Chiral Soliton Lattice in QCD

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in collaboration with

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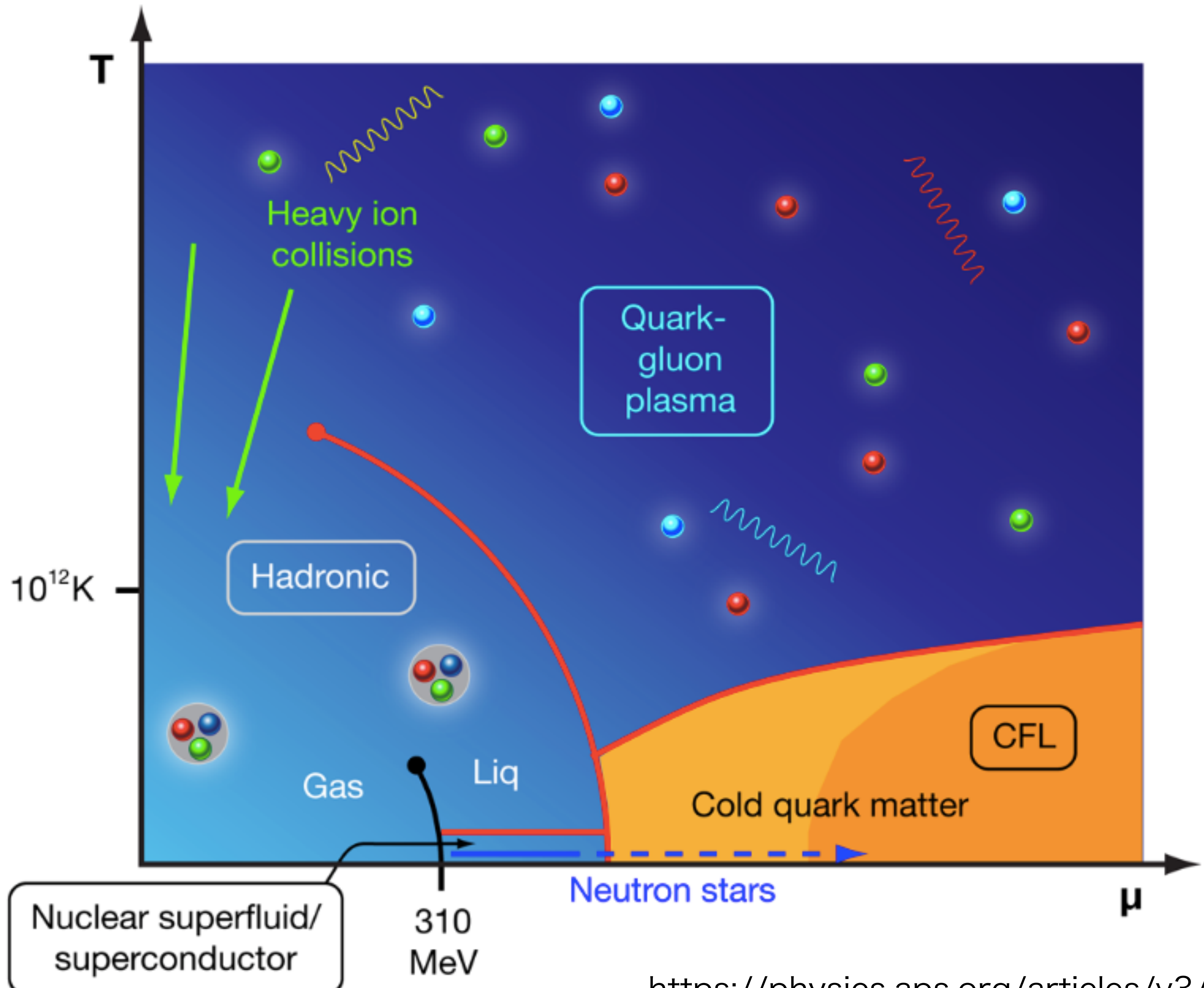
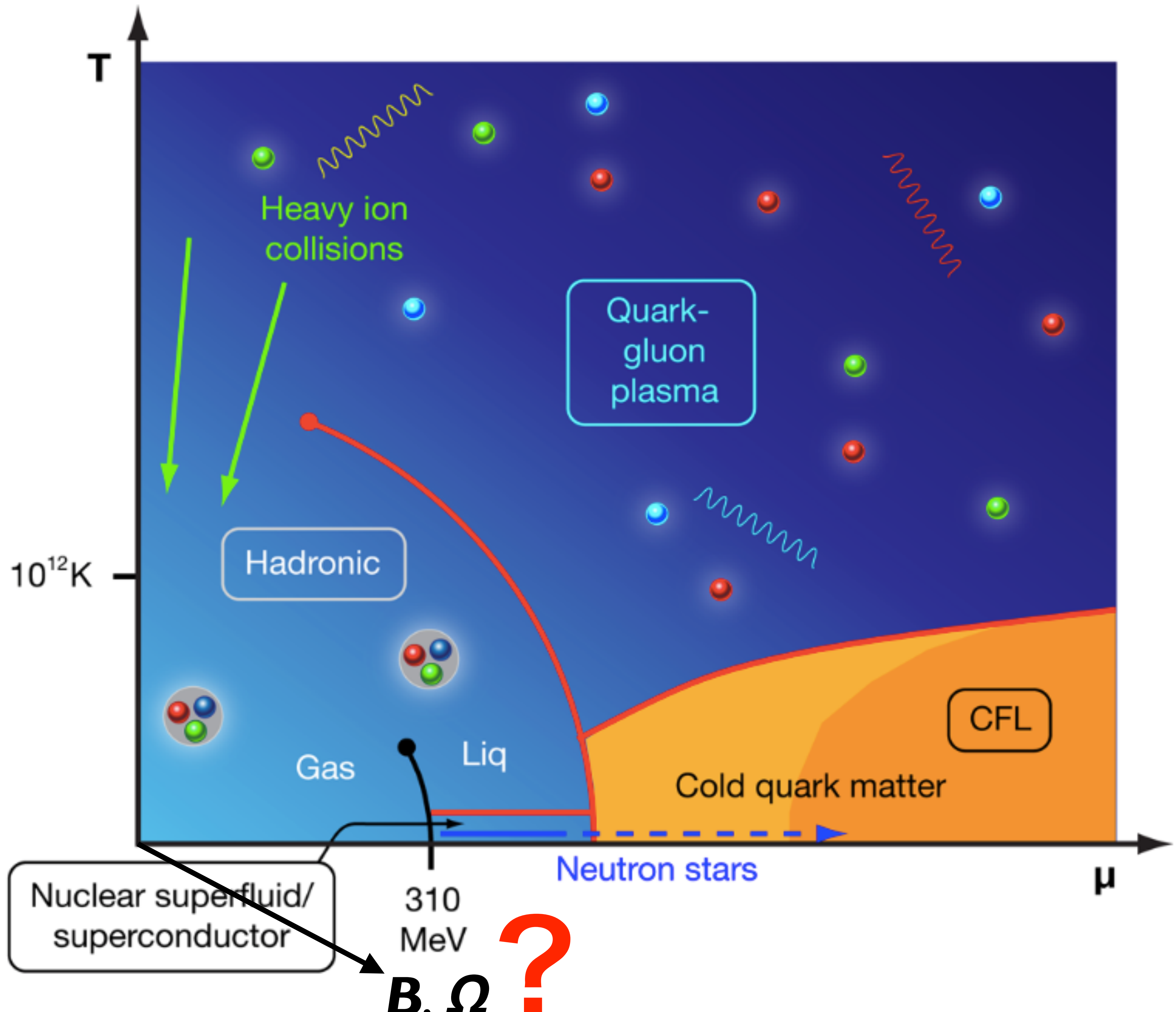
“Topological aspects of strong correlations and gauge theories”
September 8, 2021

