Exceptional topological superconductors and their Floquet analog

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Acknowledgements: A. K. Ghosh (Uppsala University, Sweden)

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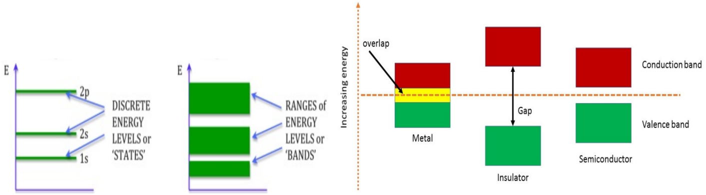
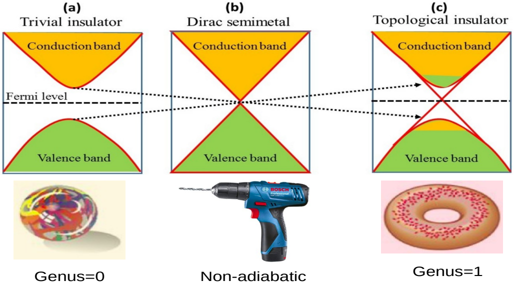
Outline

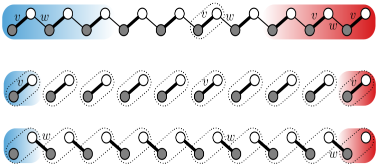
1. Introduction and motivation
   - Topologial (higher-order) systems and non-Hermitian topology
   - Periodic dynamics and its consequences
2. Question and its answer
   - Exceptional topological superconductors and their Floquet analog (PRB **106**, L140303 (2022))
3. Conclusions and experimental relevance

Understanding the basic band structure

Ind. J. Phys. **95**, 2639 (2021)



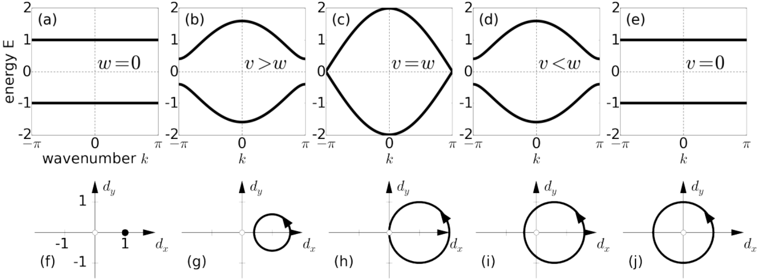




SSH Hamiltonian

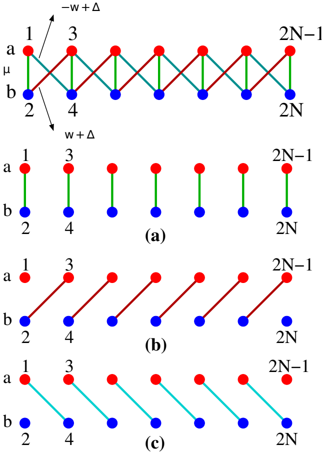
$$H = \begin{pmatrix} 0 & v + we^{ik} \\ v + we^{-ik} & 0 \end{pmatrix}$$
$$= d_x\sigma_x + d_y\sigma_y$$

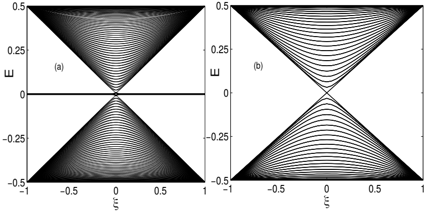
with $d_x = v + w\cos k$, $d_y = w\sin k$

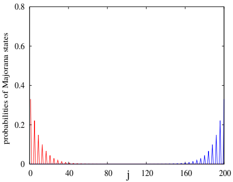


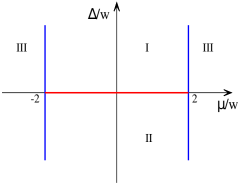
Topological superconductor in 1D: *p*-wave spinless Majorana chain Physics-Uspekhi **44**, 131 (2001)

p-wave Hamiltonian: $H = \frac{i}{2}\sum_{j=1}^{N-1}[(-w+\Delta)a_j b_{j+1} + (w+\Delta)b_j a_{j+1}] - \frac{i}{2}\sum_{j=1}^{N}\mu a_j b_j$

