Computation of Angular Momentum of Proton using Lattice QCD

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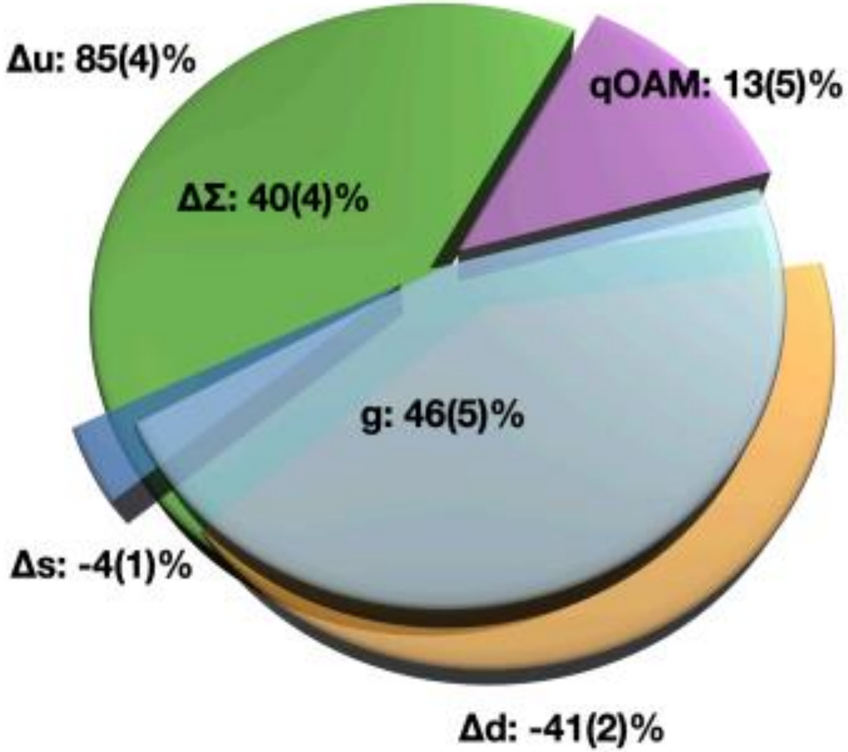
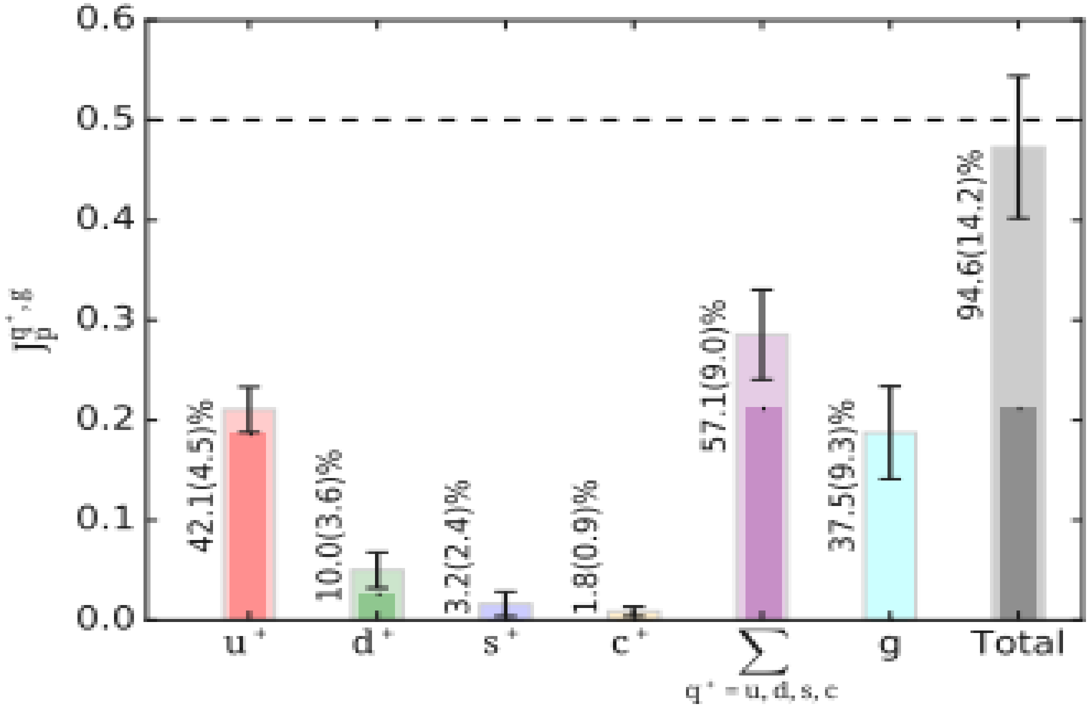
International School and Workshop on Probing Hadron Structure at the Electron-Ion Collider

ICTS, TIFR, Bengaluru



χQCD (2022)

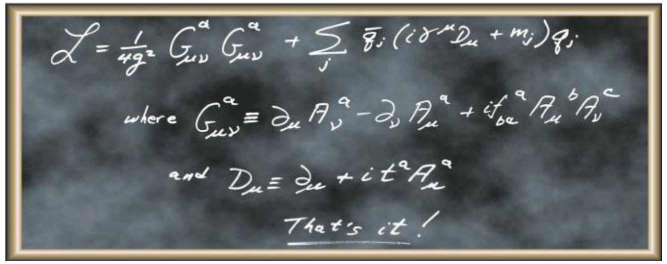
Phys. Rev. D 106, 014512 (2022)



ETMC (2020)

Phys. Rev. D 101, 094513 (2020)

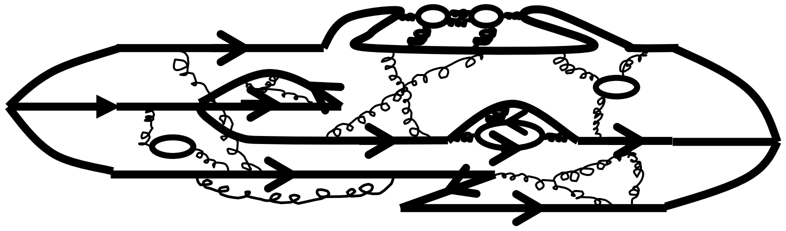
QCD



$$S_{QCD} = \int d^4x\, L_{QCD}(m_{q,}g_s)$$

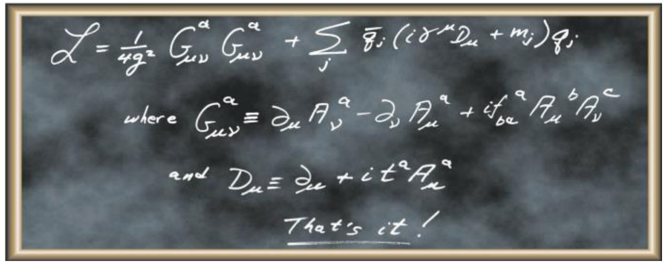
$$\langle C \rangle = \frac{\int DGDqD\bar{q}Ce^{-S_{QCD}}}{\int DGDqD\bar{q}\, e^{-S_{QCD}}}$$

$$C_{\mathcal{O}}(t_i, t_f) = \sum_{\vec{x}} e^{-i\vec{p}.\vec{x}} \langle 0|\mathcal{O}(\vec{x}_f, t_f)\bar{\mathcal{O}}(\vec{x}_i, t_i)|0\rangle$$

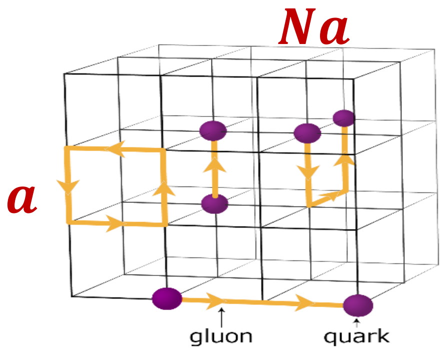


large time $\sim e^{-E_0 t}$

QCD → LQCD



Euclidean time



$$S_{QCD} = \int d^4x \, L_{QCD}(m_q, g_s)$$

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$$\langle C \rangle = \frac{\int DUDqD\bar{q}Ce^{-S^E_{QCD}}}{\int DUDqD\bar{q}\, e^{-S^E_{QCD}}} \approx \frac{1}{N} \sum_n C(D^{-1}(U_n))$$

$$\Delta C = \frac{1}{\sqrt{N}} + \text{systematics}$$

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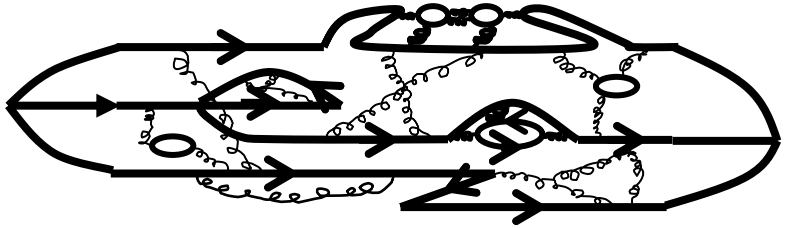

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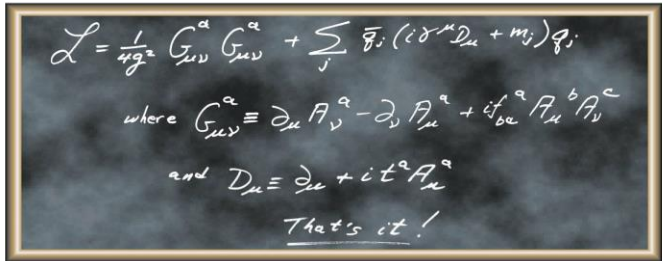
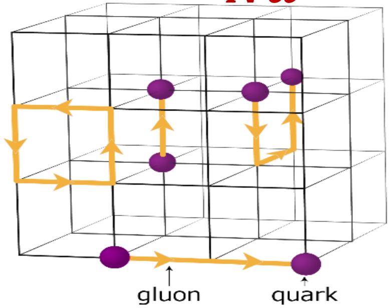


Euclidean time

Na

a



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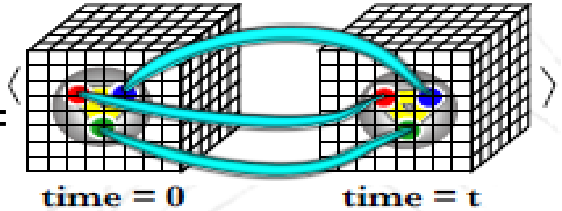
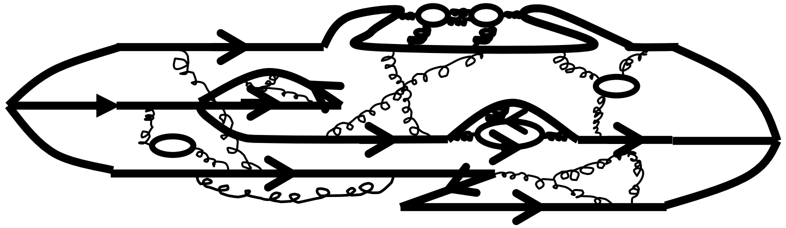
$$\left\langle C^{2pt}_{ab}(t, \vec{P}) \right\rangle =$$

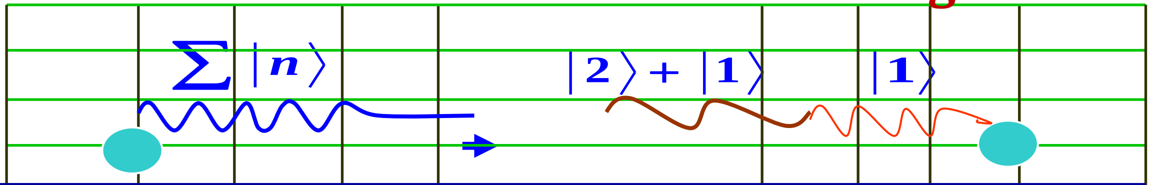


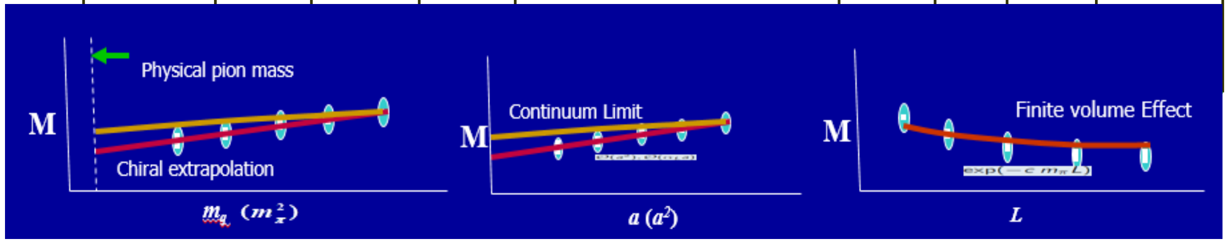
$$= \sum_n Z_{b,n} Z^\dagger_{a,n} e^{-E_n t}$$

τ large $\sim e^{-E_0 \tau}$



large time $\sim e^{-E_0 t}$





Lattice QCD Workflow

Using Monte Carlo methods
generate a set of gauge ensembles
representing QCD dynamics,
on Euclidean space-time lattices
with varying quark masses,
varying lattice sizes and
varying lattice spacings

Compute Euclidean correlation functions
corresponding to desired observables
on those lattice ensembles