QCD phase diagram

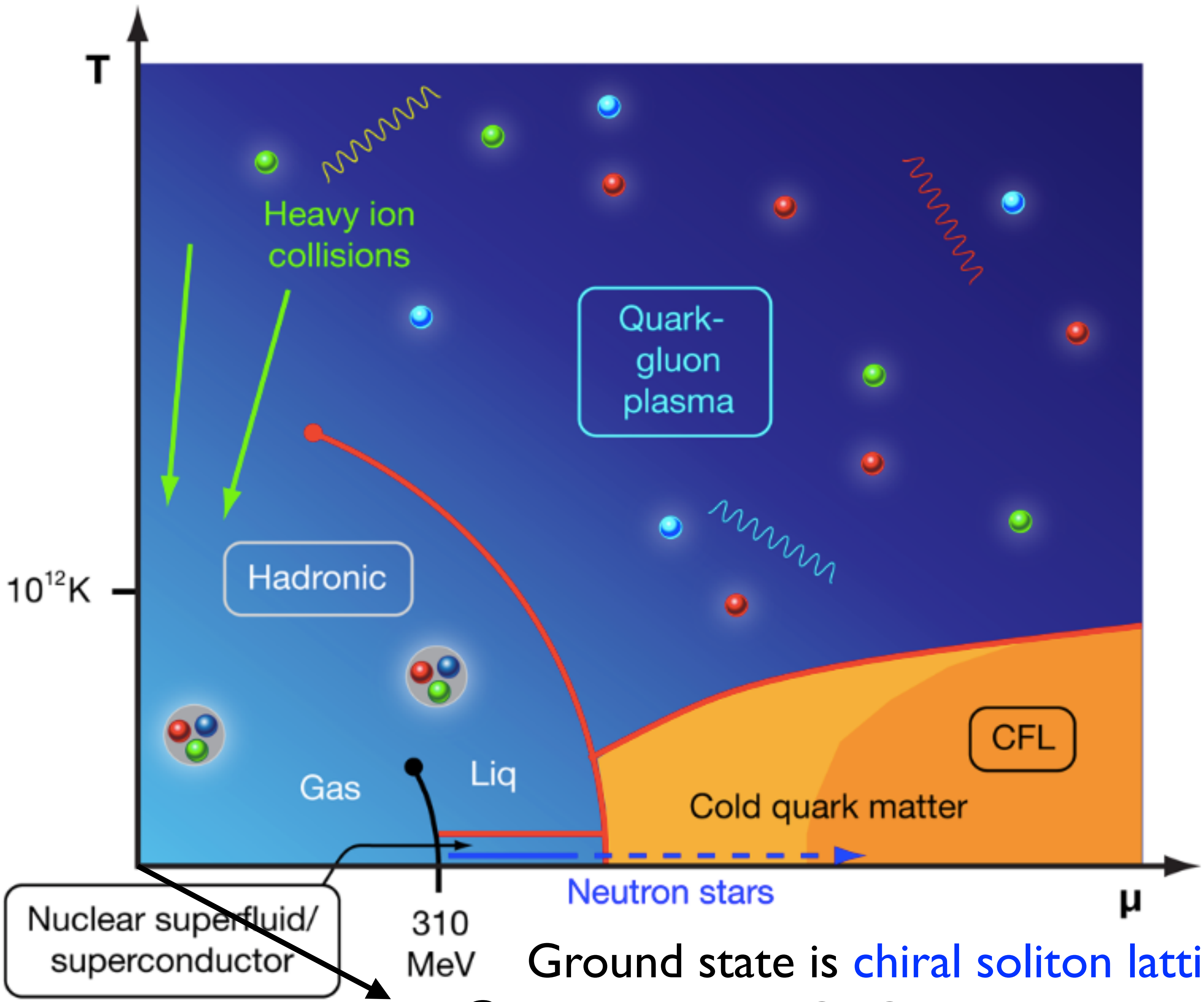


https://physics.aps.org/articles/v3/44

QCD phase diagram



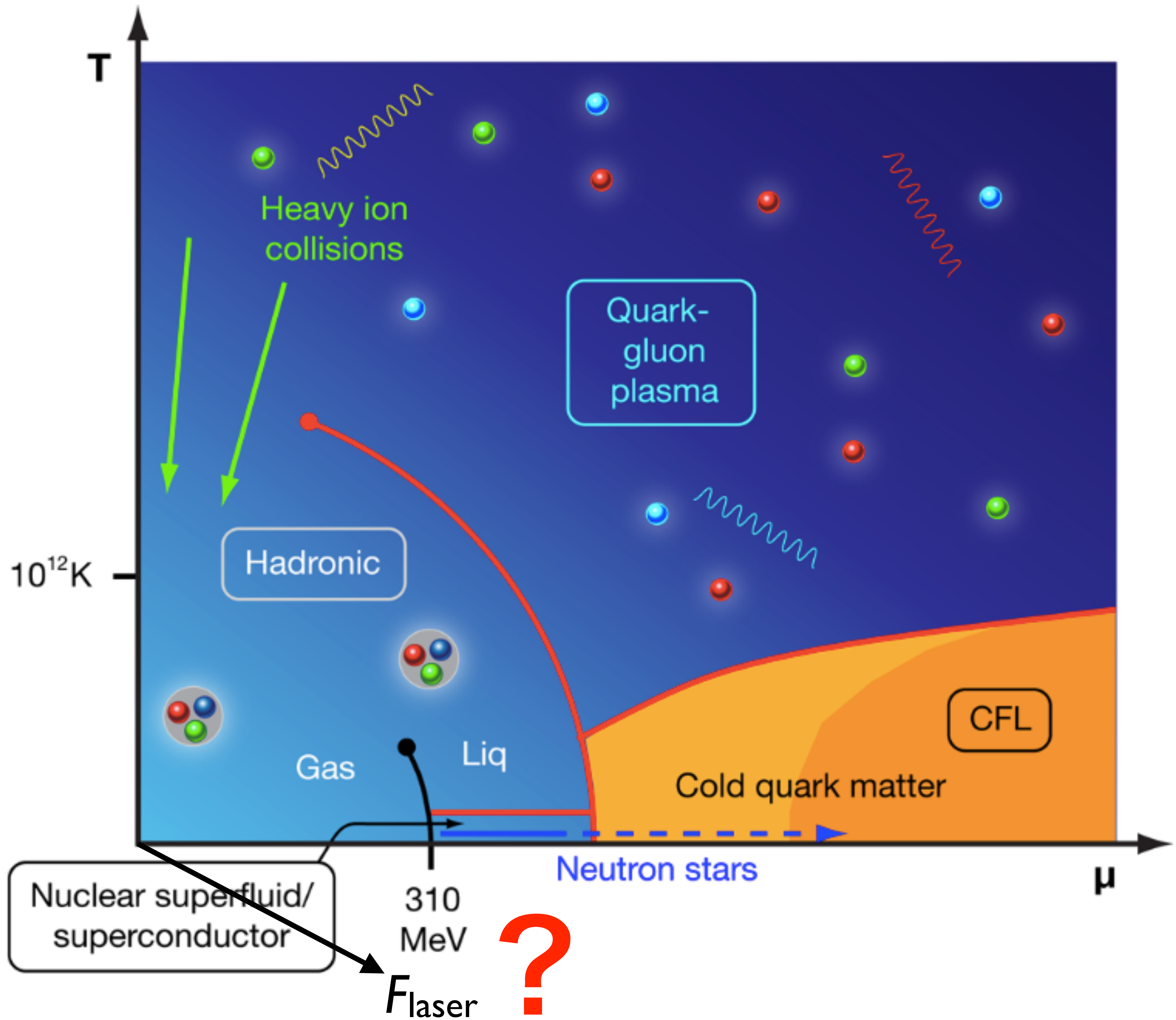
QCD phase diagram



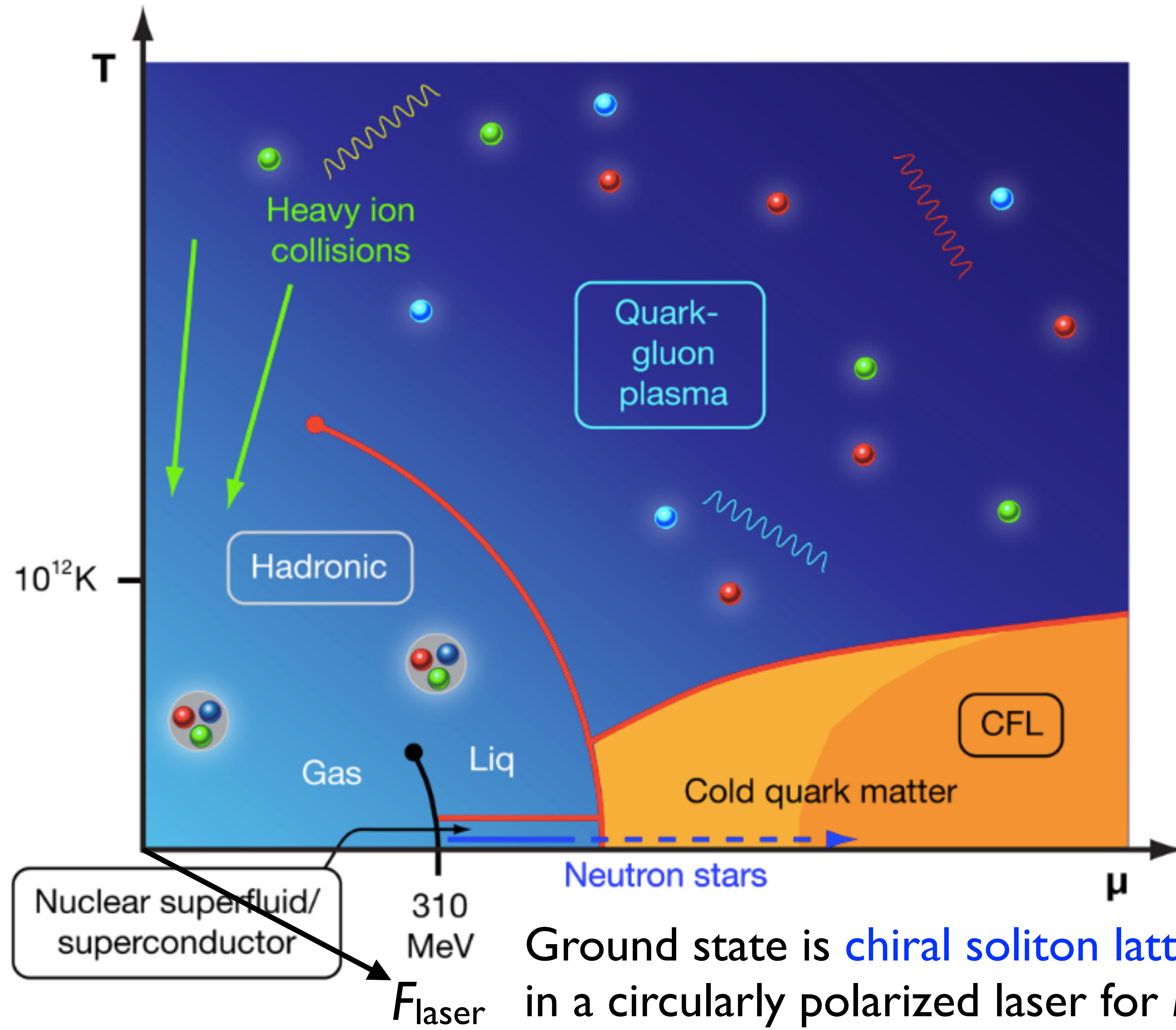
B*, *Ω

Ground state is chiral soliton lattice (CSL) for $B > B_{crit}$ or $\Omega > \Omega_{crit}$.

QCD phase diagram



QCD phase diagram



Ground state is chiral soliton lattice (CSL) in a circularly polarized laser for $F > F_{crit}$.

Chiral Soliton Lattice

- Violates parity

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- Violates parity