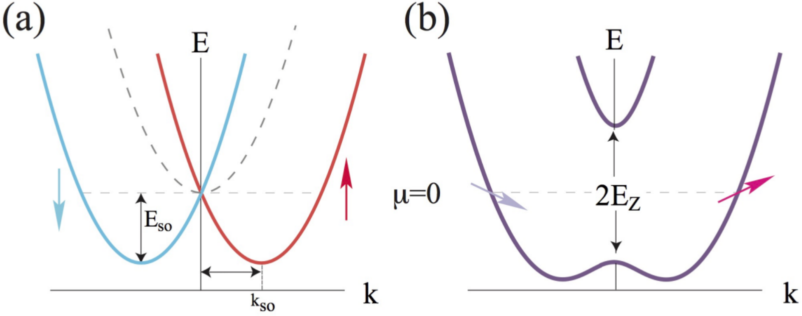






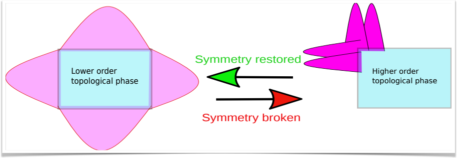


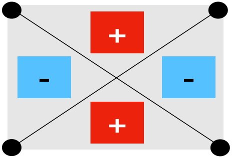
Proximity induced topological superconductor PRL **100**, 096407 (2008), PRL **104**, 040502 (2010), PRB **88**, 155420 (2013)

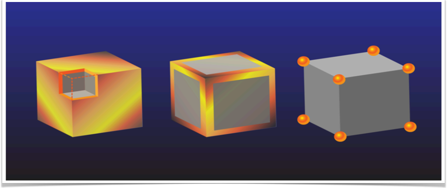


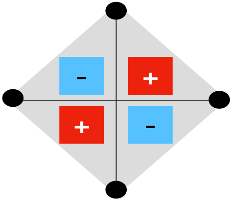
- *p*-wave Kitaev chain
- Rashba nanowire ($k_x^2 + k_x\sigma_y$) + *s*-wave proximed SC + magnetic field ($E_z\sigma_x$) ⇒ emergent *p*-wave TSC in 1D
- Surface of TI + proximity induced *s*-wave SC + magnetic field ⇒ chiral TSC in 2D
- Helical spin chain + proximity induced *s*-wave SC ⇒ emergent *p*-wave TSC

What is a higher-order topological (HOT) phase? Science **357**, 61 (2017); PRB **96**, 245115 (2017)









- $n^{\rm th}$-order topological phase in d dimensions is characterized by the existence of $n_c = (d - n)$-dimensional boundary modes
- Mutually anticommuting matrices with discrete Wilson-Dirac masses

Let's start with a 2D Hamiltonian PRB **100**, 115403 (2019)

HOTI Hamiltonian: QSHI and perturbation

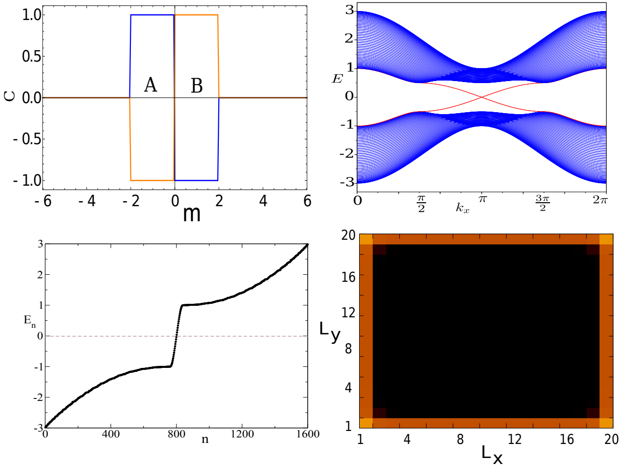
$$H_{\mathrm{QSHI}} = t_1 \sum_{j=1}^{2} \Gamma_j \sin k_j - t_0 \Gamma_3 \left[m - \sum_{j=1}^{2} \cos k_j \right]$$

$$V = \Delta \left(\cos k_x - \cos k_y \right), \quad H_{\mathrm{HOTI}} = H_{\mathrm{QSHI}} + \mathbf{V\Gamma_4}$$

with $\Gamma_1 = \sigma_3 \tau_1$, $\Gamma_2 = \sigma_0 \tau_2$, $\Gamma_3 = \sigma_0 \tau_3$, $\Gamma_4 = \sigma_1 \tau_1$ and $(\boldsymbol{\tau}, \boldsymbol{\sigma}) \in$ (sublattice, spin)

- Connecting to construction method: $t_1 \sum_{j=1}^{2} \Gamma_j \sin k_j$ in H_{QSHI} represent $m = 2$ terms $\longrightarrow$ $t_0 \Gamma_3 \left[m - \sum_{j=1}^{2} \cos k_j \right]$ is $p = 1$ term as first-order mass yielding $\underline{n = 1\text{-order regular topological phase}}$ $\longrightarrow$ $V\Gamma_4$ is $p = 2$ term as Wilson-Dirac mass yielding $\underline{n = 2\text{-order HOT phase}}$