Analyse correlation functions, and
finally take chiral, continuum and
finite volume limits (also
renormalization of lattice to
physical results)

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Angular momentum of proton:
Need appropriate operators and then
to compute their correlation functions

Compute Euclidean correlation functions
corresponding to desired observables
on those lattice ensembles

Analyse correlation functions, and
finally take chiral, continuum and
finite volume limits (also
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Operators for Angular momentum and spin sum rules

- Energy momentum tensor (Belinfante):

$$T^{\mu\nu} = \overline{T}^{\mu\nu} + \widehat{T}^{\mu\nu}$$

$$\overline{T}^{\mu\nu} = \overline{T}^{\mu\nu}_q + \overline{T}^{\mu\nu}_g$$

$$\overline{T}^{\mu\nu}_q = \overline{q} i\gamma \overleftrightarrow{D}^{\{\mu\nu\}} q \qquad \overline{T}^{\mu\nu}_g = F^{\{\mu\rho} F_\rho^{\nu\}}$$

$$\overleftrightarrow{D} = \frac{1}{2}[\overrightarrow{D} + \overleftarrow{D}] \qquad \{\,\} \Rightarrow \text{symetrization}$$

- Angular momentum density:

$$M^{\alpha\mu\nu} = \overline{T}^{\alpha\nu} x^\mu - \overline{T}^{\alpha\mu} x^\nu$$

- Angular momentum:

$$J_i = \frac{1}{2} \epsilon^{ijk} \int d^3x\, M^{0jk}(x)$$

$$\vec{J}^g = \int d^3x\, [\vec{x} \times (\vec{E} \times \vec{B})]$$

- Angular momentum density:

$$M^{\alpha\mu\nu} = \overline{T}^{\alpha\nu} x^{\mu} - \overline{T}^{\alpha\mu} x^{\nu}$$

- Angular momentum of quarks:

$$J_i = \frac{1}{2}\epsilon^{ijk} \int d^3x\, M^{0jk}(x)$$

$$\vec{J}^q(\mu) = \int d^3x \left[\bar{q}\frac{\vec{\gamma}\gamma_5}{2}q + \bar{q}(\vec{x} \times i\vec{D})q\right]$$

$$= \frac{1}{2}\Sigma_q(\mu) + \vec{L}_q(\mu)$$

Proton spin decomposition

Frame Independent (Ji)

Phys. Rev. Lett., 78:610–613, 1997

$$J_P = J_q + J_g$$

$$= \sum_{q=u,d,s,c} \left(\frac{1}{2}\Sigma_q + L_q \right) + J_g$$

Quark spin

Quark orbital angular momentum

Total gluon angular momentum

Each term is gauge invariant.

Expt: JLab, COMPASS, HERMES, J-PARC, EIC

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- $\frac{1}{2}\Sigma_q$: Quark spin
- L_q: Quark orbital angular momentum
- J_g: Total gluon angular momentum

Each term is gauge invariant.

Expt: JLab, COMPASS, HERMES, J-PARC, EIC

Infinite momentum frame

(Jaffe-Manohar) Nucl. Phys., B337:509–546, 1990

$$J_P = \sum_{q=u,d,s,c} \frac{1}{2}\Sigma_q + \Delta G + \mathcal{L}_q + \mathcal{L}_g$$

- $\frac{1}{2}\Sigma_q$: Quark spin
- ΔG: Gluon helicity $\sim \epsilon^{ij} F^{+i} A^j$
- $\mathcal{L}_q$: Quark orbital angular momentum $\bar{q}(x \times i\partial)\psi$
- $\mathcal{L}_g$: Gluon orbital angular momentum $F(x \times \partial)A$

Not gauge invariant. Use light-cone gauge. Also from GPDs, GTMD

Expt: PHENIX, STAR, COMPASS, HERMES, EIC

These decompositions are not unique. There are many ways, and each can have their legitimate meanings

$$\left\langle N(p',s') \middle| \mathcal{T}_{q,g}^{\{\mu\nu\}} \middle| N(p,s) \right\rangle = \frac{1}{2}\bar{u}_N(p',s')\Big[\boldsymbol{T_1^{q,g}}(q^2)(\gamma^\mu \bar{p}^\nu + \gamma^\nu \bar{p}^\mu)$$

$$+ \frac{1}{2m_N}\boldsymbol{T_2^{q,g}}(q^2)\{iq_\alpha(\bar{p}^\mu \sigma^{\nu\alpha} + \bar{p}^\nu \sigma^{\mu\alpha})\}$$

$$+ \boldsymbol{D_{q,g}}(q^2)\frac{q^\mu q^\nu - g_{\mu\nu}q^2}{m_N} + \boldsymbol{\bar{C}_{q,g}}(q^2) m_N g_{\mu\nu}\Big] u_N(p,s)$$

$q = p' - p$: momentum transfer $\quad \bar{p} = (p + p')/2$

$\boldsymbol{T_1, T_2, D, \bar{C}}$: Gravitational form factors

$$T_1^q(0) = \int_0^1 dx\, x(q(x) + \bar{q}(x)) \qquad T_1^g(0) = \int_0^1 dx\, xg(x)$$

Momentum fraction

$$\left\langle N(p',s') \middle| \mathcal{T}_{q,g}^{\{\mu\nu\}} \middle| N(p,s) \right\rangle = \frac{1}{2} \bar{u}_N(p',s') \Big[\boldsymbol{T_1^{q,g}}(q^2)(\gamma^\mu \bar{p}^\nu + \gamma^\nu \bar{p}^\mu) + \frac{1}{2m_N} \boldsymbol{T_2^{q,g}}(q^2)\{ i q_\alpha (\bar{p}^\mu \sigma^{\nu\alpha} + \bar{p}^\nu \sigma^{\mu\alpha})\} + \boldsymbol{D_{q,g}}(q^2) \frac{q^\mu q^\nu - g_{\mu\nu} q^2}{m_N} + \boldsymbol{\bar{C}_{q,g}}(q^2) m_N g_{\mu\nu} \Big] u_N(p,s)$$

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Anomalous gravitomagntic moment

$$+ \boldsymbol{D_{q,g}}(q^2)\frac{q^\mu q^\nu - g_{\mu\nu}q^2}{m_N} + \bar{C}_{q,g}(q^2) m_N g_{\mu\nu}\Big]\, u_N(p,s)$$

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Pressure

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Anomalous gravitomagntic moment

Pressure

Trace anomaly

$q = p' - p$: momentum transfer $\qquad \bar{p} = (p + p')/2$

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Pressure **Trace anomaly**

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At $q^2 \to 0$ $\quad \boldsymbol{J^{q,g} = \frac{1}{2}[T_1(0) + T_2(0)]^{q,g}} \quad \boldsymbol{\langle x\rangle^{q,g} = T_1(0)^{q,g}}$

Momentum fraction (second moment of the PDF)

Sum rule: $\boldsymbol{\langle x\rangle^q + \langle x\rangle^g = T_1(0)^q + T_1(0)^g = 1}$

$$\boldsymbol{J^q + J^g = \frac{1}{2} = \frac{1}{2}[T_1(0) + T_2(0)]^q + \frac{1}{2}[T_1(0) + T_2(0)]^g}$$

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$\rightarrow$ $\boldsymbol{T_2(0)^q + T_2(0)^g = 0}$

- Angular momentum :

$$\vec{J}^q(\mu) = \int d^3x \left[\bar{q}\frac{\vec{\gamma}\gamma_5}{2}q + \bar{q}(\vec{x} \times i\vec{D})q\right]$$

$$= \frac{1}{2}\,\Sigma_q(\mu) + \vec{L}_q(\mu)$$

$$\vec{J}^g = \int d^3x \left[\vec{x} \times (\vec{E} \times \vec{B})\right]$$

- **Operators:**

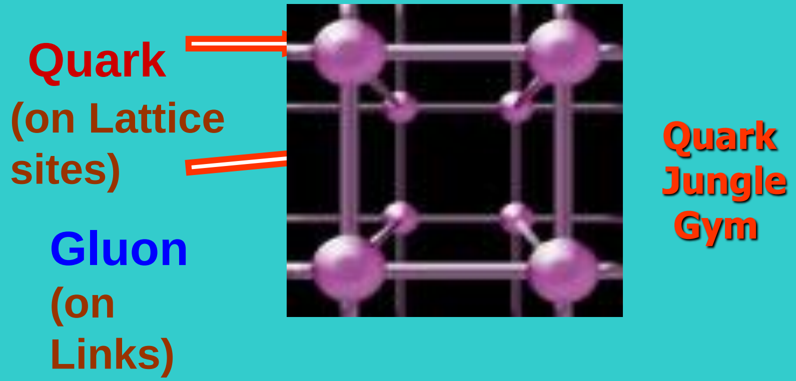
$$\langle N(p',s)|\mathcal{O}_A^{\mu}|N(p,s)\rangle \;; \qquad \mathcal{O}_A^{\mu} = \bar{q}\gamma^{\mu}\gamma_5 q$$

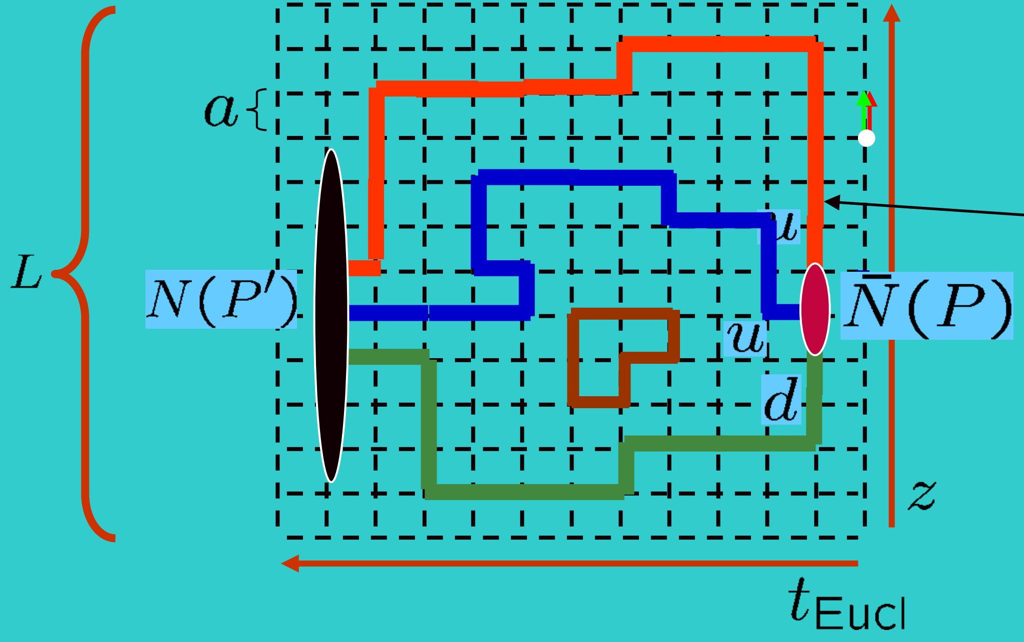
$$\langle N(p',s)|\mathcal{O}_{J_q}^{\mu\nu}|N(p,s)\rangle \;; \qquad \mathcal{O}_{J_q}^{\mu\nu} = \bar{q}\,\gamma^{\{\mu}\overleftrightarrow{D}^{\nu\}}q$$

$$\langle N(p',s)|\mathcal{O}_{J_g}^{\mu\nu}|N(p,s)\rangle \;; \qquad \mathcal{O}_{J_g}^{\mu\nu} = 2\,\mathrm{Tr}\,[\,G_{\mu\sigma}G_{\nu\sigma}]$$