- Carries topological charge

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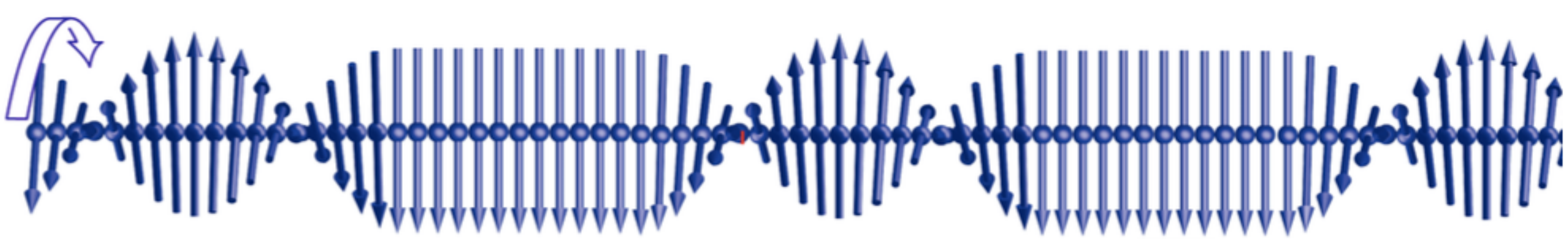
CSL known to appear in chiral magnets and liquid crystals



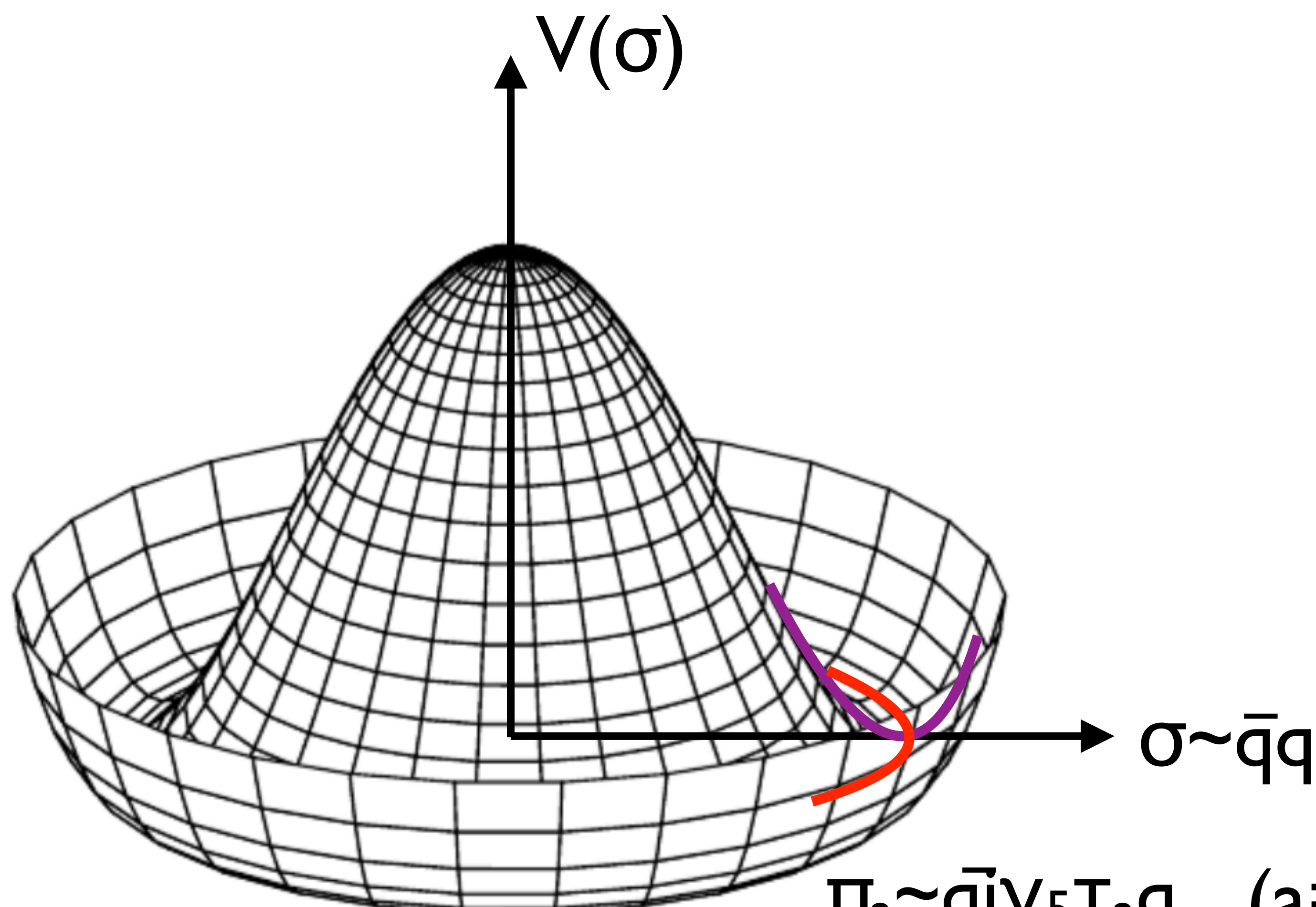
Y. Togawa, *et al.*, PRL (2012)

CSL is realized as the QCD ground state at finite μ and $\boldsymbol{B}/\boldsymbol{\Omega}$ or under a polarized laser field.

Low-energy effective theory of QCD matter

Chiral symmetry breaking

2-flavor QCD: $SU(2)_L \times SU(2)_R \rightarrow SU(2)_V$



$\pi_a \sim \bar{q} i \gamma_5 \tau_a q$ (a=1,2,3)

