weak limit of $\nabla T_n(u_\varepsilon)$.
We don't know the precise behavior of $A_\varepsilon(x, u_\varepsilon)\nabla u_\varepsilon \cdot \nabla T_n(u_\varepsilon)$
To take $h(u_\varepsilon)\varphi$, $\varphi \in \mathcal{C}^\infty_0(\Omega)$, as test function does allow to conclude.

Step 5 : passing to the limit

$h_n(u_\varepsilon)h(u)\varphi \in L^\infty(\Omega) \cap H_0^1(\Omega)$. $\nabla\big(h_n(u_\varepsilon)h(u)\varphi\big) = h_n'(u_\varepsilon)h(u)\varphi\nabla u_\varepsilon + h_n(u_\varepsilon)h'(u)\varphi\nabla u$
$+ h_n(u_\varepsilon)h(u)\nabla\varphi$

For any $n > 0$, we denote by h_n the function given by the graph

$|h_n'(s)| = \dfrac{\mathbb{1}_{\{n<|s|<2n\}}}{n}$ a.e. in $\mathbb{R}$.



Let $h \in W^{1,\infty}(\mathbb{R})$ with compact support and $\varphi \in C_0^\infty(\Omega)$. In view of the regularity of u_ε, u and φ, and since h has a compact support, $h_n(u_\varepsilon)h(u)\varphi \in L^\infty(\Omega) \cap H_0^1(\Omega)$ is then an admissible test function in the approximate problem :

$\forall \varepsilon > 0$
$\forall n \geq 1$.

$$\lambda\int u_\varepsilon h_n(u_\varepsilon)h(u)\varphi + \int A_\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon \nabla\big(h_n(u_\varepsilon)h(u)\varphi\big) = \int f h_n(u_\varepsilon)h(u)\varphi$$

- $\left|\int_\Omega h_n'(u_\varepsilon)h(u)\varphi A_\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\cdot\nabla u_\varepsilon\right| \leq \frac{\|h\|_\infty\|\varphi\|_\infty}{n}\int_{\{|u_\varepsilon|<2n\}} A_\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\cdot\nabla u_\varepsilon$

and Step 4 $\lim_{n\to+\infty}\overline{\lim_{\varepsilon\to 0}}\;(\;) = 0$.

- $h_n(u_\varepsilon)A_\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon = h_n(u_\varepsilon)A_\varepsilon(x,T_{2n}(u_\varepsilon))\nabla T_{2n}(u_\varepsilon) \longrightarrow h_n(u)A(x,T_{2n}(u))\nabla T_{2n}(u)$

$$\int_\Omega u_\varepsilon h_n(u_\varepsilon)h(u)\varphi + \int_\Omega h_n(u_\varepsilon)h(u)A_\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\cdot\nabla\varphi$$
$$+\int_\Omega h'(u)h_n(u_\varepsilon)\varphi A_\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\cdot\nabla u$$
$$+\int_\Omega h_n'(u_\varepsilon)h(u)\varphi A_\varepsilon(x,u_\varepsilon)\nabla u_\varepsilon\cdot\nabla u_\varepsilon = \int_\Omega f_\varepsilon h_n(u_\varepsilon)h(u)\varphi$$

We now study the behavior of each term, first as ε goes to zero and then as n goes to infinity.

- $u_\varepsilon h_n(u_\varepsilon) \longrightarrow u\,h_n(u)$ a.e., L^∞-weak $*$ (support$(h_n) = [-2n,2n]$)

$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon h_n(u_\varepsilon)h(u)\varphi = \int_\Omega u\,h_n(u)h(u)\varphi \underset{n \text{ large}}{=} \int_\Omega u\,h(u)\varphi$ supp $h \subset [-n,n]$

- $f_\varepsilon \longrightarrow f$ strongly L^1

$h_n(u_\varepsilon)h(u)\varphi \longrightarrow h_n(u)h(u)\varphi$ a.e., L^∞-weak $*$ $\Big\} \Rightarrow \lim_{\varepsilon\to 0}\int_\Omega f_\varepsilon h_n(u_\varepsilon)h(u)\varphi$