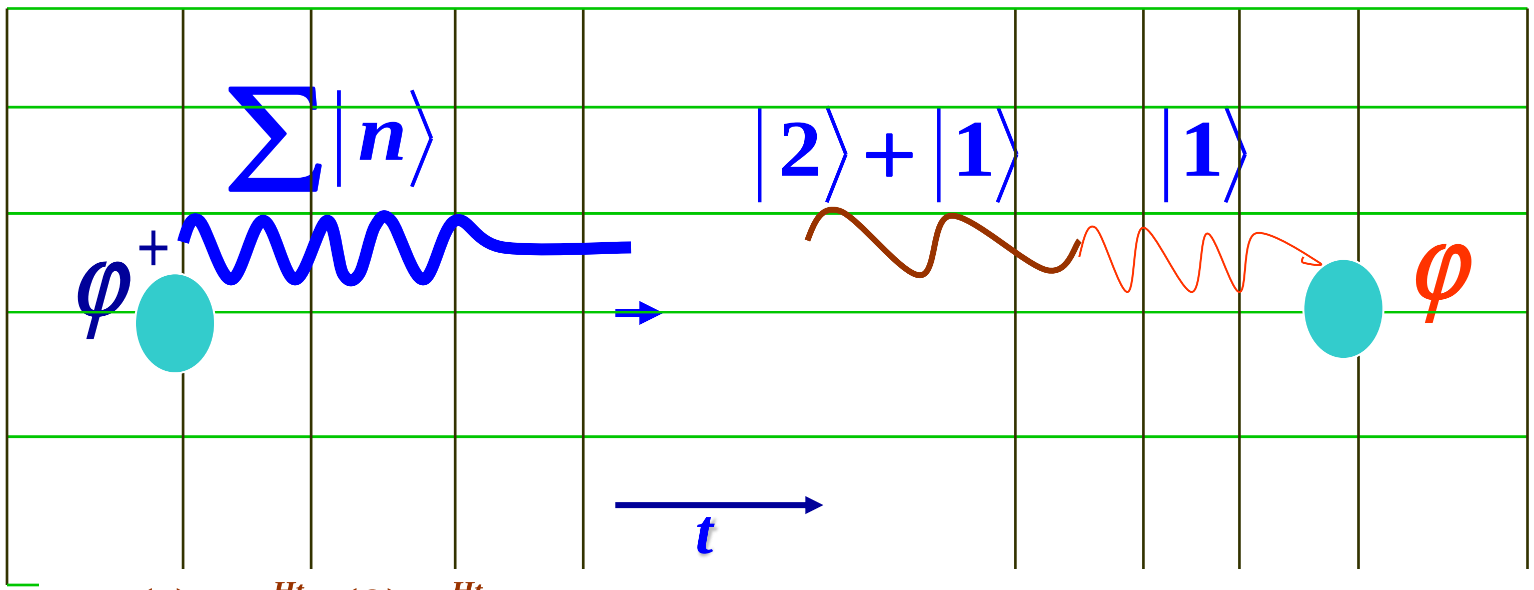
Physical observables

$\langle x \rangle_{q,g}$	Momentum fraction	$\gamma_\mu \overleftrightarrow{D}_\nu$
$\langle x^2 \rangle_{q,g}$	Second moment	$\gamma_\mu \overleftrightarrow{D}_\nu \overleftrightarrow{D}_\delta$
$\Delta u - \Delta d = g_A$	Axial charge	$\gamma_5 \gamma_\mu$
$\Delta u + \Delta d = \Delta\Sigma_{u,d}$	Spin content	$\gamma_5 \gamma_\mu$
$\delta u - \delta d = g_T$	Tensor charge	$\gamma_5 \sigma_{\mu\nu}$





quark propagators :
Inverse of very large matrix of space-time, spin and color



$$\varphi(t) = e^{Ht}\varphi(0)e^{-Ht}$$

$$G(t,\vec{p}) = \sum_{\vec{x}} e^{-i\vec{p}.(\vec{x}-\vec{x}_0)} \sum_{n,\vec{q}} \langle 0|\varphi(x)|n,\vec{q}\rangle\langle n,\vec{q}|\varphi(x_0)|0\rangle$$

$$= \sum_{\vec{x}} e^{-i\vec{p}.(\vec{x}-\vec{x}_0)} \sum_{n,\vec{q}} \langle 0|e^{H(t-t_0)-i\vec{p}'.(\vec{x}-\vec{x}_0)}\varphi(x_0)e^{-H(t-t_0)+i\vec{p}'.(\vec{x}-\vec{x}_0)}|n,\vec{q}\rangle\langle n,\vec{q}|\varphi(x_0)|0\rangle$$

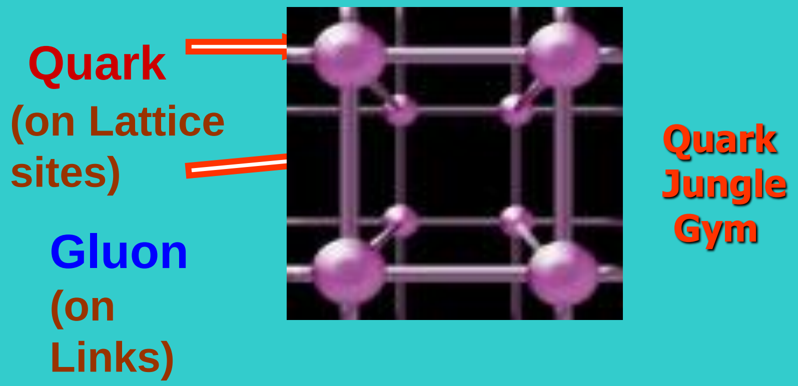
$$= \sum_{\vec{x}} e^{-i\vec{p}.(\vec{x}-\vec{x}_0)} \sum_{n,\vec{q}} e^{i\vec{q}.(\vec{x}-\vec{x}_0)-E_q^n(t-t_0)} \langle 0|\varphi(x_0)|n,\vec{q}\rangle\langle n,\vec{q}|\varphi(x_0)|0\rangle$$

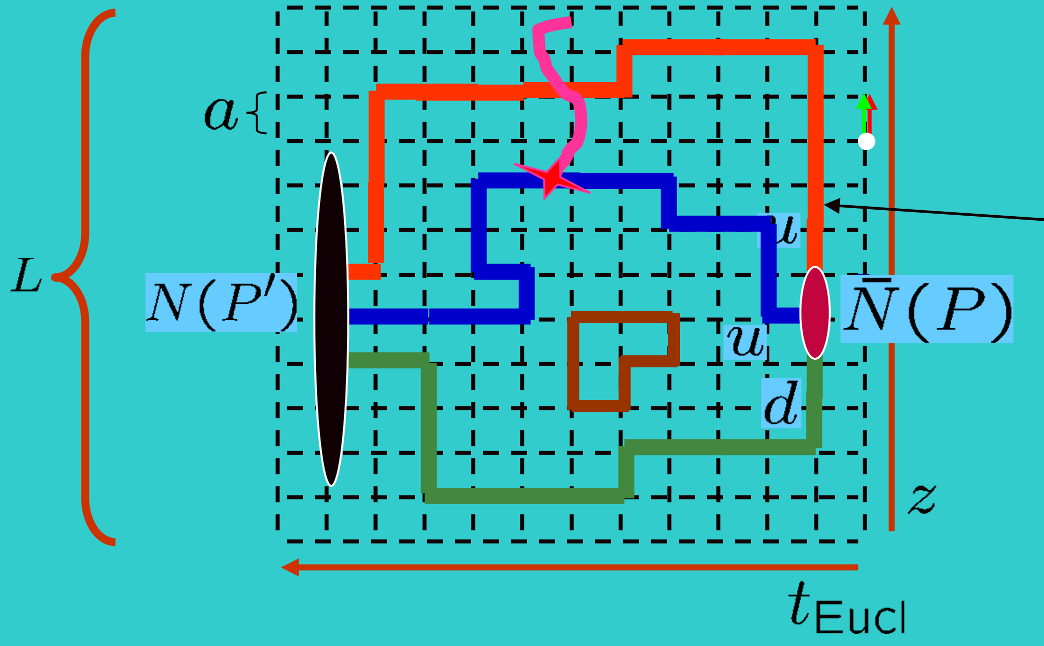
$$\approx \sum_{n,\vec{q}} \delta(\vec{p}-\vec{q})e^{i(\vec{p}-\vec{q}).\vec{x}_0-E_q^n(t-t_0)} \langle 0|\varphi(x_0)|n,\vec{q}\rangle\langle n,\vec{q}|\varphi(x_0)|0\rangle$$

$$= \sum_{n} e^{-E_p^n(t-t_0)} \left|\langle 0|\varphi(x_0)|n,\vec{p}\rangle\right|^2$$

Determines how effectively this operator interpolates states '*n*' from the vacuum

$$= \sum_{n} W_n e^{-E_p^n(t-t_0)} \xrightarrow[t\to\infty]{} W_1 e^{-E_1^n(t-t_0)}$$





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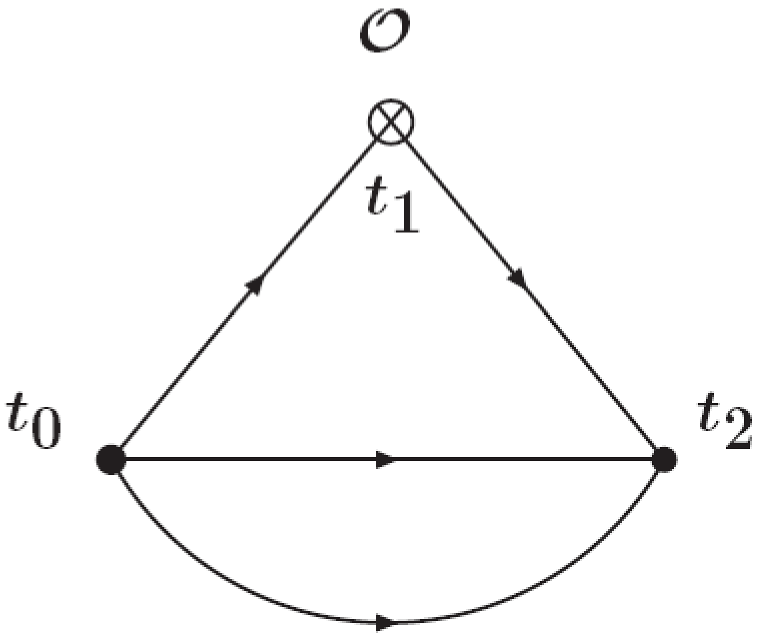
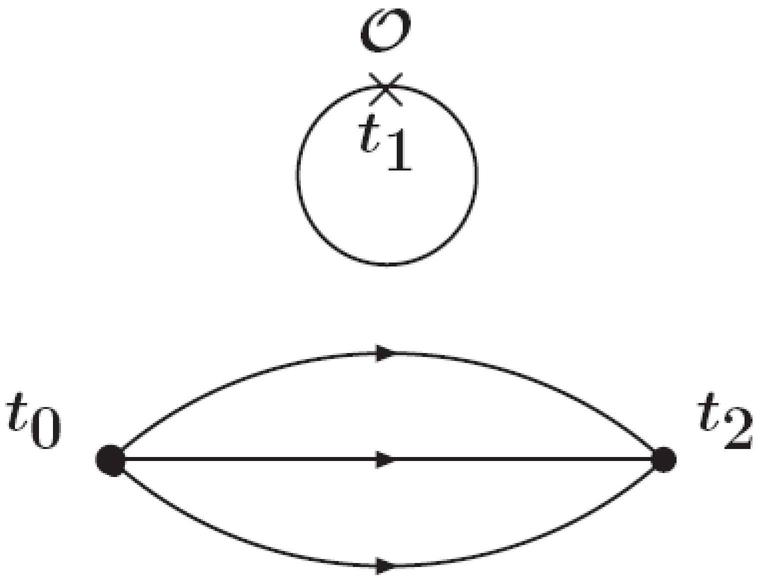
Inverse of very large matrix of space-time, spin and color

$$G^{\alpha\beta}_{N\mathcal{O}N}(t_2, t_1, \vec{p}) \;=\; \sum_{\vec{x}_2, \vec{x}_1} e^{-i\vec{p}.(\vec{x}_2 - \vec{x}_0)} \,\langle\, 0 \,|\, \mathrm{T}\,(\chi^{\alpha}(x_2)\, \mathcal{O}(x_1)\, \bar{\chi}^{\beta}(x_0))\,|\, 0\,\rangle$$