~ Nambu-Goldstone (NG) modes

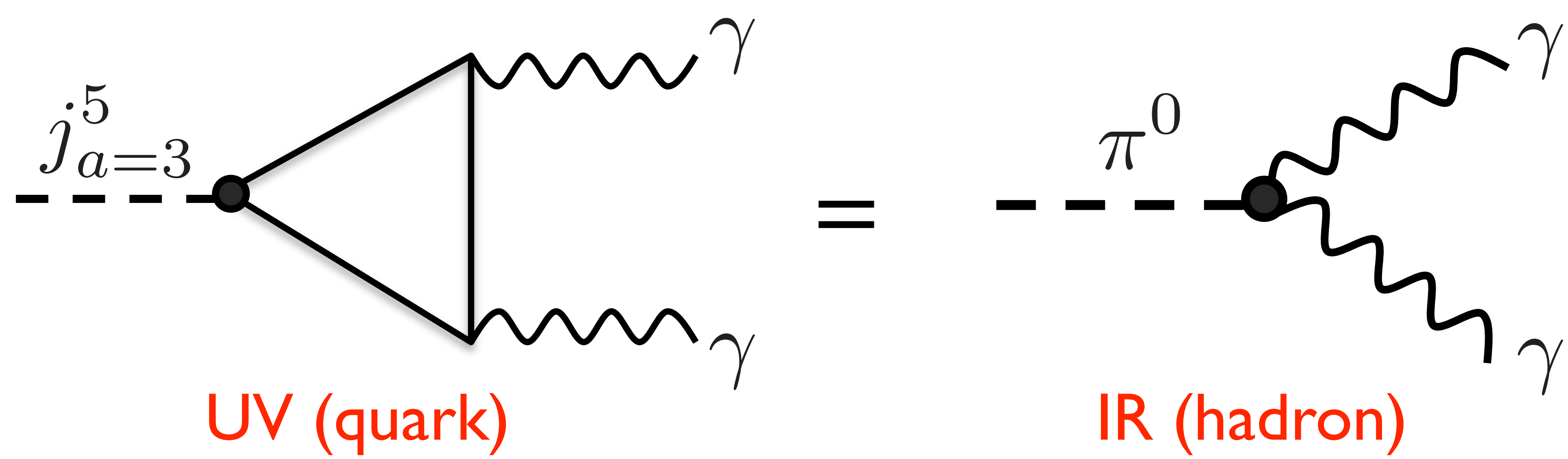
Chiral Perturbation Theory (ChPT)

- Low-energy effective theory for NG modes in QCD
- NG modes parametrized by $SU(2)_L \times SU(2)_R / SU(2)_V$: $\Sigma \sim \mathrm{e}^{\mathrm{i}\pi_a \tau_a}$
- Systematic expansion in derivatives and quark masses
- Lagrangian at the leading order:

$$\mathcal{L} = \frac{f_\pi^2}{4} \left[\mathrm{Tr}(\partial_\mu \Sigma^\dagger \partial^\mu \Sigma) + m_\pi^2 (\Sigma + \Sigma^\dagger) \right]$$

Chiral anomaly

- Chiral anomaly is energy independent (anomaly matching)



axial vector

$$j_3^{\mu 5} = \bar{q}\gamma^\mu\gamma^5\tau_3 q, \quad \partial_\mu j_3^{\mu 5} = \frac{1}{2\pi^2}\boldsymbol{E}\cdot\boldsymbol{B} \qquad\qquad \mathcal{L}_{\pi_0\gamma\gamma} = \frac{1}{4\pi^2}\pi_0\boldsymbol{E}\cdot\boldsymbol{B}$$

isospin

Wess-Zumino-Witten term

Son, Zhitnitsky; PRD (2004); Son, Stephanov, PRD (2008)

- Anomalous term in the vacuum: $\mathcal{L}_{\mathrm{anom}} = C\pi^0 \boldsymbol{E}\cdot\boldsymbol{B}, \quad C = \dfrac{1}{4\pi^2}$
- Anomalous term at finite μ_B and $\boldsymbol{B}$: $\mathcal{L}_{\mathrm{anom}} = C\mu_{\mathrm{B}}\boldsymbol{\nabla}\pi_0\cdot\boldsymbol{B}$

$$\because \quad S_{\mathrm{anom}} = C\int \mathrm{d}^4x\ \pi_0(-\boldsymbol{\nabla}\phi)\cdot\boldsymbol{B} \sim C\int \mathrm{d}^4x\ \phi\boldsymbol{\nabla}\pi_0\cdot\boldsymbol{B}$$

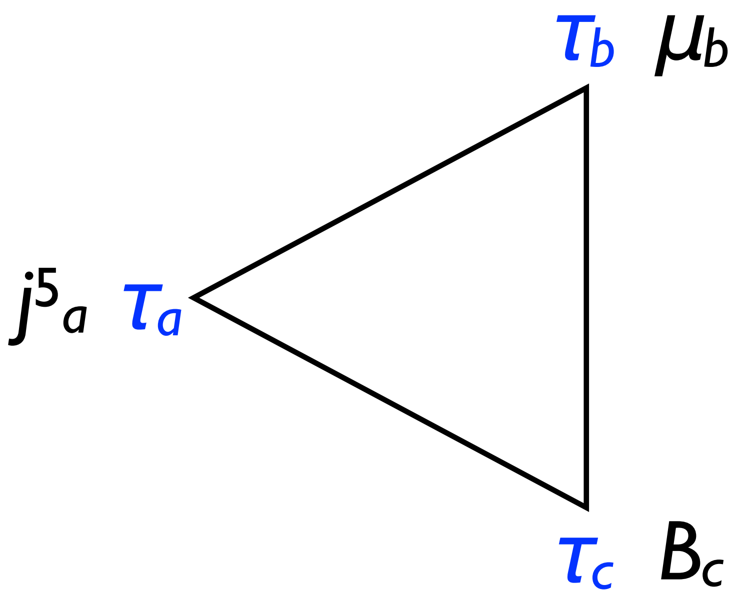
scalar potential

This can also be understood as UV-IR matching of CME.

Chiral magnetic effect

- CME for a single chiral fermion: $\boldsymbol{j} = \pm\frac{\mu}{4\pi^2}\boldsymbol{B}$
- CME for generic charges in QCD: see, e.g., Landsteiner, arXiv:1610.04413

$$\boldsymbol{j}_a^5 = N_\mathrm{c} d_{abc}\frac{\mu_b}{2\pi^2}\boldsymbol{B}_c, \quad d_{abc} = \frac{1}{2}\mathrm{Tr}\left[\tau_a\{\tau_b, \tau_c\}\right]$$



UV-IR matching of CME

- Nonzero CME at finite μ_B and $\boldsymbol{B}$: $\boldsymbol{j}^5_{a=3} = \frac{\mu_B}{2\pi^2}\boldsymbol{B}$

- Consider a local axial rotation: $q \to e^{-i\theta_a \gamma_5 \tau_a} q, \quad \pi_a \to \pi_a + 2\theta_a$

- Matching of action due to CME between UV and IR:

$$\delta S_{\mathrm{QCD}} = \int \mathrm{d}^4 x \boldsymbol{\nabla} \theta_3 \cdot \boldsymbol{j}^5_3 \longleftrightarrow \delta S_{\mathrm{EFT}} = \int \mathrm{d}^4 x \boldsymbol{\nabla} \left(\frac{\pi_0}{2} \right) \cdot \boldsymbol{j}^5_3$$

- This leads to the anomalous term:

$$\mathcal{L}_{\mathrm{anom}} = \frac{\mu_B}{4\pi^2} \boldsymbol{\nabla} \pi_0 \cdot \boldsymbol{B}$$

Chiral vortical effect

- CVE for a single fermion: $\boldsymbol{j} = \pm\left(\frac{\mu^2}{4\pi^2} + \frac{T^2}{12}\right)\boldsymbol{\Omega}$

- CVE for generic charges in QCD: see, e.g., Landsteiner, arXiv:1610.04413

$$\boldsymbol{j}^5_a = N_\mathrm{c}\left(d_{abc}\frac{\mu_b\mu_c}{2\pi^2} + b_a\frac{T^2}{6}\right)\boldsymbol{\Omega}$$

$$d_{abc} = \frac{1}{2}\mathrm{Tr}\left[\tau_a\{\tau_b,\tau_c\}\right], \quad b_a = \mathrm{Tr}(\tau_a)$$

UV-IR matching of CVE

Huang, Nishimura, Yamamoto, JHEP (2018)

- Postulate the CVE matching between UV and IR:

$$\delta S_{\mathrm{QCD}} = \int \mathrm{d}^4x \boldsymbol{\nabla}\theta_3 \cdot \boldsymbol{j}_3^5 \longleftrightarrow \delta S_{\mathrm{EFT}} = \int \mathrm{d}^4x \boldsymbol{\nabla}\left(\frac{\pi_0}{2}\right) \cdot \boldsymbol{j}_3^5$$

- This leads to the anomalous term under rotation:

$$\mathcal{L}_{\mathrm{anom}} = N_{\mathrm{c}}\left(d_{abc}\frac{\mu_b\mu_c}{4\pi^2} + b_a\frac{T^2}{12}\right)\boldsymbol{\nabla}\pi_a \cdot \boldsymbol{\Omega}$$

- Anomalous term in QCD w/ $\boldsymbol{\Omega}$ at finite μ_B, μ_I (T=0):

$$\mathcal{L}_{\mathrm{anom}} = \frac{\mu_{\mathrm{B}}\mu_{\mathrm{I}}}{2\pi^2}\boldsymbol{\nabla}\pi_0 \cdot \boldsymbol{\Omega}$$

(N_f=3) $$\mathcal{L}_{\mathrm{anom}} = \frac{\mu_{\mathrm{B}}^2}{4\pi^2}\sqrt{\frac{2}{3}}\boldsymbol{\nabla}\eta' \cdot \boldsymbol{\Omega}$$

Chiral Soliton Lattice in QCD matter

Brief summary

- QCD at finite μ and ***B*** ($\boldsymbol{B} = B\hat{\boldsymbol{z}}$):

$$\mathcal{H} = \frac{f_\pi^2}{2}(\partial_z \pi_0)^2 - \underbrace{C\mu_{\mathrm{B}} B \partial_z \pi_0}_{\text{anomalous}} - \underbrace{m_\pi^2 f_\pi^2 \cos \pi_0}_{\text{mass}} + \text{const.}$$

QCD vs. chiral magnet

- QCD at finite μ and ***B*** ($\boldsymbol{B} = B\hat{\boldsymbol{z}}$):

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- Chiral magnets: Kishine, Ovchinnikov, Solid State Phys. **66** (2015)

$$\mathcal{H} = JS^2 a \left[\frac{1}{2}(\partial_z \phi)^2 - \underbrace{q_0 \partial_z \phi}_{\text{Dzyaloshinskii-Moriya}} - \underbrace{m^2 \cos \phi}_{\text{Zeeman}} \right] + \text{const.}$$

Two Hamiltonians are mathematically equivalent.