First-order topological phases PRB **100**, 115403 (2019)



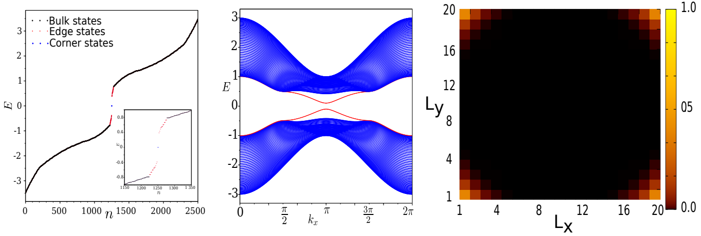
- Gap vanishes at $\boldsymbol{k} = (0,0)$, (π,π) when $m = \pm 2$ and $\boldsymbol{k} = (\pi,0)$, $(0,\pi)$ when $m = 0$.

HOTI Hamiltonian: QSHI and perturbation

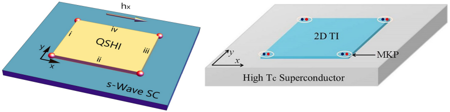
$$H_{\rm QSHI} = t_1 \sum_{j=1}^{2} \Gamma_j \sin k_j - t_0 \Gamma_3 \left[m - \sum_{j=1}^{2} \cos k_j \right]$$

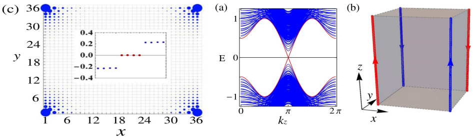
$$V = \Delta \left(\cos k_x - \cos k_y \right), \quad H_{\rm HOTI} = H_{\rm QSHI} + \mathbf{V}\boldsymbol{\Gamma}_4$$

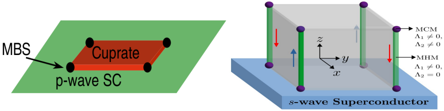
with $\Gamma_1 = \sigma_3\tau_1$, $\Gamma_2 = \sigma_0\tau_2$, $\Gamma_3 = \sigma_0\tau_3$, $\Gamma_4 = \sigma_1\tau_1$ and $(\boldsymbol{\tau}, \boldsymbol{\sigma}) \in$ (sublattice, spin)

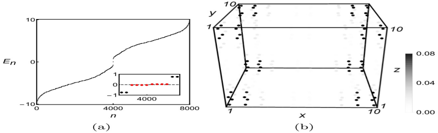


Higher-order topological superconductor (HOTSC) PRB **98**, 165144 (2018), PRL **124**, 227001 (2020), PRB **104**, 134508 (2021), PRB **104**, L180503 (2021)

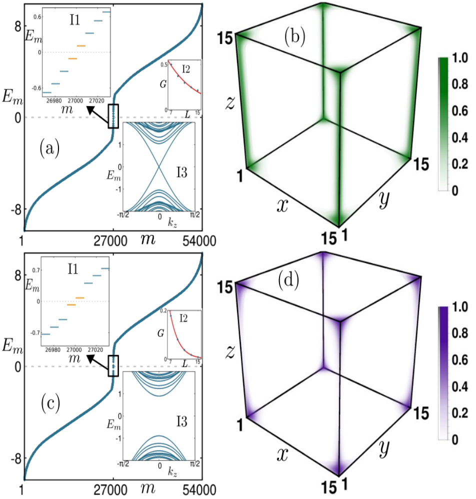








- QSHI with SOC + proximity induced *s*-wave SC + uniaxial magnetic field ⇒ corner modes in 2D (TRS broken)
- QSHI with SOC + proximity induced *d*-wave SC ⇒ Corner modes in 2D (TRS preserved)
- Hopping + $p + id$-wave SC ⇒ Corner modes in 2D ($C_4\mathcal{T}$ preserved)
- 3D TI + proximity induced *s*-wave SC + magnetic field + Wilson-Dirac mass terms ⇒ hinge and corner modes in 3D
- Hopping + SOC+ $d_1 + id_2$-wave SC ⇒ Corner modes in 3D



- Second-order Wilsonian mass term: $d_{x^2-y^2}$ SC gap term $\Delta_1(\cos k_x - \cos k_y)\tau_x$ (8 MHMs) **or** $\Lambda_1(\cos k_x - \cos k_y)\mu_x\sigma_y$ (16 MHMs) in the TI + s-wave proximed SC

- Third-order Wilsonian mass term: $d_{3z^2-r^2}$ SC gap term $\Delta_2(2\cos k_z - \cos k_x - \cos k_y)\tau_y$ (8 MCMs) **or** $\Lambda_1(2\cos k_z - \cos k_x - \cos k_y)\mu_z$ (16 MCMs) in the TI + s-wave proximed SC