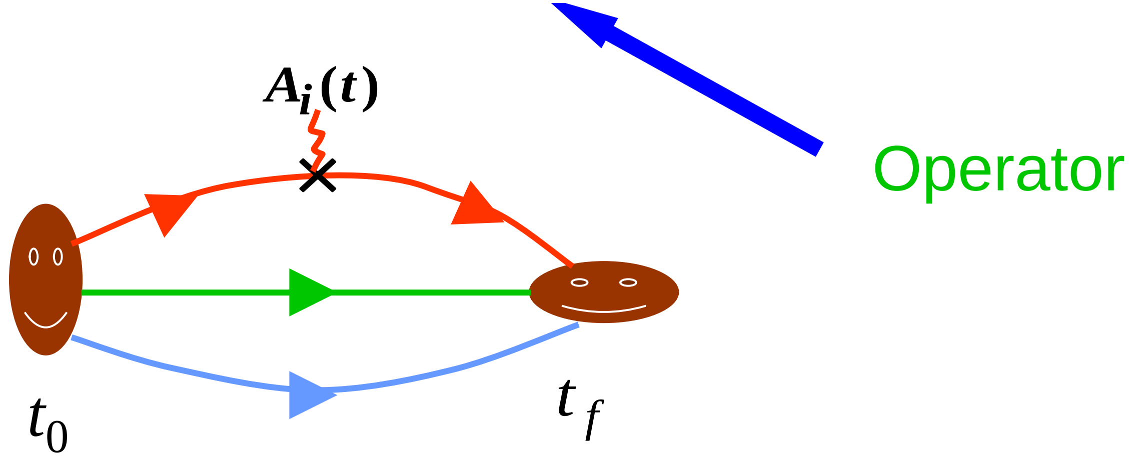
Connected insertion



Disconnected insertion

Three Point Correlation Function

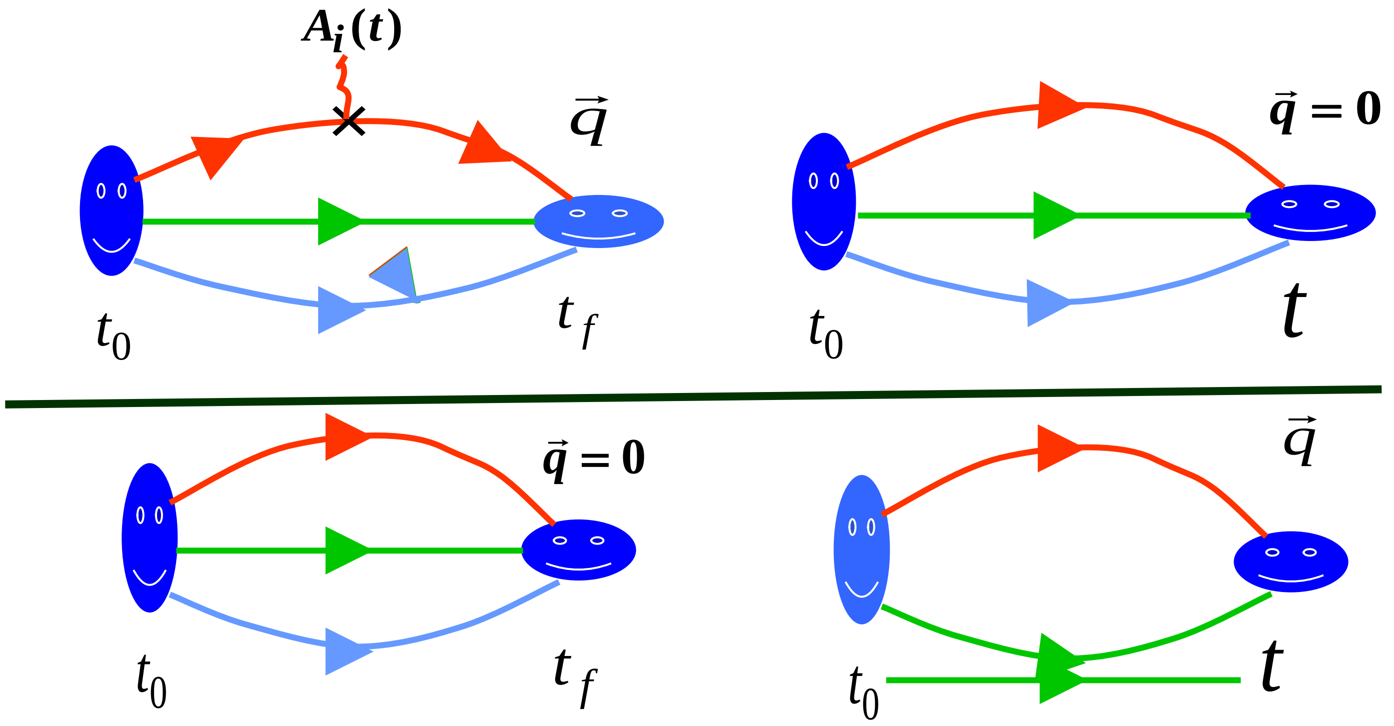
$$G^{\alpha\beta}_{PT_{\mu\nu}P}(t_2,t_1,\vec{p},\vec{p'}) = \sum_{\vec{x_2},\vec{x_1}} e^{-i\vec{p}\cdot\vec{x_2}} e^{-i\vec{q}\cdot\vec{x_1}} < 0|\, T\left(\chi^\alpha(x_2)\, O_{\mu\nu}(x_1)\, \bar{\chi}^\beta(0)\right) |0 >$$



$$\Gamma_{\alpha\beta}\; G^{\beta\alpha}_{NA_iN}\,(t_f,t,t_0,\bar{p},\vec{q}) \equiv \Gamma_{\alpha\beta}\sum_{\vec{x},\vec{x}_f} e^{i\vec{q}\cdot\vec{x}} \left\langle T(\chi^\alpha(x_f)\, A_i\, \bar{\chi}^\beta(x_0)\right\rangle$$

$$\xrightarrow{t_f-t,\,t-t_0>>1,} \frac{E_q+m}{E_q}|\phi|^2\, e^{-m(t_f-t)-E_q(t-t_0)}\left[g_A(q^2)-h_A(q^2)\frac{q_i^2}{E_q+m}\right] \;;\; \Gamma = \begin{pmatrix}\sigma_i & 0\\ 0 & 0\end{pmatrix}$$

Form Factors



The combined ratios leads to the form factors

$$g_A(q^2) - h_A(q^2)\frac{q_i^2}{E_q + m} \xrightarrow{q_{i=0}} g_A(q^2)$$

Quark spin contribution $\Delta\Sigma$

- Flavor-singlet axial vector current

$$A_\mu^0 = \sum_{f=u,d,s,c} \bar{q}_f i\gamma_\mu\gamma_5 q_f$$

- On the lattice we need to compute the matrix element of the flavor-singlet axial vector current

$$\langle N(p,s)|A_\mu^0|N(p,s)\rangle = s_\mu g_A^0$$

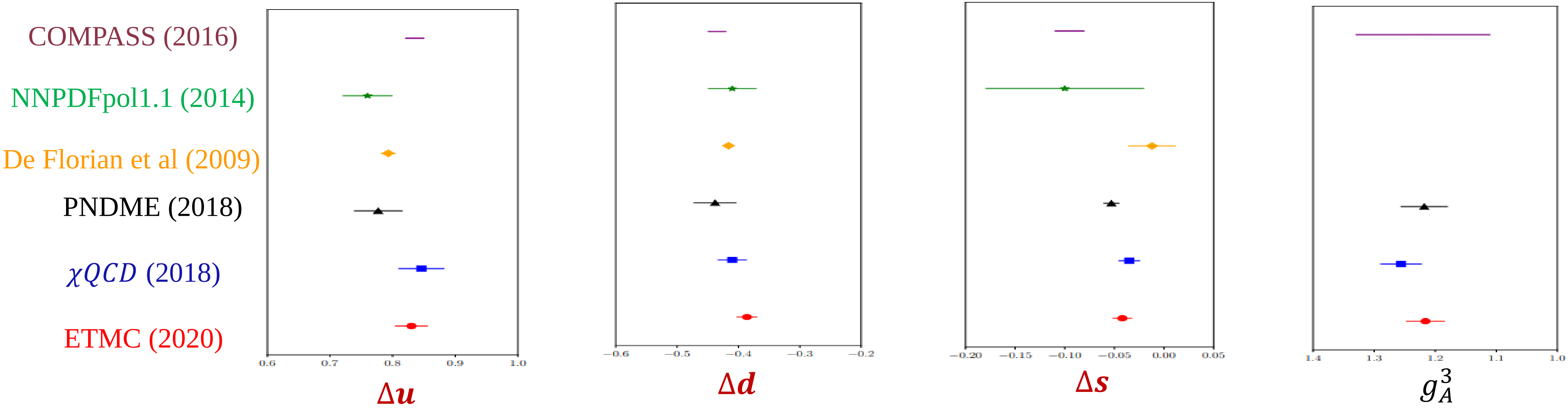
s_μ Polarization vector

$$g_A^0 = \Delta\Sigma = \Delta u + \Delta d + \Delta s + \Delta c$$

Quark spin contribution of the *u*,*d*,*s* and *c* quarks

$$= \Delta(u+d)_{\mathrm{CI}} + \Delta(u+d)_{\mathrm{DI}} + \Delta s_{\mathrm{D}I} + \Delta c_{\mathrm{D}I}$$

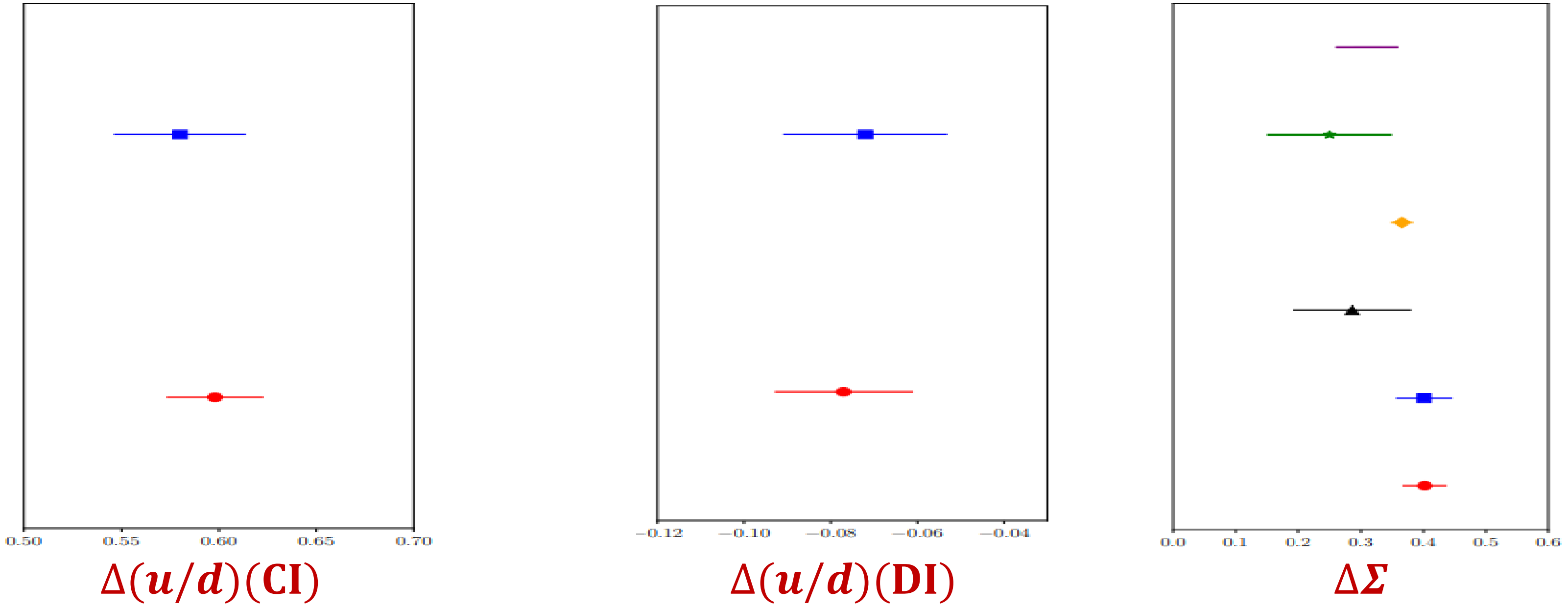
Quark spin contribution



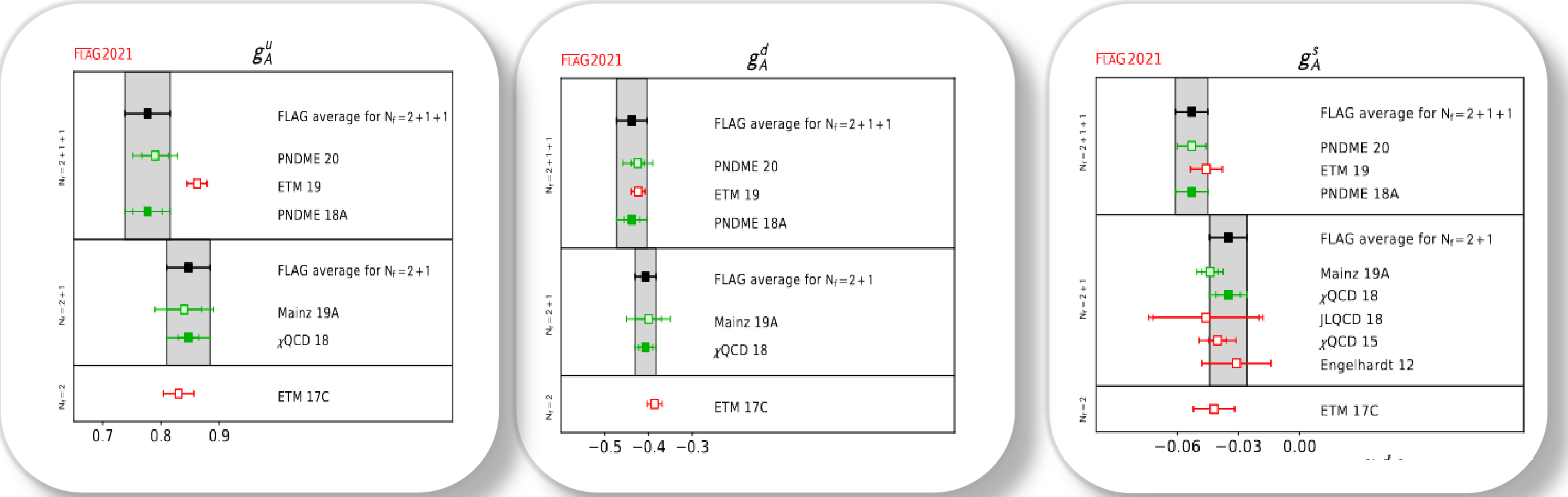
De Florian et al (2009):
Phys. Rev., D80:034030, 2009

NNPDFpol1.1 (2014):
Nucl. Phys. B, 887:276–308, 2014

COMPASS (2016)
Phys. Lett. B, 753:18–28, 2016



Summary from: K. F. Liu :arXiv 2112.08416



FLAG2021
g_A^u
FLAG average for $N_f=2+1+1$
PNDME 20
ETM 19
PNDME 18A
FLAG average for $N_f=2+1$
Mainz 19A
χQCD 18
ETM 17C
$N_f=2+1+1$
$N_f=2+1$
$N_f=2$
0.7
0.8
0.9
FLAG2021
g_A^d
FLAG average for $N_f=2+1+1$
PNDME 20
ETM 19
PNDME 18A
FLAG average for $N_f=2+1$
Mainz 19A
χQCD 18
ETM 17C
−0.5
−0.4
−0.3
FLAG2021
g_A^s
FLAG average for $N_f=2+1+1$
PNDME 20
ETM 19
PNDME 18A
FLAG average for $N_f=2+1$
Mainz 19A
χQCD 18
JLQCD 18
χQCD 15
Engelhardt 12
ETM 17C
−0.06
−0.03
0.00