QCD vs. chiral magnet

- QCD at finite μ and $\boldsymbol{B}$ ($\boldsymbol{B} = B\hat{\boldsymbol{z}}$):

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- K … F

$$H_j = -J \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}$$

$$H_j^{\mathrm{DM}} = \boldsymbol{D} \cdot (\boldsymbol{S}_j \times \boldsymbol{S}_{j+1})$$

$$H_j^{\mathrm{Z}} = -\boldsymbol{H} \cdot \boldsymbol{S}_j$$

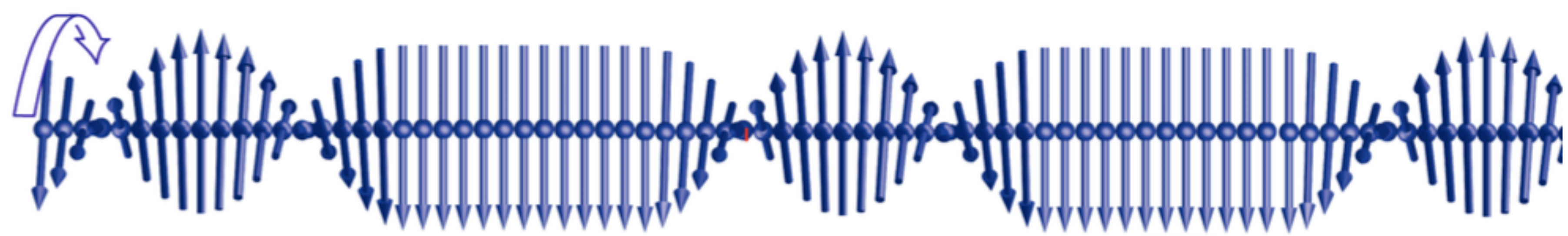
$$\mathcal{H} = JS^2 a \left[\frac{1}{2}(\partial_z \phi)^2 - \underbrace{q_0 \partial_z \phi}_{\text{Dzyaloshinskii-Moriya}} - \underbrace{m^2 \cos\phi}_{\text{Zeeman}}\right] + \text{const.}$$

Two Hamiltonians are mathematically equivalent.

Chiral Soliton Lattice (CSL)

1D lattice of topological solitons w/ parity and translational symm. breaking

cf) chiral magnet



Y. Togawa, *et al.*, PRL (2012)

- Equation of motion: $\partial_z^2 \pi_0 = m_\pi^2 \sin \pi_0$

- Solution: $\cos \dfrac{\pi_0(\bar{z})}{2} = \mathrm{sn}(\bar{z}, k)\,, \quad \bar{z} \equiv \dfrac{z m_\pi}{k}$

Jacobi elliptic function

elliptic modulus

Rough picture

$$\mathcal{H} = \frac{f_\pi^2}{2}(\partial_z \pi_0)^2 - C\mu_{\mathrm{B}} B \partial_z \pi_0 - \cancel{m_\pi^2 f_\pi^2 \cos\pi_0} + \mathrm{const.}$$

- **2nd term ≫ 3rd term:** $\langle \pi_0 \rangle = \dfrac{C\mu_{\mathrm{B}} B}{f_\pi^2} z + \mathrm{const.}$

Rough picture

$$\mathcal{H} = \frac{f_\pi^2}{2}(\partial_z \pi_0)^2 - \cancel{C\mu_{\rm B} B \partial_z \pi_0} - m_\pi^2 f_\pi^2 \cos \pi_0 + \text{const.}$$

- 2nd term ≫ 3rd term: $\langle \pi_0 \rangle = \dfrac{C\mu_{\rm B} B}{f_\pi^2} z + \text{const.}$
- 2nd term ≪ 3rd term: $\langle \pi_0 \rangle = 2\pi n$

Rough picture

$$\mathcal{H} = \frac{f_\pi^2}{2}(\partial_z \pi_0)^2 - C\mu_{\rm B} B \partial_z \pi_0 - m_\pi^2 f_\pi^2 \cos\pi_0 + {\rm const.}$$

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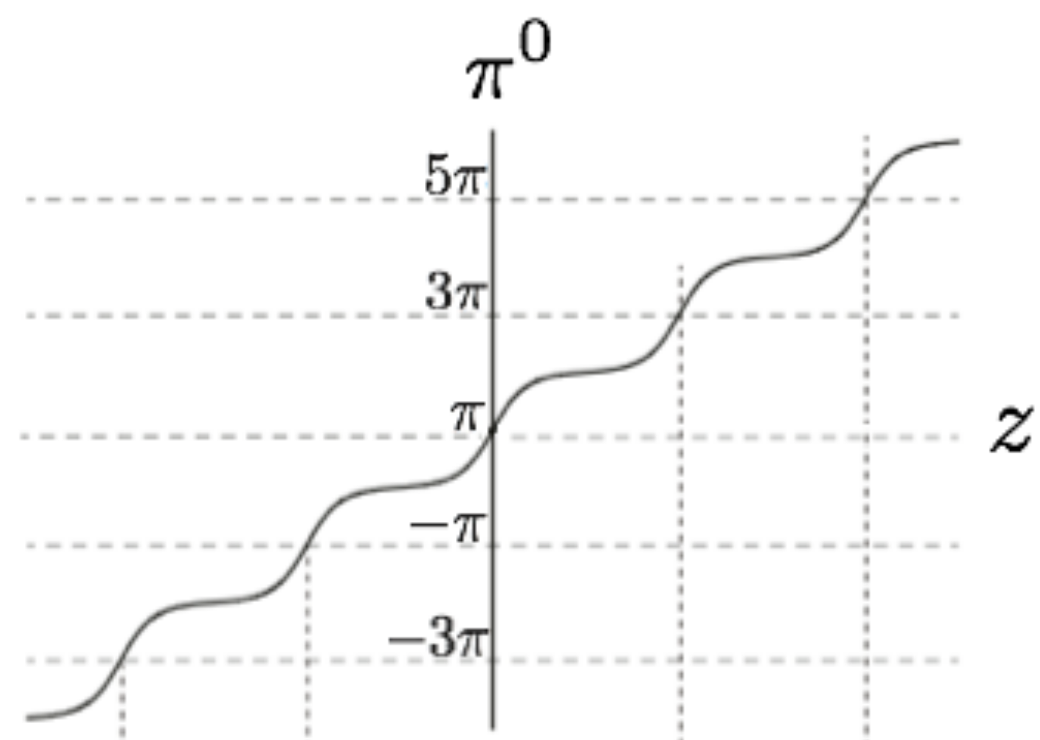
$$\cos\frac{\pi_0(\bar{z})}{2} = {\rm sn}(\bar{z}, k)\,, \quad \bar{z} \equiv \frac{z m_\pi}{k}$$



Topological charges

$$\mathcal{H} = \frac{f_\pi^2}{2}(\partial_z \pi_0)^2 - C\mu_{\mathrm{B}} B \partial_z \pi_0 - m_\pi^2 f_\pi^2 \cos \pi_0 + \mathrm{const.}$$

- CSL carries topological charge densities:
  - Baryon number: $n_{\mathrm{B}}(z) = -\dfrac{\partial \mathcal{H}}{\partial \mu_{\mathrm{B}}} = CB\partial_z \pi_0(z)$
  - Magnetization: $m(z) = -\dfrac{\partial \mathcal{H}}{\partial B} = C\mu_{\mathrm{B}} \partial_z \pi_0(z)$
  - Angular momentum: $j(z) = -\dfrac{\partial \mathcal{H}}{\partial \Omega} = 2C\mu_{\mathrm{B}}\mu_{\mathrm{I}} \partial_z \pi_0(z)$

Son, Stephanov, PRD (2008); Huang, Nishimura, Yamamoto, JHEP (2018)