Microscopics to an effective non-Hermitian model

- Explicit presence of gain and loss terms (optical meta-materials)
- Interacting systems (electron-electron, electron-phonon) leading to quasiparticles of finite lifetime i.e., $H \to H + \Sigma$ where self-energy Σ is complex
- Quantum systems in contact with reservoir (Lindblad master equation):

$$\boxed{\partial_t \rho(t) = \mathcal{L}_t \rho(t) = -i[H, \rho(t)] + \sum_j (L_j \rho(t) L_j^\dagger - \frac{1}{2}\{L_j L_j^\dagger, \rho(t)\})}$$

Redefining the Hamiltonian $H_e = H - i \sum j L_j^\dagger L_j$ leads to

$$\boxed{\partial_t \rho(t) = \mathcal{L}_t \rho(t) = -i[H_e \rho(t) - \rho(t) H_e^\dagger] + \sum_j L_j \rho(t) L_j^\dagger}$$ where $\sum_j L_j \rho(t) L_j^\dagger$ is the jump term

Non-Hermitian Hamiltonian: $H \neq H^\dagger$

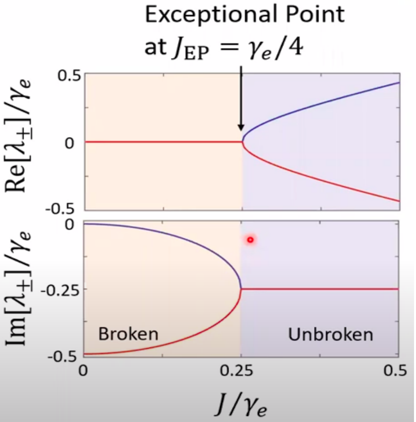
$$H|\psi_n^R\rangle = E|\psi_n^R\rangle,\ \langle\psi_n^L|H = E\langle\psi_n^L| \Rightarrow H^\dagger|\psi_n^L\rangle = E^*|\psi_n^L\rangle$$

$$H_{\mathrm{eff}} = \begin{pmatrix} 0 & J \\ J & -i\frac{\gamma_e}{2} \end{pmatrix}$$

Eigenvalues & eigenvectors

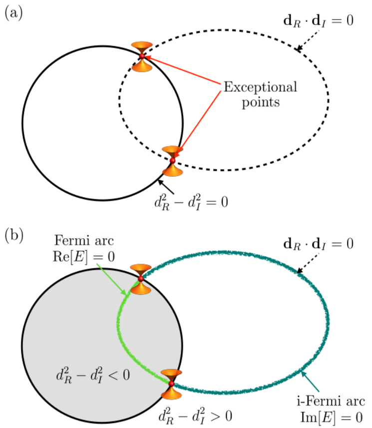
$$\lambda_\pm = -i\frac{\gamma_e}{4} \pm \sqrt{J^2 - \frac{\gamma_e^2}{16}}$$

$$|\psi_\pm\rangle \propto \begin{pmatrix} \lambda_\pm \\ J \end{pmatrix}$$



$$H = J\sigma_x + \frac{i\gamma_e}{4}\sigma_z - \alpha\frac{i\gamma_e}{4}\sigma_0 \rightarrow \text{PT-symmetric if } \alpha = 0$$

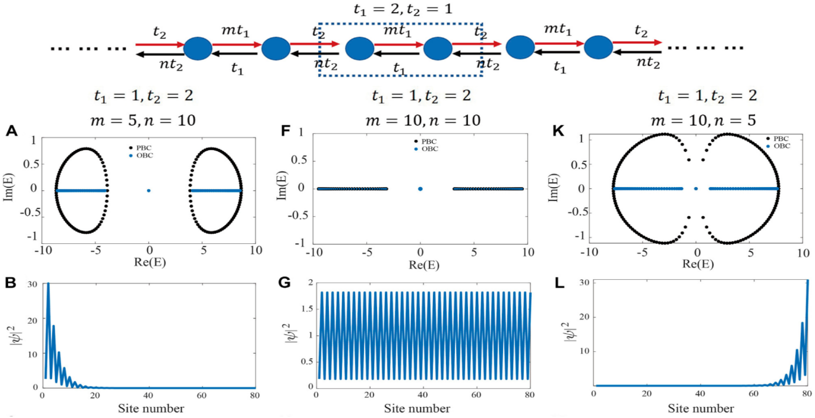
Non-Hermitian degeneracies RMP **93**, 15005 (2021)