Axial ward identity

$$\partial_\mu A^0_\mu = \sum_{f=u,d,s} 2m_f P_f - 2iN_f q$$

***χQCD* (2017-24)**

$$P_f = \bar{\psi}_f i\gamma_5 \psi_f$$

Pseudoscalar density

$$q = \frac{1}{16\pi^2} G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}$$

Topological charge

From the review by K. F. Liu :
arXiv 2112.08416

Nucleon matrix element of this is satisfied by CI and DI separately

Topological charge contribution is part of the quark spin.

$$\left\langle N(p',s') \middle| \mathcal{T}_{q,g}^{\{\mu\nu\}} \middle| N(p,s) \right\rangle = \frac{1}{2} \bar{u}_N(p',s') \Big[T_1^{q,g}(q^2)(\gamma^\mu \bar{p}^\nu + \gamma^\nu \bar{p}^\mu) + \frac{1}{2m_N} T_2^{q,g}(q^2)\{ i q_\alpha (\bar{p}^\mu \sigma^{\nu\alpha} + \bar{p}^\nu \sigma^{\mu\alpha})\} + D_{q,g}(q^2) \frac{q^\mu q^\nu - g_{\mu\nu} q^2}{m_N} + \bar{C}_{q,g}(q^2) m_N g_{\mu\nu} \Big] u_N(p,s)$$

Anomalous gravitomagntic moment

Pressure

Trace anomaly

$q = p' - p$: momentum transfer $\quad \bar{p} = (p + p')/2$

$\boldsymbol{T_1, T_2, D, \bar{C}}$: Gravitational form factors

$$\boldsymbol{J^{q,g} = \frac{1}{2}[T_1(0) + T_2(0)]^{q,g}}$$

PHYSICAL REVIEW D, VOLUME 62, 114504

Quark orbital angular momentum from lattice QCD

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[3]*SLAC, P.O. Box 4349, Stanford, California 94309*
[4]*Copernicus Astronomical Center, ul. Bartycka 18, PL-00-716 Warsaw, Poland*
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(Received 10 December 1999; published 25 October 2000)

On an Euclidean space-time lattice,

$$\mathcal{T}_{4i}^{q(E)} = (-1)\frac{i}{4}\sum_f \bar{\psi}_f[\gamma_4\vec{D}_i + \gamma_i\vec{D}_4 - \gamma_4\overleftarrow{D}_i - \gamma_i\overleftarrow{D}_4]\psi_f$$

$$\vec{D}_\mu\psi(x) = \frac{1}{2a}[U_\mu(x)\psi(x+a_\mu) - U_\mu^\dagger(x-a_\mu)\psi(x-a_\mu)],$$

$$\bar{\psi}(x)\overleftarrow{D}_\mu = \frac{1}{2a}[\bar{\psi}(x+a_\mu)U_\mu^\dagger(x) - \bar{\psi}(x-a_\mu)U_\mu^\dagger(x-a_\mu)]$$

$$\mathcal{T}_{4i}^{g(E)} = (+i)\left[-\frac{1}{2}\sum_{k=1}^{3} 2\mathrm{Tr}^{\mathrm{color}}[G_{4k}G_{ki} + G_{ik}G_{k4}]\right]$$

$$G_{\mu\nu}^{(E)}(x) = \frac{1}{8}(P_{\mu\nu}(x) - P_{\mu\nu}^\dagger(x))$$

$$\begin{aligned} P_{\mu\nu} = {} & U_\mu(x)U_\nu(x+\mu)U_\mu^\dagger(x+\nu)U_\nu^\dagger(x) \\ & + U_\nu(x)U_\mu^\dagger(x-\mu+\nu)U_\nu^\dagger(x-\mu)U_\mu(x-\mu) \\ & + U_\mu^\dagger(x-\mu)U_\nu^\dagger(x-\mu-\nu)U_\mu(x-\mu-\nu)U_\nu(x-\nu) \\ & + U_\nu^\dagger(x-\nu)U_\mu(x-\nu)U_\nu(x-\nu+\mu)U_\mu^\dagger(x) \end{aligned}$$

Correlation functions for OAM

- Time ordered two-point correlation function of nucleon:

$$G_{\alpha\beta}^{NN}(\vec{p},t) = \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} \langle 0|T[\chi_\alpha(\vec{x},t)\bar{\chi}_\beta(\vec{0},0)]|0\rangle$$

Nucleon interpolating field:

$$\chi_\alpha(x) = \epsilon_{abc} u(x)_\alpha^a [u(x)^b \tilde{\mathcal{C}} d(x)^c]$$

$$\tilde{\mathcal{C}} = \mathcal{C}\gamma_5 \quad \mathcal{C} \equiv \gamma_2\gamma_4$$

$$C_{2\text{pt}}(\vec{p},t) \equiv \text{Tr}[\Gamma_0 G^{NN}(\vec{p},t)] \xrightarrow{t\gg 1} \frac{Z_p^2}{(La)^3}\frac{E_p+m}{E_p} e^{-E_p(t-t_0)} + Ae^{-E_p^1(t-t_0)}$$

$$\Gamma_0 = P_+ = \frac{1+\gamma_4}{2}$$

- Matrix element of tensor current can be obtained using three-point correlation functions:

$$G_{\alpha\beta}^{T_{4i}^{q,g}}(t_\text{f},\tau,\vec{p}_\text{f},\vec{p}_\text{i}) = \sum_{\vec{x}_\text{f},\vec{z}} e^{-i\vec{p}_\text{f}\cdot(\vec{x}_\text{f}-\vec{z})} e^{i\vec{p}_\text{i}\cdot\vec{z}} \langle 0|T[\chi_\alpha(\vec{x}_\text{f},t_\text{f}) \mathcal{T}_{4i}^{q,g}(\vec{z},\tau)\bar{\chi}_\beta(\vec{0},0)]|0\rangle$$

- With a definition of $C_{3\text{pt},\Gamma_\alpha}^{4i}(t_\text{f},\tau,\vec{p}_\text{f},\vec{p}_\text{i}) \equiv \text{Tr}[\Gamma_\alpha G^{T_{4i}^{q,g}}(t_\text{f},\tau,\vec{p}_\text{f},\vec{p}_\text{i})]$

$$R_{\Gamma_\alpha}^{4i}(t_\text{f},\tau,\vec{p}_\text{f},\vec{p}_\text{i}) \equiv \frac{C_{3\text{pt},\Gamma_\alpha}^{4i}(t_\text{f},\tau,\vec{p}_\text{f},\vec{p}_\text{i})}{C_{2\text{pt}}(\vec{p}_\text{f},t_\text{f})} \sqrt{\frac{C_{2\text{pt}}(\vec{p}_\text{i},t_\text{f}-\tau)C_{2\text{pt}}(\vec{p}_\text{f},\tau)C_{2\text{pt}}(\vec{p}_\text{f},t_\text{f})}{C_{2\text{pt}}(\vec{p}_\text{f},t_\text{f}-\tau)C_{2\text{pt}}(\vec{p}_\text{i},\tau)C_{2\text{pt}}(\vec{p}_\text{i},t_\text{f})}}$$

Required ratios

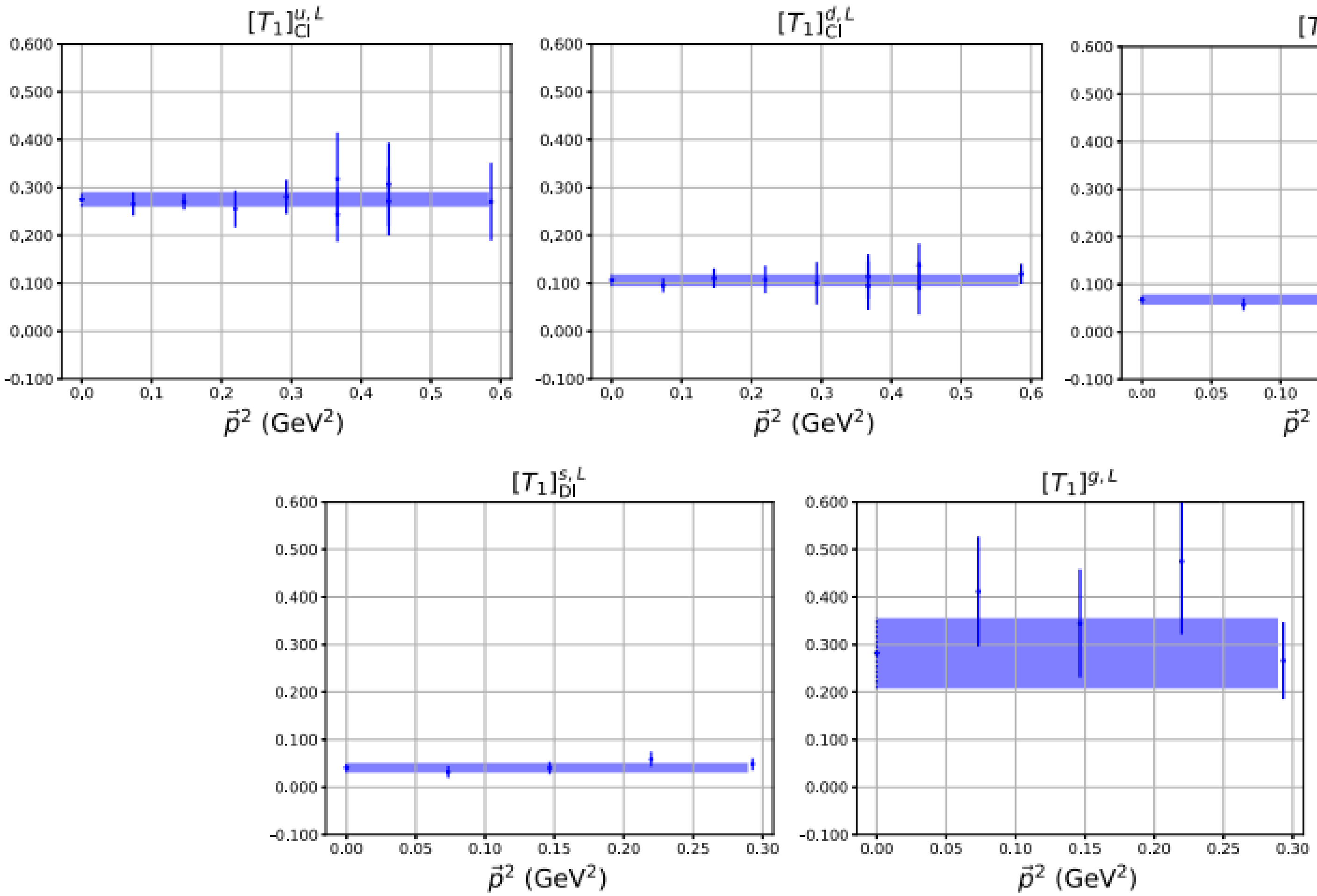
$$\xrightarrow[t_\text{f}-t\gg 1]{t_\text{f}\gg 1} \frac{a_1 T_1(Q^2) + a_2 T_2(Q^2) + a_3 D(Q^2)}{4\sqrt{E_{p'}(E_{p'}+m)E_p(E_p+m)}}$$