Topological charges

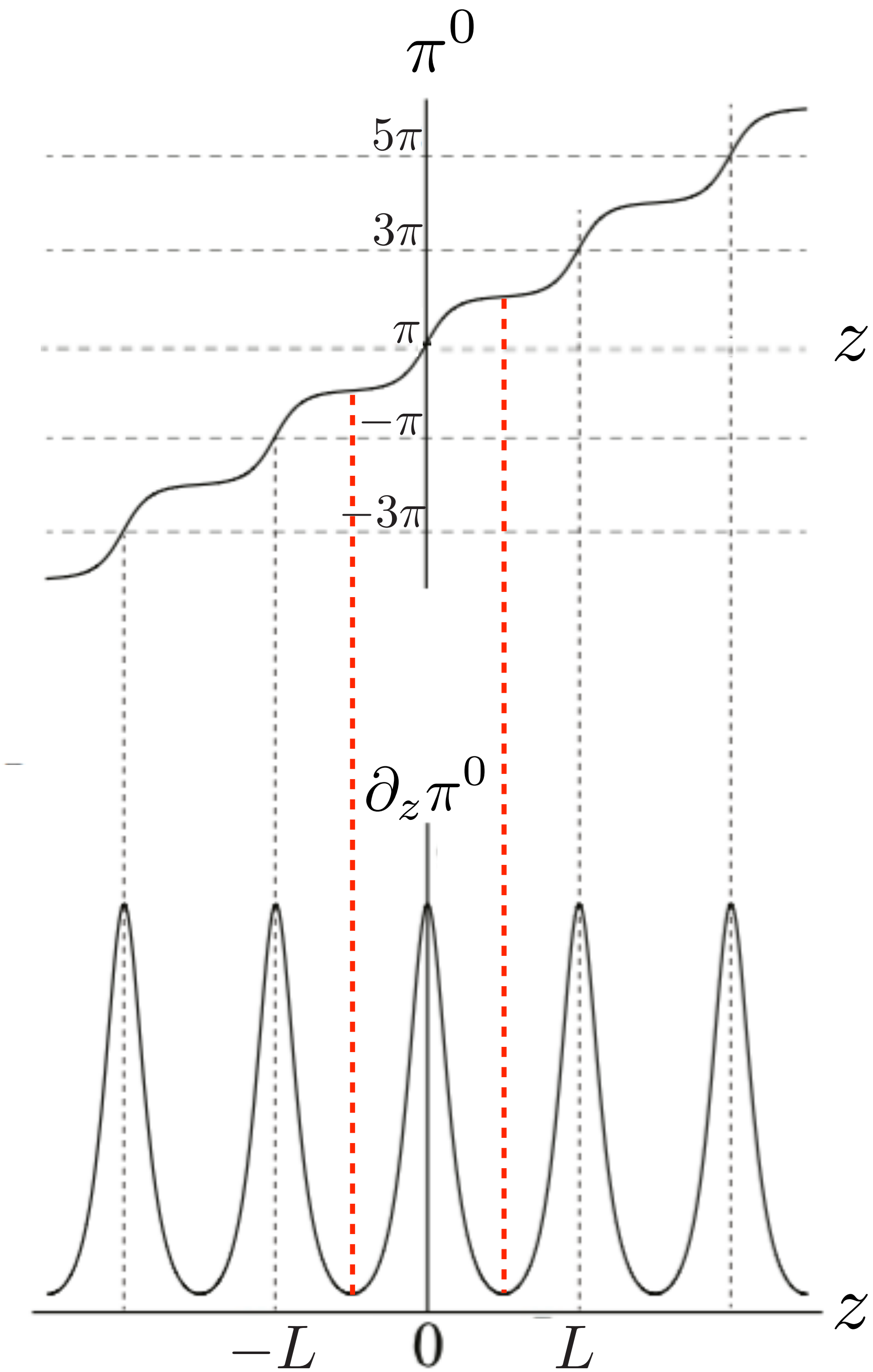
- Each lattice has baryon charge:

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} n_{\rm B}(z)\mathrm{d}z = CB\left[\underbrace{\pi_0\left(\frac{L}{2}\right)}_{2\pi} - \underbrace{\pi_0\left(-\frac{L}{2}\right)}_{0}\right] = \frac{B}{2\pi}$$

- Baryon number, magnetization, angular momentum

$$\frac{N_{\rm B}}{S} = \frac{B}{2\pi}, \qquad \frac{M}{S} = \frac{\mu_{\rm B}}{2\pi}, \qquad \frac{J}{S} = \frac{\mu_{\rm B}\mu_{\rm I}}{\pi}$$

independent of the detailed config. of π^0



Ground state

- Analytic solution for CSL:

$$\cos\frac{\pi^0(\bar{z})}{2} = \mathrm{sn}(\bar{z}, k)\,, \quad \bar{z} = \frac{z m_\pi}{k}$$

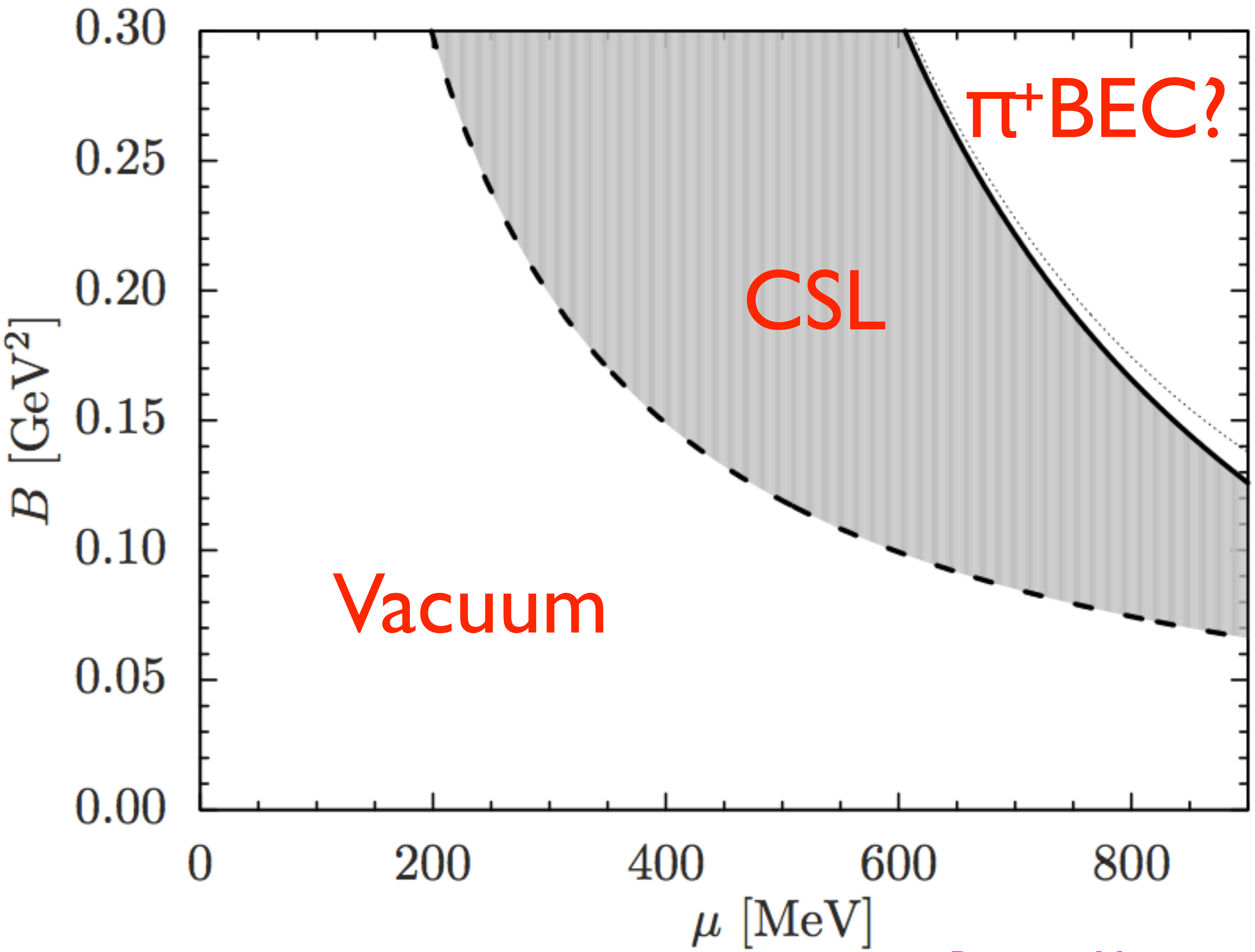
- CSL is favored than vacuum for $B > B_{\mathrm{CSL}}$ (Ω=0) or $\Omega > \Omega_{\mathrm{CSL}}$ (B=0)

$$\frac{E(k)}{k} = \frac{\mu_{\mathrm{B}} B}{16\pi m_\pi f_\pi^2} \qquad \therefore\ B_{\mathrm{CSL}} = \frac{16\pi m_\pi f_\pi^2}{\mu_{\mathrm{B}}}\,, \quad \Omega_{\mathrm{CSL}} = \frac{8\pi m_\pi f_\pi^2}{\mu_{\mathrm{B}}|\mu_{\mathrm{I}}|}$$

$$E(k) = \int_0^{\pi/2} \mathrm{d}\phi \sqrt{1 - k^2 \sin^2\phi}$$: complete elliptic integral of the 2nd kind

Brauner, Yamamoto, JHEP (2017); Huang, Nishimura, Yamamoto, JHEP (2018)

Phase diagram (T=0, Ω=0)



Brauner, Yamamoto, JHEP (2017)

Thermal fluctuations

- 1D lattice is typically destabilized by thermal fluctuations.