- Two-level Hamiltonian $H(k) = \boldsymbol{d}(k).\boldsymbol{\sigma} + d_0(k)\sigma_0$ with $\boldsymbol{d} = \boldsymbol{d}_R + i\boldsymbol{d}_I$
- Hermitian case with $\boldsymbol{d}_I = 0$: $E_\pm = \pm\sqrt{d_x^2 + d_y^2 + d_z^2}$ vanishes for $d_x = 0, d_y = 0, d_z = 0 \rightarrow$ three constraints for a two band crossing
- Non-Hermitian case with $\boldsymbol{d}_I \neq 0$: Complex-energy spectrum $E_\pm = d_0 \pm \sqrt{d_R^2 - d_I^2 + 2i\boldsymbol{d}_R \cdot \boldsymbol{d}_I}$
- Degeneracies occur more frequently when $d_R^2 - d_I^2 = 0$ and $\boldsymbol{d}_R \cdot \boldsymbol{d}_I = 0 \rightarrow$ two constraints for a two band crossing

Modified SSH model:

$$H = \begin{pmatrix} 0 & t_1 + t_2 e^{ik} \\ m t_1 + n t_2 e^{-ik} & 0 \end{pmatrix}$$



Main questions

- Can we engineer non-Hermitian exceptional HOTSC phase **using** d-**wave paring?**
- How is the bulk boundary correspondence modified? How to characterize the MZMs?
- How can one engineer the anomalous Floquet HOTSC phase for the NH case?

Hermitian HOTSC model PRB **105**, 155406 (2022)

Underlying model for our problem

$$H_{\rm dD}(\boldsymbol{k}) = \begin{pmatrix} H_{\rm TI}^{\rm dD}(\boldsymbol{k}) - \mu & \Delta \\ \Delta^* & \mu - \mathcal{T}^{-1} H_{\rm TI}^{\rm dD}(-\boldsymbol{k}) \mathcal{T} \end{pmatrix}$$

where Pauli matrices $(\boldsymbol{\sigma}, \boldsymbol{s}, \boldsymbol{\tau})$ operate on the (sublattice/orbital, spin, particle-hole) degrees of freedom

- For 2D: $H_{\rm TI}^{\rm 2D} = (m - \cos k_x - \cos k_y)\sigma_z + \sin k_x \sigma_x s_z + \sin k_y \sigma_y$ preserving TRS $\mathcal{T} = i s_y \mathcal{K}$, $M_x = \sigma_0 s_x$ and $M_y = \sigma_z s_x$ and $\Delta = \Delta(\cos k_x - \cos k_y)\sigma_0 s_0$, PHS: $\mathcal{P} = \tau_y \sigma_x s_z \mathcal{K}$, CS: $\mathcal{C} = \tau_z \sigma_x s_y$
- For 3D: $H_{\rm TI}^{\rm 3D} = (m - \cos k_x - \cos k_y - \cos k_z)\sigma_z + \sin k_x \sigma_x s_x + \sin k_y \sigma_x s_y + \sin k_z \sigma_x s_z$ preserving TRS $\mathcal{T} = i s_y \mathcal{K}$, $M_x = \sigma_z s_x$, $M_y = \sigma_z s_y$, $M_z = \sigma_z s_z$ and $\Delta = \Delta_1(\cos k_x - \cos k_y)\sigma_0 s_0 + i\Delta_2(2\cos k_z - \cos k_x - \cos k_y)\sigma_0 s_0$, PHS: $\mathcal{P} = \tau_y \sigma_y s_y \mathcal{K}$, CS: $\mathcal{C} = \tau_z \sigma_y$

Underlying model for our problem

$$\mathcal{H}(\mathbf{k}) = \begin{pmatrix} H_{\rm TI}(\mathbf{k}) - \mu & \Delta(\cos k_x - \cos k_y)\sigma_0 s_0 \\ \Delta(\cos k_x - \cos k_y)\sigma_0 s_0 & \mu - U_{\mathcal{T}}^{-1} H_{\rm TI}^{*}(-\mathbf{k}) U_{\mathcal{T}} \end{pmatrix},$$