$= \int_\Omega f\,h_n(u)h(u)\varphi$

Similarly $h_n(u_\varepsilon) h(u) A_\varepsilon(x, u_\varepsilon) \nabla u_\varepsilon \xrightarrow[\varepsilon \to 0]{} h_n(u) h(u) A(x, u) \nabla u$ in $(L^2(\Omega))^N$

$\varepsilon \to 0$.

supp h is compact
supp h' is compact

$$\lambda \int u\, h_n(u) h(u) \varphi + \int h_n(u) h(u) A(x, u) \nabla u \cdot \nabla \varphi + \int h_n(u) h'(u) \varphi A(x, u) \nabla u \cdot \nabla \varphi + \left(\lim_{\varepsilon \to 0} \left(\quad \right) \right) = \int f h_n(u) h(u) \varphi$$

$\underline{|n \to +\infty|}$ with Step 4.

since n large $h_n(u) h(u) = h(u)$, $h_n(u) h'(u) = h'(u)$ a.e.

we have

$$\lambda \int u\, h(u) \varphi + \int h(u) A(x, u) \nabla u \cdot \nabla \varphi + \int h'(u) A(x, u) \nabla u \cdot \nabla u \, \varphi = \int f \varphi h(u)$$

<u>Conclusion</u> u is a renormalized solution.

<u>Rmq</u>. $\varphi \in L^\infty \cap H^1_0$

Renormalized solution and weak (variational) solution

If f is more regular, that is $f \in L^2(\Omega)$ then we recover that any renormalized solution belongs to $H_0^1(\Omega)$. More precisely

Proposition 3

Assume that $f \in L^2(\Omega)$ and $A(x,s) \in (L^\infty(\Omega \times \mathbb{R}))^{N^2}$. Then u is a renormalized solution of (1) iff $u \in H_0^1(\Omega)$ is a weak solution of (1).

Proposition 3 is a consequence of the following

Proposition 4

h(u) φ as test function supp h compact

For any $k > 0$ we have

$$\lambda \int_\Omega u T_k(u) + \int_\Omega A(x,u)\nabla u \cdot \nabla T_k(u) = \int_\Omega f T_k(u)$$

Formally $T_k(u)$ is an admissible test function for equation (1) and $u \in L^1(\Omega)$.

if $\underline{\lambda > 0}$

Proof of Proposition 4.

$\in H_0^1(\Omega)\cap L^\infty(\Omega)$

We use in the renormalized formulation h_n in place of h and $T_k(u)$ as test function in (8) :

$$\lambda\int_\Omega u h_n(u)T_k(u) + \int_\Omega h_n(u)A(x,u)\nabla u\cdot\nabla T_k(u)$$

$\forall n \geq 1$

$$+\int_\Omega h_n'(u)T_k(u)A(x,u)\nabla u\cdot\nabla u = \int_\Omega f h_n(u)T_k(u)$$

We now derive a priori estimates independent of n. □

- $\lim_{n\to+\infty}\left|\int h_n'(u)T_k(u)A(x,u)\nabla u\cdot\nabla u\right| \leq \overline{\lim_{n\to+\infty}}\frac{k}{n}\int_{\{|u|<2n\}} A(x,u)\nabla u\cdot\nabla u = 0$
- u finite a.e. in Ω $\quad h_n(u)\longrightarrow 1$ a.e., L^∞ weak$\star$ $\quad \lim\int f h_n(u)T_k(u) = \int f T_k(u)$
- n large $(n>k)$ $\quad \int_\Omega h_n(u)A(x,u)\nabla u\cdot\nabla T_k(u) = \int_\Omega A(x,u)\nabla u\cdot\nabla T_k(u)$
- Fatou-Lemma + a.e. $\quad \lambda u\in L^1$ . . .

Conclude. $\quad \lambda\int u T_k(u) + \int A(x,u)\nabla u\cdot\nabla T_k(u) = \int f T_k(u)$