  Baym, Friman, Grinstein, NPB (1982); Hidaka, Kamikado, Kanazawa, Noumi, PRD (2015)

  - Dispersion of phonons: $\omega^2 = Ap_z^2 + Bp_\perp^4$
  - Thermal fluctuations $\sim \int_{\Lambda_{\mathrm{IR}}} p_\perp \mathrm{d}p_\perp \int \mathrm{d}p_z \frac{1}{Ap_z^2 + Bp_\perp^4} \sim \ln \Lambda_{\mathrm{IR}}$

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- With ***B***, explicit breaking of the rotational symmetry changes it to

$$\sim \int p_\perp \mathrm{d}p_\perp \int \mathrm{d}p_z \frac{1}{A'p_z^2 + B'p_\perp^2} = \mathrm{IR\ finite}$$

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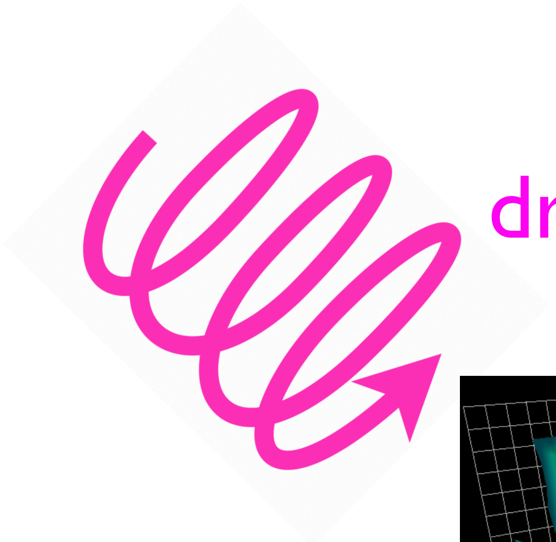
- CSL is favored over the normal phase by thermal fluct. of phonons.

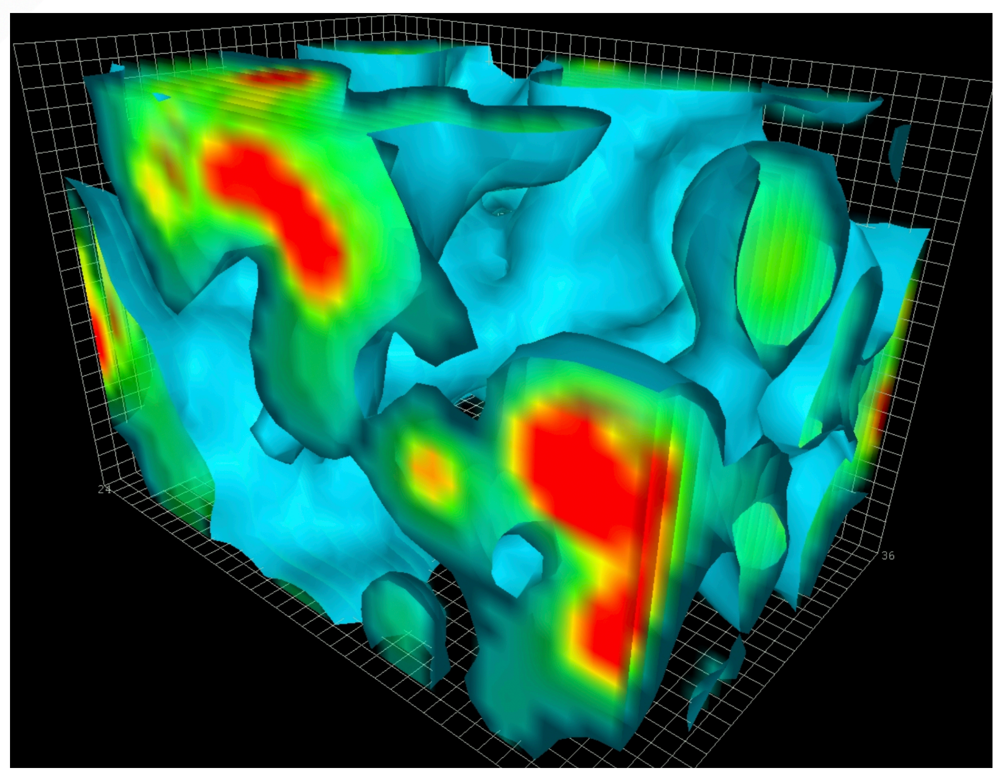
Brauner, Kolešová, Yamamoto, arXiv:2108.10044

Floquet vacuum engineering

Question

What happens to the QCD vacuum under a time-periodic circularly polarized laser with a sufficiently large intensity & frequency?





QCD vacuum D. Leinweber

Floquet theory

- Floquet theorem: temporal version of Bloch theorem for crystals

  For a Hamiltonian *H* with $H(t+T) = H(t)$,

$$|\Psi(t)\rangle = \mathrm{e}^{-\mathrm{i}\varepsilon t}|\Phi(t)\rangle, \quad |\Phi(t+T)\rangle = |\Phi(t)\rangle$$

- Discrete Fourier transform ($T = 2\pi/\Omega$):

$$H(t) = \sum_n \mathrm{e}^{-\mathrm{i}n\Omega t} H_n, \quad |\Phi(t)\rangle = \sum_n \mathrm{e}^{-\mathrm{i}n\Omega t}|\Phi_n\rangle$$

- Time-dep. *H* is mapped to a time-indep. *H* with infinite # of states (~Kaluza-Klein reduction for the S^1 time direction)

$$\sum_m (H_{m-n} - m\Omega\delta_{mn})|\Phi_n\rangle = \epsilon|\Phi_n\rangle$$

Shirley, PRB (1965); Sambe, PRA (1973)

Floquet Hamiltonian

- High-frequency expansion:

  e.g, Goldman, Dalibard, PRX (2014)

$$H_{\text{eff}} = H_{\text{eff}}^{(0)} + H_{\text{eff}}^{(1)} + O\left(\frac{1}{\Omega^2}\right),$$

$$H_{\text{eff}}^{(0)} = H_0, \quad H_{\text{eff}}^{(1)} = \frac{1}{\Omega}\sum_{n>0}\frac{[H_{-n}, H_n]}{n}$$

Floquet Hamiltonian

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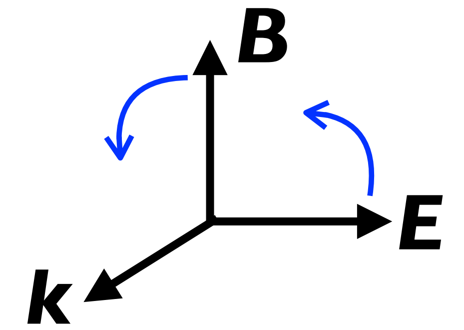
- Right-handed laser field: $\boldsymbol{A}(t,\boldsymbol{x}) = \frac{F}{\Omega}(\cos\Omega(t-z), \sin\Omega(t-z), 0)$



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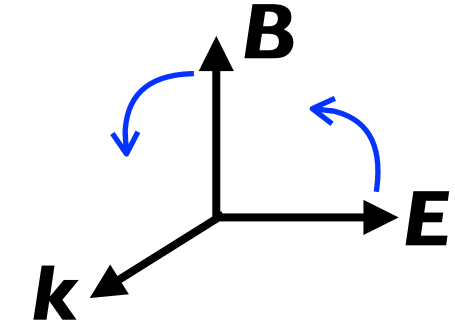


- For two-flavor QCD,

$$\mathcal{H}_{\mathrm{Floquet}} = -\frac{1}{\Omega^3}\sum_{\mathrm{f=u,d}}(eQ_{\mathrm{f}}F)^2 j_{\mathrm{f}}^{5z} = -\frac{e^2F^2}{\Omega^3}\left(\frac{1}{3}j_3^{5z} + \frac{5}{18}j^{5z}\right)$$