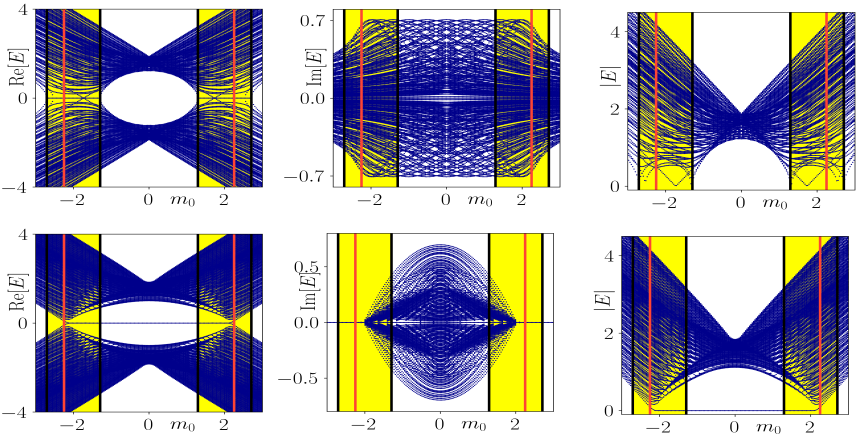
- Ingredients: $H_{\rm TI}(\mathbf{k})=$ $(\lambda_x \sin k_x + i\gamma_x)\sigma_x s_z + (\lambda_y \sin k_y + i\gamma_y)\sigma_y s_0 + (m_0 - t_x \cos k_x - t_y \cos k_y)\sigma_z s_0 =$ $H_{\rm TI}^{\rm H}(\mathbf{k}) + i\gamma_x \sigma_x s_z + i\gamma_y \sigma_y s_0$
- $\mathcal{H}(\mathbf{k})$ preserves ramified (time-reversal symmetry) TRS: $U_{\mathcal{T}} \mathcal{H}_{\rm TI}^{*}(\mathbf{k}) U_{\mathcal{T}}^{-1} = \mathcal{H}_{\rm TI}(-\mathbf{k})$ with $U_{\mathcal{T}} = \sigma_0 s_y$
- $\mathcal{H}(\mathbf{k})$ preserves ramified (particle-hole symmetry) $\text{PHS}^{\dagger}$: $U_{\mathcal{C}} \mathcal{H}_{\rm TI}^{*}(\mathbf{k}) U_{\mathcal{C}}^{-1} = -\mathcal{H}_{\rm TI}(-\mathbf{k})$ with $U_{\mathcal{C}} = \sigma_x s_0$
- $H_{\rm TI}^{\rm H}(\mathbf{k})$ preserves TRS $\mathcal{T} = iU_{\mathcal{T}}\mathcal{K}$ and PHS $\mathcal{C} = U_{\mathcal{C}}\mathcal{K}$

The model and crystalline symmetries

The compact form $\mathcal{H}(\mathbf{k}) = \mathbf{N} \cdot \mathbf{\Gamma}$; where, $\mathbf{N} = \{\lambda_x \sin k_x + i\gamma_x, \lambda_y \sin k_y + i\gamma_y, m_0 - t_x \cos k_x - t_y \cos k_y, \Delta(\cos k_x - \cos k_y)\}$, $\mathbf{\Gamma} = \{\tau_z\sigma_x s_z, \tau_z\sigma_y s_0, \tau_z\sigma_z s_0, \tau_x\sigma_0 s_0\}$ with the Pauli matrices $\boldsymbol{\tau}$, $\boldsymbol{\sigma}$, and $\boldsymbol{s}$ act on PH (e, h), orbital (α, β), and spin $(\uparrow, \downarrow)$ degrees of freedom, respectively.

- For $t_x = t_y$, $\lambda_x = \lambda_y$, and $|\gamma_x| = |\gamma_y| \neq 0$, $\mathcal{H}(\mathbf{k})$ breaks the following crystalline symmetries four-fold rotation with respect to z, $C_4 = \tau_z e^{-\frac{i\pi}{4}\sigma_z s_z}$, mirror-reflection along x, $\mathcal{M}_x = \tau_x\sigma_x s_0$ and mirror-reflection along y, $\mathcal{M}_y = \tau_x\sigma_y s_0$ while $\mathcal{H}^{\rm H}(\mathbf{k})$ respects the above symmetries
- $\mathcal{H}(\mathbf{k})$ and $\mathcal{H}^{\rm H}(\mathbf{k})$ both preserve mirror-rotation I $\mathcal{M}_{xy} = C_4\mathcal{M}_y$ for $\gamma_x = \gamma_y \neq 0$ $[\mathcal{M}_{xy}\mathcal{H}(k_x, k_y)\mathcal{M}_{xy}^{-1} = \mathcal{H}(k_y, k_x)]$, and mirror-rotation II $\mathcal{M}_{x\bar{y}} = C_4\mathcal{M}_x$ for $\pm\gamma_x = \mp\gamma_y \neq 0$ $[\mathcal{M}_{x\bar{y}}\mathcal{H}(k_x, k_y)\mathcal{M}_{x\bar{y}}^{-1} = \mathcal{H}(-k_y, -k_x)]$, sublattice/ chiral symmetry $\mathcal{S} = \tau_y\sigma_0 s_0$ $[\mathcal{S}\mathcal{H}(\mathbf{k})\mathcal{S}^{-1} = -\mathcal{H}(\mathbf{k})]$.