1. $$R^{4i}_{\Gamma_0}(t_{\mathrm{f}}, \tau, \vec{p}, \vec{p}) = p_i T_1(0)$$

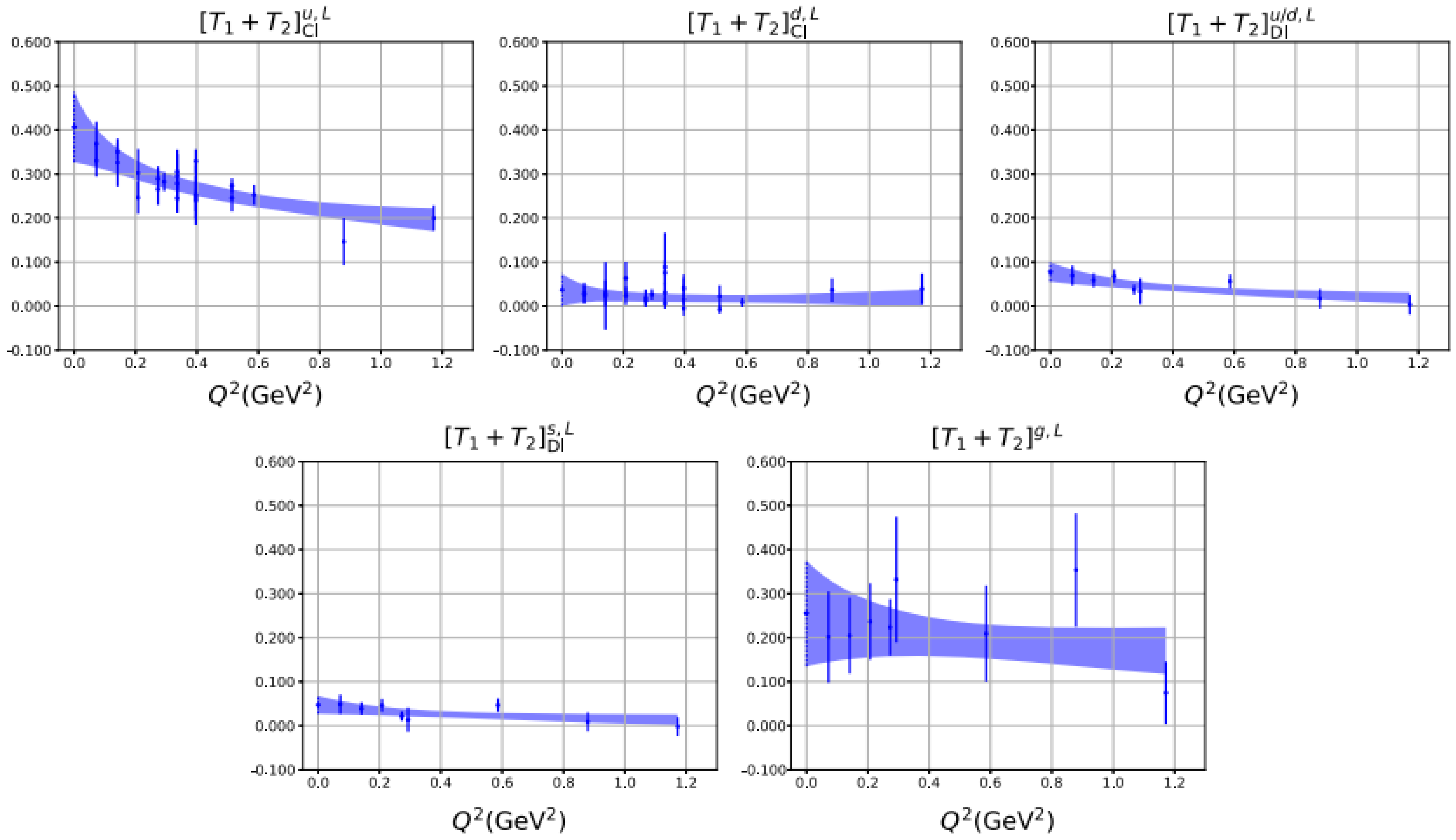
2. $$R^{4i}_{\Gamma_j}(t_{\mathrm{f}}, t, \vec{p}, \vec{0}) = \frac{-i}{4}\sqrt{\frac{E_p + m}{2E_p}}\epsilon_{i,j,k} p_k [T_1 + T_2](Q^2)$$

3. $$R^{4i}_{\Gamma_j}(t_{\mathrm{f}}, t, \vec{0}, \vec{p}) = \frac{-i}{4}\sqrt{\frac{E_p + m}{2E_p}}\epsilon_{i,j,k} p_k [T_1 + T_2](Q^2)$$

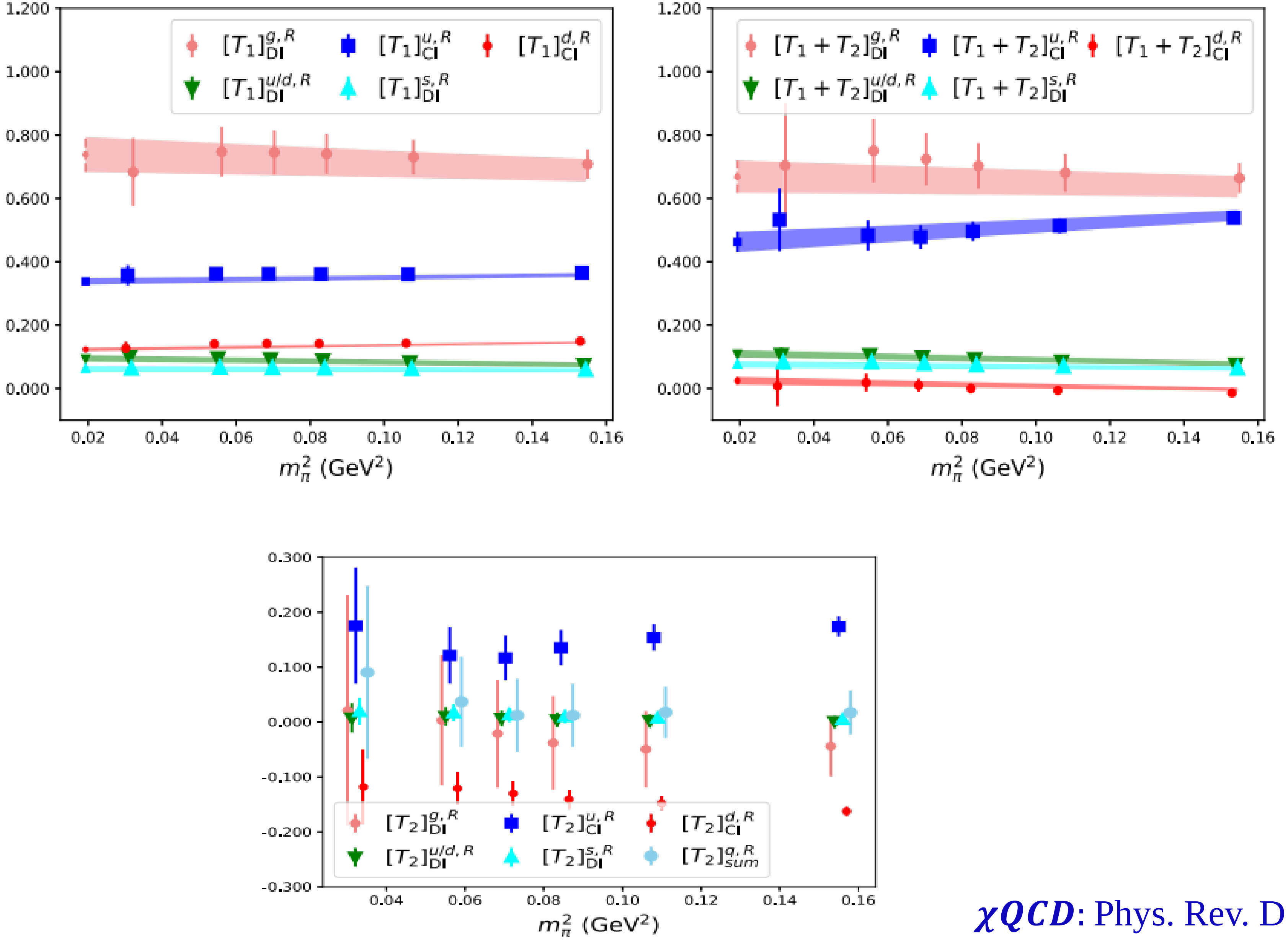
4. $$R^{4i}_{\Gamma_j}(t_{\mathrm{f}}, t, \vec{p}, -\vec{p}) = \frac{-i}{2}\epsilon_{i,j,k} p_k [T_1 + T_2](Q^2)$$



χQCD: Phys. Rev. D 106, 014512 (2022)



χQCD: Phys. Rev. D 106, 014512 (2022)



χQCD: Phys. Rev. D 106, 014512 (2022)

Renormalization

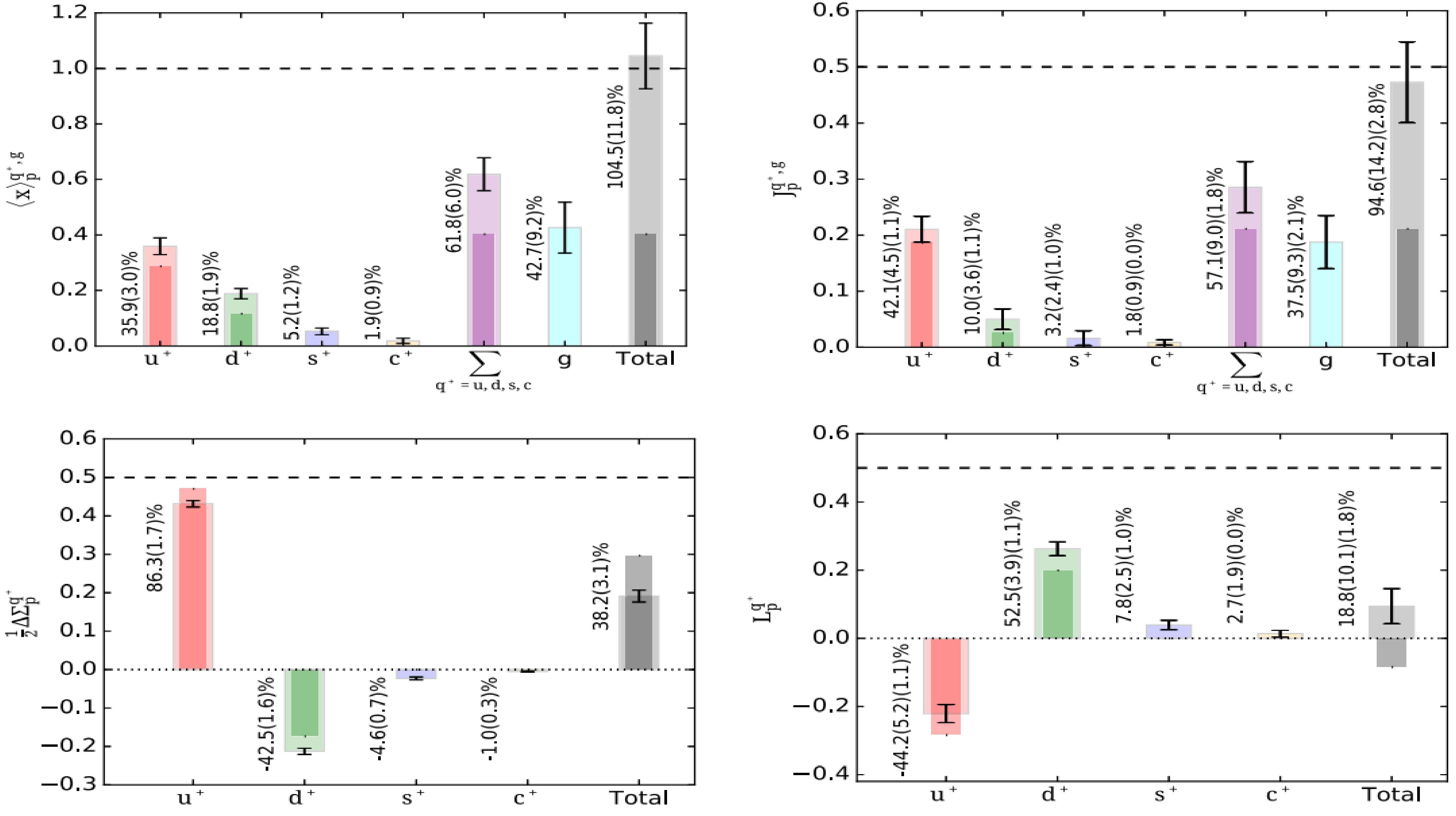
$$\mathcal{O}^{S}(M) = Z^{S}_{\mathcal{O};\mathrm{bare}}(M)\mathcal{O}_{\mathrm{bare}},$$

$$\mathcal{O}^{\mathrm{RGI}} = \Delta Z^{S}_{\mathcal{O}}(M)\mathcal{O}^{S}(M) \equiv Z^{\mathrm{RGI}}_{\mathcal{O}}\mathcal{O}_{\mathrm{bare}}.$$

For example:

$$\langle x\rangle_R^{q^+} = Z_{qq}\langle x\rangle_B^{q^+} + Z_{qg}\langle x\rangle_B^{g}$$

$$\langle x\rangle_R^{g} = Z_{gg}\langle x\rangle_B^{g} + Z_{gq}\langle x\rangle_B^{q^+}$$