$j_3^{5z} = f_\pi^2 \partial_z \pi_0$ at IR

Laser-driven CSL

- Total effective Hamiltonian (in z direction):

$$\mathcal{H}^z_{\text{total}} = \frac{f_\pi^2}{2}(\partial_z \pi_0)^2 - \frac{e^2 F^2 f_\pi^2}{3\Omega^3}\partial_z \pi_0 - m_\pi^2 f_\pi^2 \cos \pi_0 + \text{const.}$$

- Ground state becomes CSL when $F \geq F_{\text{CSL}} \equiv \sqrt{\frac{12 m_\pi \Omega^3}{\pi e^2}}$

(For Ω~1 GeV, F_{CSL}~1 GeV^2)

QCD vacuum can be altered to CSL by a strong polarized laser
(Floquet vacuum engineering)

Yamada, Yamamoto, arXiv:2107.07074

Summary

- Chiral soliton lattice (CSL) is a novel state of matter that appears universally in cond-mat physics and QCD.
- QCD CSL has close a connection to anomaly & CME/CVE.
- 3-flavor QCD under rotation: η' CSL Nishimura, Yamamoto, JHEP (2020)
- Possible observations? Application to Weyl/Dirac semimetals

Backup slides

Chiral magnets

Kishine, Ovchinnikov, Solid State Phys. **66** (2015)

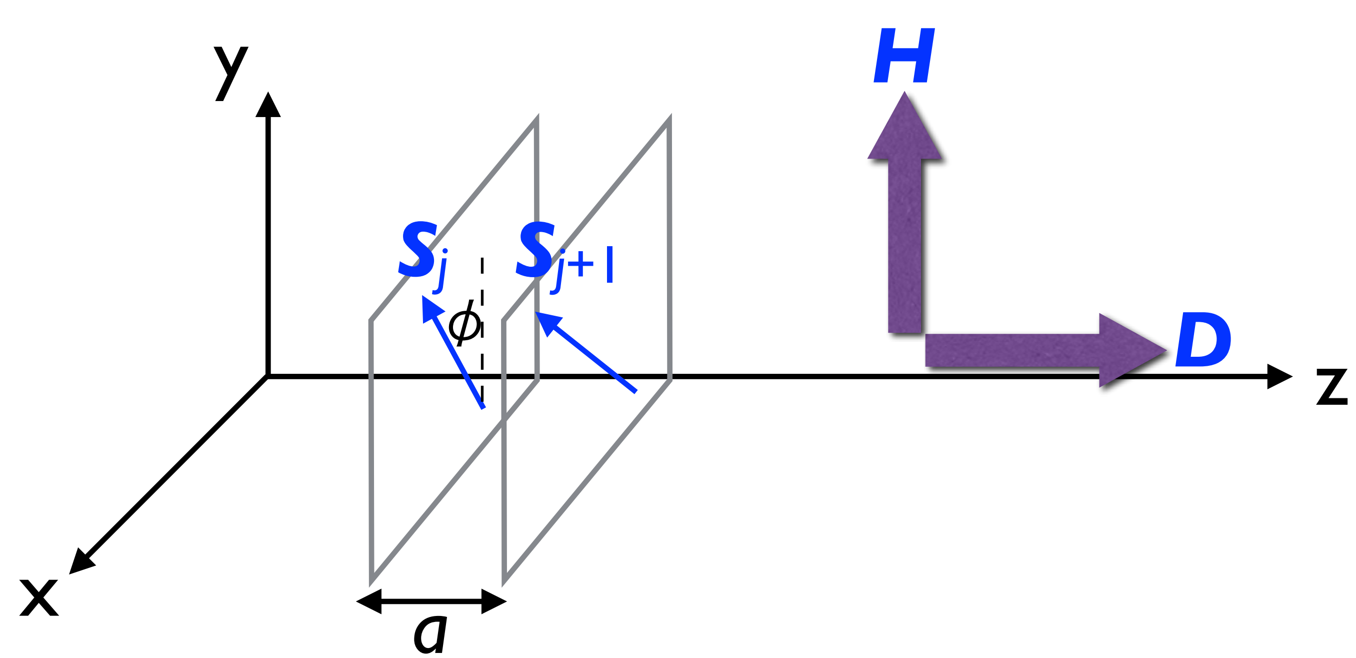
- Heisenberg Hamiltonian:

$$H_j = -J\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} - \boldsymbol{H} \cdot \boldsymbol{S}_j \simeq \frac{1}{2}JS^2a^2(\partial_z\phi)^2 - HS\cos\phi + \text{const.}$$

$$z = ja$$

- Dzyaloshinskii-Moriya (DM) interaction:

$$H_j^{\text{DM}} = \boldsymbol{D} \cdot (\boldsymbol{S}_j \times \boldsymbol{S}_{j+1}) \simeq -DS^2a\partial_z\phi$$



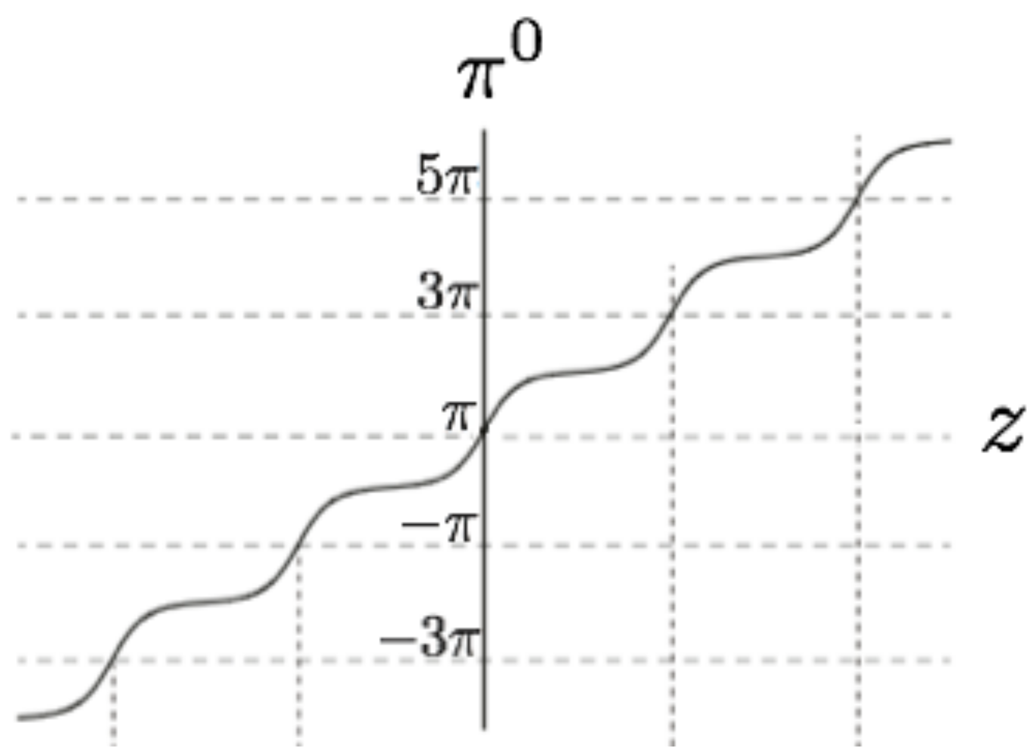
Nonrelativistic photons in the QCD CSL

Electro-magnetism in CSL

- Effective theory for electromagnetic fields:

$$\mathcal{L} = \frac{\varepsilon}{2}\boldsymbol{E}^2 - \frac{1}{2\mu}\boldsymbol{B}^2 + C\langle\pi^0\rangle\boldsymbol{E}\cdot\boldsymbol{B} - j^\mu A_\mu$$

CSL solution



Axion electrodynamics

- Modified Maxwell's equations:

$$\epsilon \boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho - C \langle \boldsymbol{\nabla} \pi^0 \rangle \cdot \boldsymbol{B},$$

$$\frac{1}{\mu} \boldsymbol{\nabla} \times \boldsymbol{B} = \epsilon \partial_t \boldsymbol{E} + \boldsymbol{j} + \underbrace{C \langle \boldsymbol{\nabla} \pi^0 \rangle \times \boldsymbol{E}}_{\text{Anomalous Hall effect}}$$

Dispersion relations of photons are modified

Non-relativistic photons

- For helicity +1, $\omega \sim \dfrac{f_\pi^2}{\mu_{\rm B} B_{\rm ex}} k^2$: non-relativistic gapless photon
- For helicity -1, $\omega \sim \dfrac{\mu_{\rm B} B_{\rm ex}}{f_\pi^2}$: gapped photon

Yamamoto, PRD (2016); Ozaki, Yamamoto, JHEP (2017);
Qiu, Cao, Huang, PRD (2017); Brauner, Kadam, JHEP (2017)

Non-relativistic photon = type-B NG mode of 1-form symmetry

see also Sogabe, Yamamoto, PRD (2019); Hidaka, Yokokura, Hirono, PRL (2021)