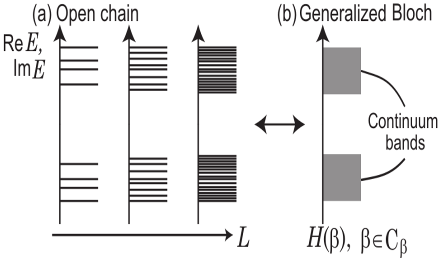
Energy spectrum with PBC and OBC

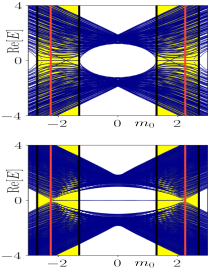


The EPs $m_0^{s,\pm} = s(t_x + t_y) \pm \sqrt{\gamma_x^2 + \gamma_y^2}$ for $(k_x, k_y) = (0,0)$ and (π, π) with $s = \pm$ are marked by black lines within which $\mathrm{Re}[E(k)]$ associated with $\mathcal{H}(\mathbf{k})$ remains gapless as designated by yellow-shaded region

Gap closing lines PRL **123**, 066404 (2019)

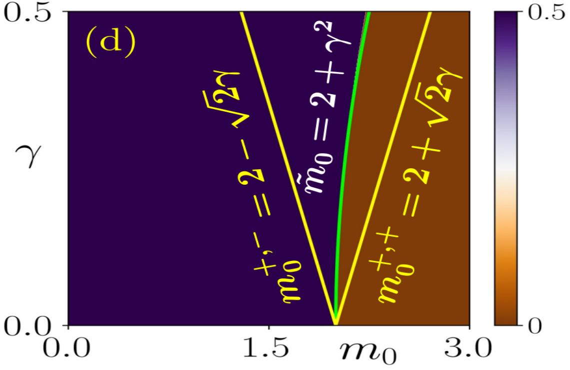
How is the bulk boundary correspondence modified?





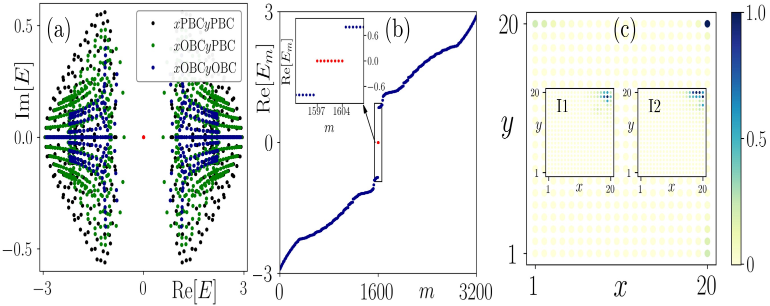
- Hermitian $\mathcal{H}^{\mathrm{H}}(\mathbf{k})$: topologial phase appears $m_0 < |t_x + t_y|$, trivial phase appears $m_0 > |t_x + t_y|$
- Non-Hermitian $\mathcal{H}(\mathbf{k})$: $m_0^{s,\pm} = s(t_x + t_y) \pm \sqrt{\gamma_x^2 + \gamma_y^2} \rightarrow$ topogical gapped phase from PBC $m_0^{-,+} < m_0 < m_0^{+,-}$
- Non-Bloch momentum $\mathbf{k} \rightarrow \mathbf{k'} + i\beta$ with $\beta_i = \gamma_i/\lambda_i$ $(i = x, y)$ to get satisfy bulk-boundary correspondence for small γ
- Topological gapped phase from OBC sustains for $m > m_0^{+,-}$ and $m < m_0^{-,+}$

Can we engineer non-Hermitian exceptional HOTSC phase?



- The topologial phase (PBC and OBC) boundary with Non-Bloch momentum $\mathbf{k} \to \mathbf{k'} + i\boldsymbol{\beta}$ is $m = \pm(t_x + t_y + \gamma_x^2/2\lambda_x^2 + \gamma_y^2/2\lambda_y^2)$
- Non-Hermiticity induced topological phase that exists beyond $m = \pm(t_x + t_y)$.

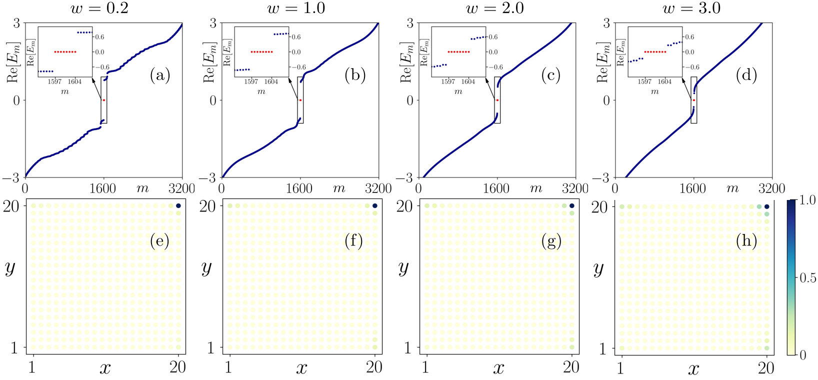
Another consequence of non-Bloch momentum: Skin modes