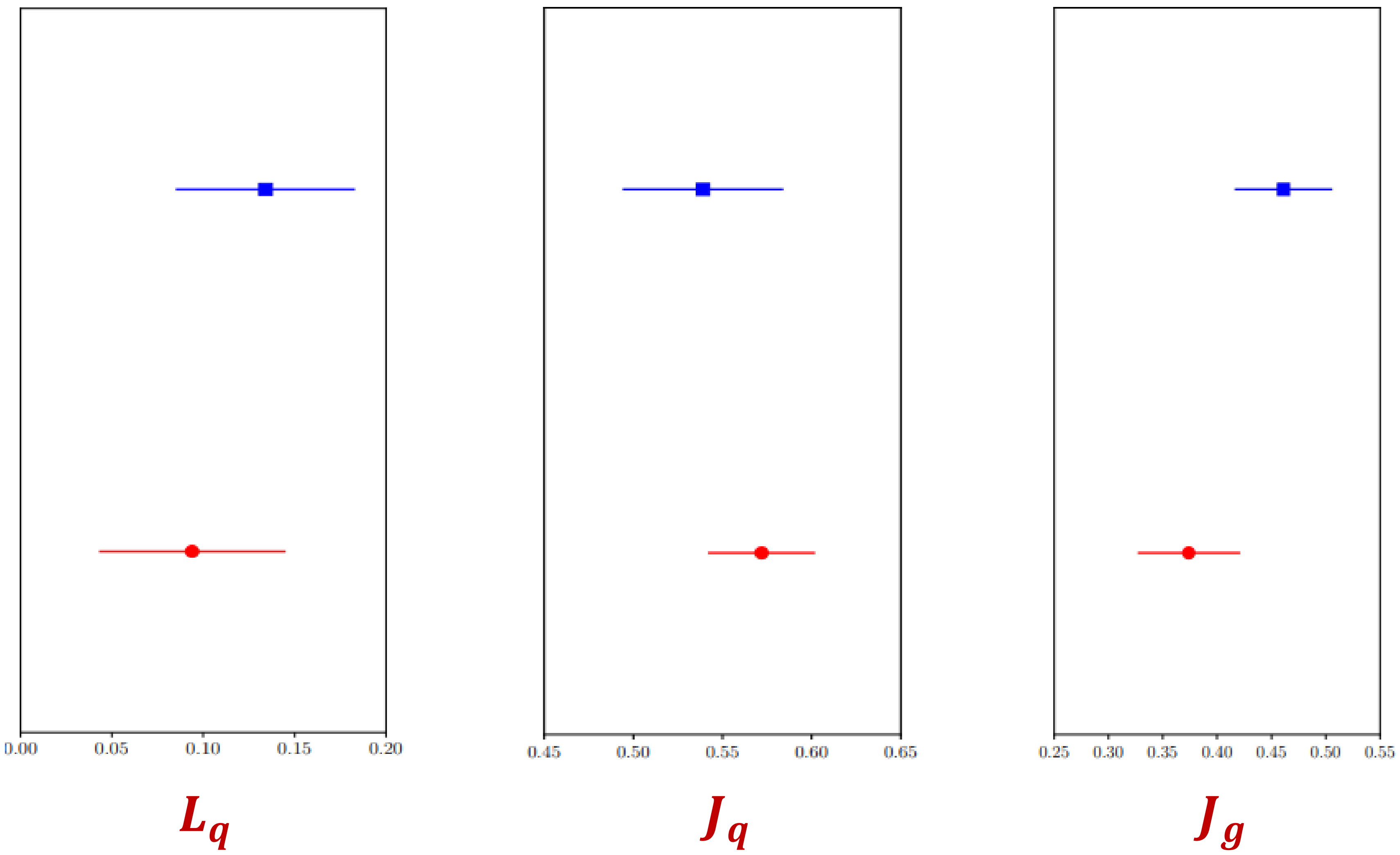


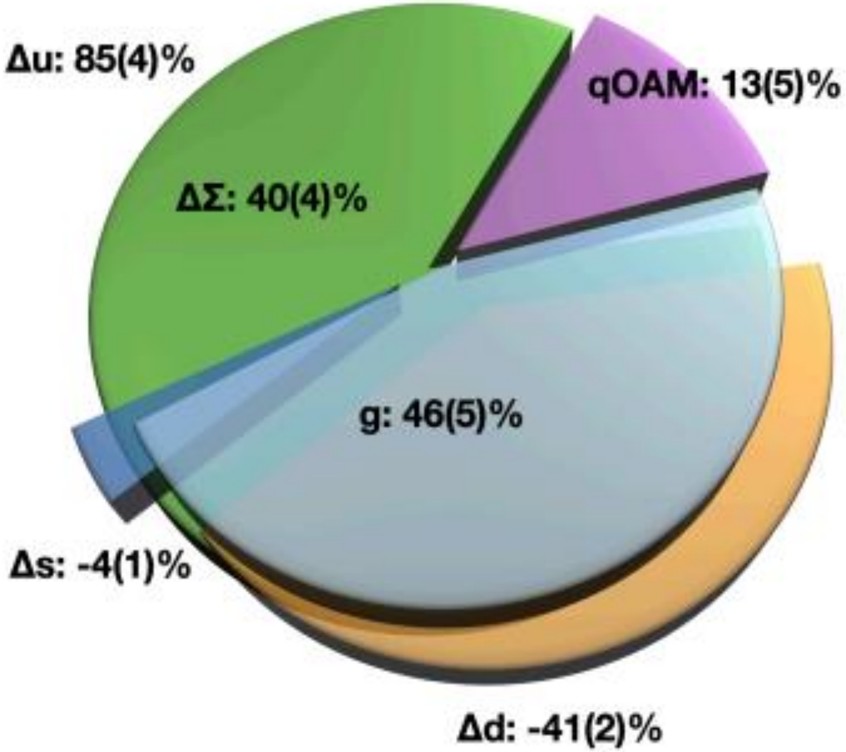
ETMC: Phys. Rev. D 101, 094513 (2020)

Angular momentum components from LQCD calculations

χQCD (2022)
PRD 106, 014512 (2022)

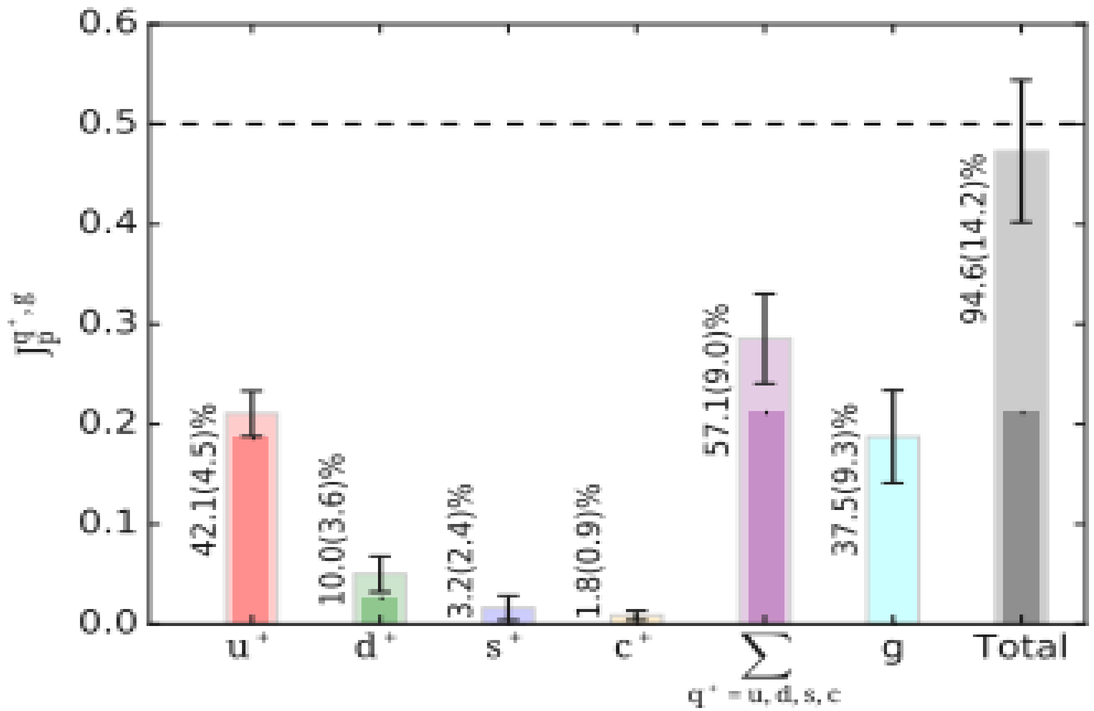
ETMC (2020)
PRD 101, 094513 (2020)





χQCD (2022)

Phys. Rev. D 106, 014512 (2022)



ETMC (2020)

Phys. Rev. D 101, 094513 (2020)

OAM from GTMD

Longitudinal quark OAM in *z*-dir $L_3^U = \int dx \int d^2 b_T \int d^2 k_T (b_T \times k_T)_3 \mathcal{W}^U(x, b_T, k_T)$

$$L_3^U = -\int dx \int d^2 k_T \frac{k_T^2}{m^2} F_{14}\Big|_{\Delta_T=0}$$

C. Lorcé and B. Pasquini, Phys. Rev. D 84, 014015 (2011).

$$\frac{L_3}{n} = \frac{1}{a}\epsilon_{ij} \frac{\frac{\partial}{\partial \Delta_{T,j}} (\Phi(a\vec{e}_i) - \Phi(-a\vec{e}_i))}{\Phi(a\vec{e}_i) + \Phi(-a\vec{e}_i)}\Bigg|_{\Delta_T=0}$$

$$\Phi(z_T) = \langle P + \Delta_T/2, S = \vec{e}_3 | \bar{\psi}(-z_T/2)\gamma^+ U[-z_T/2, z_T/2]\psi(z_T/2) | P - \Delta_T/2, S = \vec{e}_3 \rangle$$

M. Engelhardt, PRD 95, 094505 (2017)

n : number of valence quarks

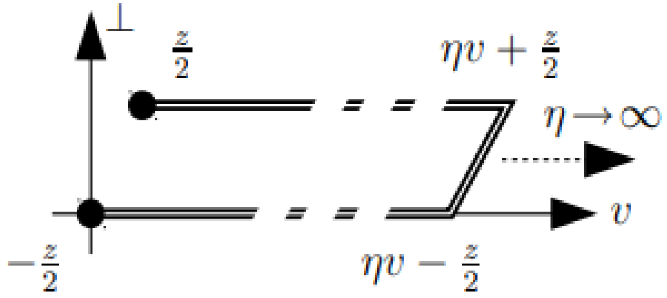
e_3 : Unit vector in the longitudinal direction

e_i : Unit vector in the transverse direction

Δ_T : Momentum transfer

z_T : Operator separation

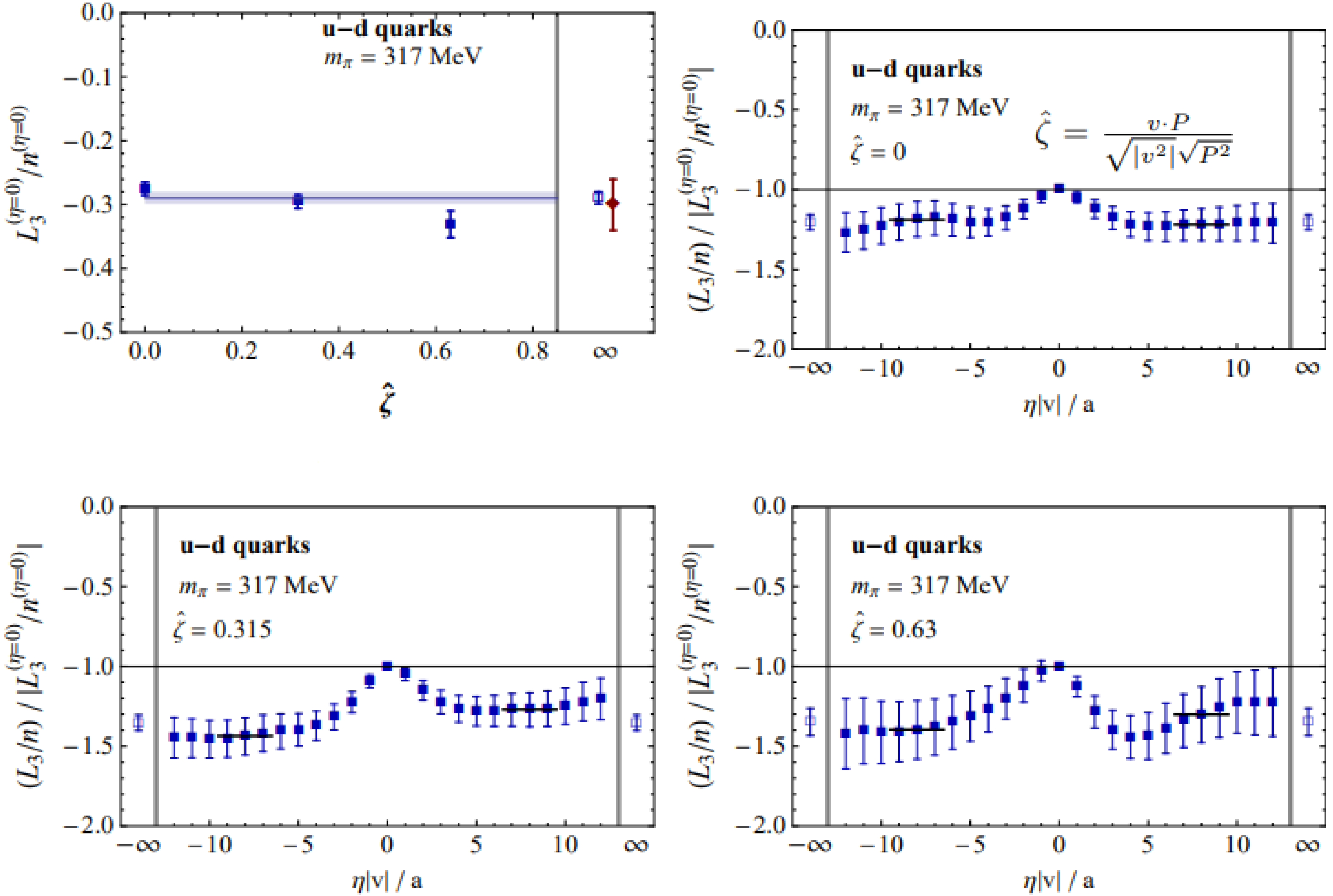
U : Wilson line between ψ and $\bar{\psi}$



- Straight $U\left[-\frac{z}{2}, \frac{z}{2}\right]$ → Ji OAM
- Staple-shaped $U\left[-\frac{z}{2}, \frac{z}{2}\right]$ → JM OAM
- Torque accumulation due to final state interaction causes the difference

M. Burkardt, Phys. Rev. D 88, 014014 (2013)

OAM from GTMD



M. Engelhardt, PRD 95, 094505 (2017)   Phys. Rev. D, 102(7) 074505, 2020

OAM from GTMD

Quark OAM can also be calculated using the second Mellin moment of the twist-3 generalized parton distribution in the forward limit

Engelhardt@SPIN 2023

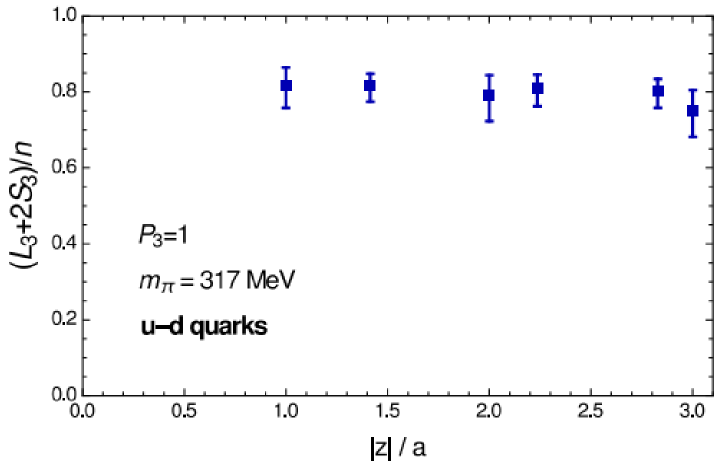
$$L_3 = (L_3 + 2S_3) - 2S_3 = -\int dx\, x\tilde{E}_{2T} - \int dx\, \tilde{H}$$

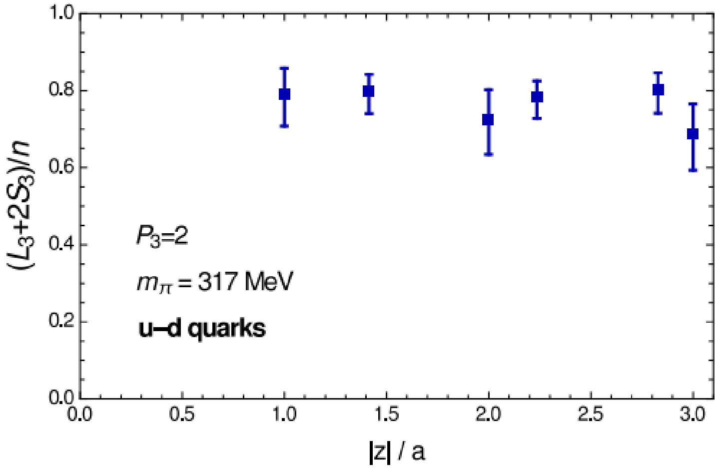
$$L_3 + 2S_3 = \epsilon_{ij} \frac{1}{2} \frac{\partial}{\partial(z \cdot P)} \frac{\partial}{\partial \Delta^i} \langle P + \Delta_T/2, +| \bar{\psi}(-z/2)\gamma^j \mathcal{U} \psi(z/2) | P - \Delta_T/2, +\rangle|_{z^+=z^-=0, \Delta_T=0, z_T \to 0}$$

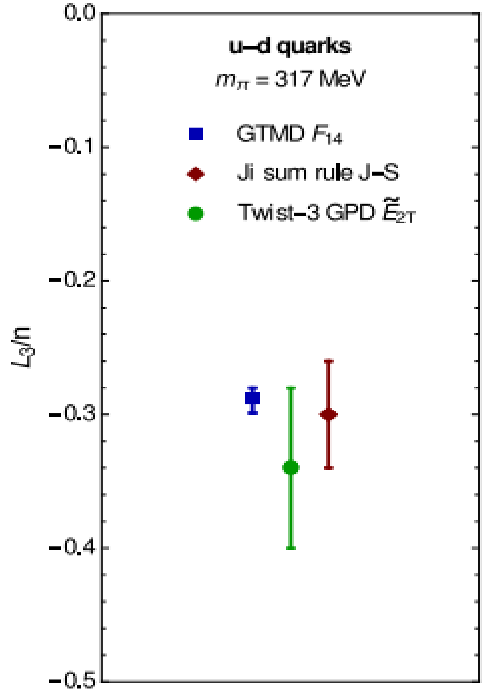
$$2\, P^j\, n = \langle P, +| \bar{\psi}(-z/2)\gamma^j \mathcal{U} \psi(z/2) | P, +\rangle\Big|_{z^+=z^-=0, z_T \to 0}$$

Original frame: $z^+ = 0, \quad z \cdot P = z^- P^+, \quad z^2 = -z_T^2$

Lattice frame: $z_0 = 0, \quad z \cdot P = -z_3 P_3, \quad z^2 = -z_3^2 - z_T^2$







Gluon helicity from Lattice QCD

- ΔG: High energy PP collision

  $\mathcal{L}_q, \mathcal{L}_g$: Generalized parton distributions (GPDs)
  Wigner distributions (GTMD)

Phys. Rev. Lett., 91:062001, 2003
Phys. Rev. D, 69:074014, 2004
JHEP, 08:056, 2009, JHEP, 05:041, 2011

- Matrix elements of appropriate equal-time local operator operators
  → boost to the infinite momentum frame
  ~ non-local gauge invariant from light-cone
  Lattice + LMET

Phys. Rev. Lett., 111:112002, 2013
Phys. Rev. D, 89(8):085030, 2014
Phys. Lett., B743:180–183, 2015
Phys. Rev., D93(5):054006, 2016

- $\Delta G \sim \vec{E} \times \vec{A}_{phys}$

$$\mathcal{D}^i A^i_{phys} \equiv \partial^i A^i_{phys} - ig\left[A^i, A^i_{phys}\right] = 0$$

$$A^i = A^i_{phys} + A^i_{ngi}$$

A^i_{phys}: similar to the transverse gauge-invariant part of the gauge potential $A_\perp$ in QED

ΔG From Lattice QCD

- First moment of the gluon helicity distribution:

$$\Delta G = \int dx \frac{i}{2xP^+} \int \frac{d\xi^-}{2\pi} e^{-ixP^+\xi^-} \times \langle PS | F_a^{+\alpha}(\xi^-) \mathcal{L}^{ab}(\xi^-, 0) \tilde{F}^+_{\alpha,b}(0) | PS \rangle$$

A. Manohar, Phys. Lett. B 255, 579 (1991)

$\xi^\pm = (\xi^0 \pm \xi^3)/\sqrt{2}$ Light-front coordinates

$\mathcal{L}(\xi^-, 0) = P\exp[-ig \int_0^{\xi^-} A^+(\eta^-, 0_\perp) d\eta^-]$

Light-cone gauge link

- A gauge-invariant gluon helicity operator in a nonlocal form

$$\tilde{S}_g = \left[\vec{E}^a(0) \times \left(\vec{A}^a(0) - \frac{1}{\nabla^+} (\vec{\nabla} A^{+,b}) \mathcal{L}^{ba}(\xi^-, 0) \right) \right]^z$$

$\nabla^+ = \partial / \partial \xi^-$

Y. Hatta, Phys. Rev. D 84, 041701 (2011).
X. Ji, J.-H. Zhang, and Y. Zhao, Phys. Rev. Lett. 111, 112002 (2013)

$A^+ = 0$

LMET

$\vec{E} \times \vec{A}$

$$\vec{S}_g = 2 \int d^3x \mathrm{Tr}(\vec{E}_c \times \vec{A}_c)$$

Coulomb gauge: $\vec{\partial} \cdot \vec{A} = 0$

S. Chen, X.-F. Lu, W.-M. Sun, F. Wang, and T. Goldman, Phys. Rev. Lett. 100, 232002 (2008).
X.-S. Chen, W.-M. Sun, X.-F. Lu, F. Wang, and T. Goldman, Phys. Rev. Lett. 103, 062001 (2009).
C. Lorce, Phys. Rev. D 87, 034031 (2013).
Y. Zhao, K.-F. Liu, and Y. B. Yang, Phys. Rev. D 93, 054006 (2016).

$\vec{S}_g$: not Lorentz invariant, frame dependent
Needs to calculate in rest and moving frames
and needs to be matched after renormalization.
Also calculations have to be in Coulomb gauge

Lattice QCD calculation for ΔG

χ**QCD**: PRL 118, 102001 (2017)

- **Gauge fixed potential:**

$$A_{c,\mu} = \left(\frac{U^c_\mu(x) - U^{c\dagger}_\mu(x) + U^c_\mu(x-a\hat{\mu}) - U^{c\dagger}_\mu(x-a\hat{\mu})}{4iag} \right)_{\text{traceless}}$$

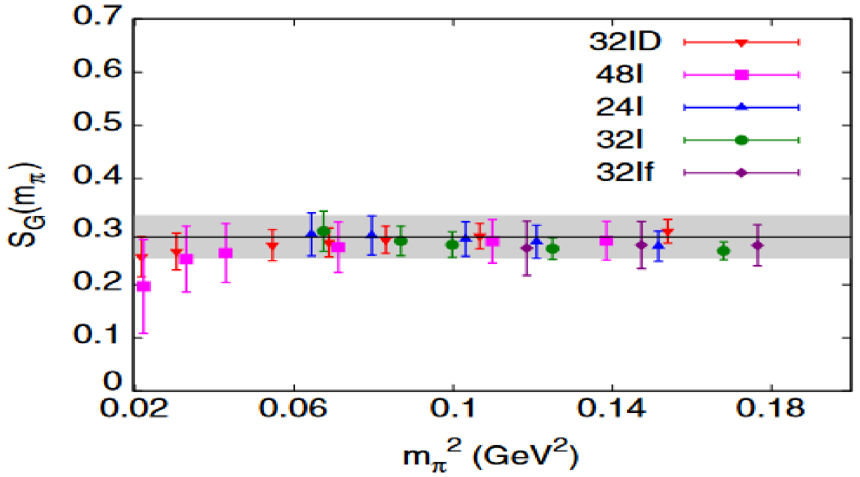
- **Coulomb gauge:**

$$\sum_{\mu=x,y,z} [U^c_\mu(x) - U^c_\mu(x-a\hat{\mu})] = 0$$

- **Chromoelectric field:**

$$F^c_{\mu\nu} = \frac{i}{8a^2 g}(\mathcal{P}_{\mu,\nu} - \mathcal{P}_{\nu,\mu} + \mathcal{P}_{\nu,-\mu} - \mathcal{P}_{-\mu,\nu} + \mathcal{P}_{-\mu,-\nu} - \mathcal{P}_{-\nu,-\mu} + \mathcal{P}_{-\nu,\mu} - \mathcal{P}_{\mu,-\nu})$$

$$\mathcal{P}_{\mu,\nu} = U^c_\mu(x) U^c_\nu(x+a\hat{\mu}) U^{c\dagger}_\mu(x+a\hat{\nu}) U^{c\dagger}_\nu(x).$$



$$\Delta G\,(\mu^2 = 10\ \text{GeV}^2) \approx S_G(\infty, \mu^2 = 10\ \text{GeV}^2)$$
$$= 0.251(47)(16)$$
$$\rightarrow 50(9)(3)\%\ \textbf{of proton spin}$$

Caveat: convergence problem for perturbative series in LMET systematics is not under control and more work is necessary.

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Further improvement requires better ground state overlap without excited state contamination, more statistics
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bigger size lattices with large number of configurations

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Extending these calculations for nuclei?
Need: Bigger volume lattices and new methodologies for controlling systematics

Conclusions

- **Lattice calculations can provide first-principles non-perturbative quantitative results with controlled precision for the helicity and orbital angular momentum content, structure functions, PDFs, GPDss, TMDs, DA, wave-functions, etc. of hadrons.**

- **These observables are necessary to understand the non-perturbative structure of hadrons.**

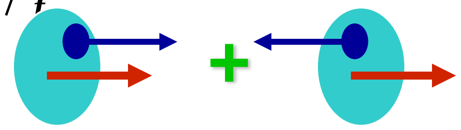
- **Lattice calculations are playing important roles in studying the spin content and the emergence of mass of a proton. These calculations are showing a substantial contribution from gluon angular momentum and also from quark orbital angular momentum.**

- **In the ongoing exascale era or supercomputing it is envisaged that the results from these calculations will be more precise.**
- **Similar to one nucleon case, these calculations may possibly be performend for low lying nuclei too!**
- **These calculations will be very important for achieving EIC science goals**

In partonic interpretation, v_n^f is the *(n-1)th* moment of the momentum fraction carried by quarks with flavor f at scale μ

i. e. ,

$$v_n^f(\mu) = \int_0^1 dx\, x^{n-1}[f(x) + (-1)^n \bar{f}(x)] = \langle x^{n-1} \rangle_f$$

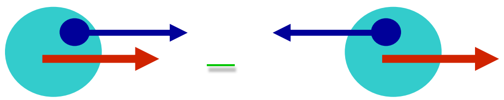


$$v_n^g(\mu) = \int_0^1 dx\, x^{n-1} g(x)$$

$$a_n^{(f)} = 2\Delta^{(n)} q^{(f)} = 2\int_0^1 \mathrm{d}x\, x^n \frac{1}{2}[q_\uparrow^{(f)}(x) - q_\downarrow^{(f)}(x)],\ f = u, d$$

$q_\uparrow(x),\ q_\downarrow(x)$ are the quark distribution function with spin parallel or antiparallel to the direction of motion

$$a_0^{(u)} = 2\Delta u, \quad a_0^{(d)} = 2\Delta d$$



where, Δf is the fraction quark spin carried by quarks with flavor *f.*

For g_2, a_n^f : twist 2

d_n^f : twist 3 Wandzura-Wilczek contribution