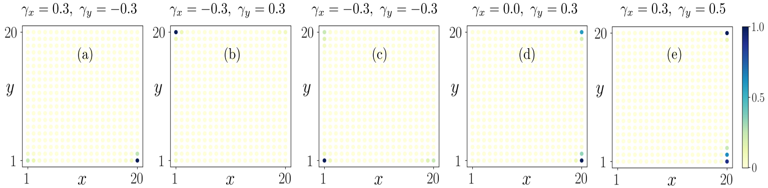
- Non-Bloch form of momentum in non-Hermitian system leads to skin modes which are otherwise Bloch band for Hermitian system

Disorder stability

- The onsite disorder potential of the form $V(i,j) = \sum_{i,j} V_{ij}\Gamma_3$ that preserves the chiral and mirror-rotation I symmetry. Here, V_{ij} is randomly distributed in the range $V_{ij} \in \left[-\frac{w}{2}, \frac{w}{2}\right]$



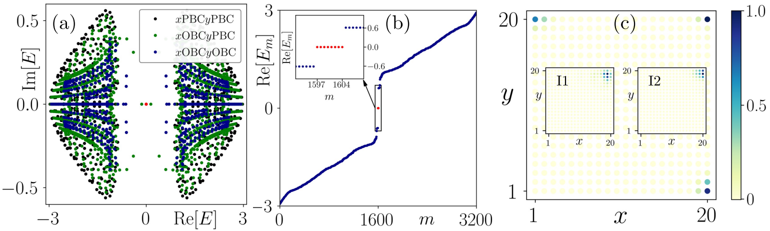
Breaking of mirror-rotation symmetries and localization at multiple corners



- Mirror rotation I with $\mathcal{M}_{xy} = C_4\mathcal{M}_y$ constraints single corner localization: $\mathcal{M}_{xy}\mathcal{H}(k_x, k_y)\mathcal{M}_{xy}^{-1} = \mathcal{H}(k_y, k_x)$, if $\gamma_x = \gamma_y \neq 0$ while $t_x = t_y$ and $\lambda_x = \lambda_y$
- Mirror rotation II with $\mathcal{M}_{x\bar{y}} = C_4\mathcal{M}_x$ constraints single corner localization: $\mathcal{M}_{x\bar{y}}\mathcal{H}(k_x, k_y)\mathcal{M}_{x\bar{y}}^{-1} = \mathcal{H}(-k_y, -k_x)$, if $\pm\gamma_x = \mp\gamma_y \neq 0$ while $t_x = t_y$ and $\lambda_x = \lambda_y$,
- Sublattice/ chiral symmetry with $\mathcal{S} = \tau_y\sigma_0 s_0$ is preserved: $\mathcal{S}\mathcal{H}(\mathbf{k})\mathcal{S}^{-1} = -\mathcal{H}(\mathbf{k})$.

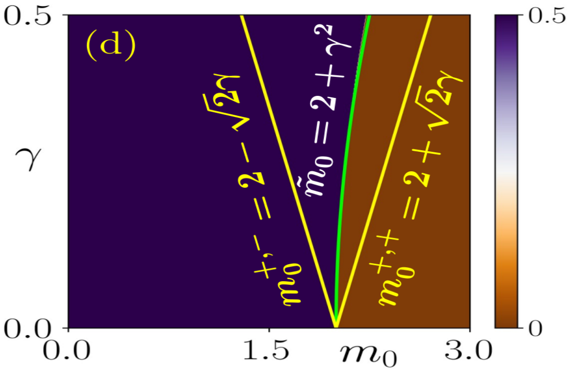
s-wave NH HOTSC

- The Hamiltonian is given by $\mathcal{H}(\mathbf{k}) = \mathbf{N}\cdot\mathbf{\Gamma}$; where, $\mathbf{N} = \{\lambda_x \sin k_x + i\gamma_x, \lambda_y \sin k_y + i\gamma_y, m_0 - t_x \cos k_x - t_y \cos k_y, \Delta_s, \Lambda(\cos k_x - \cos k_y)\}$, $\mathbf{\Gamma} = \{\tau_z\sigma_x s_z, \tau_z\sigma_y s_0, \tau_z\sigma_z s_0, \tau_x\sigma_0 s_0, \tau_0\sigma_x s_x\}$. The last term proportional to Λ represents C_4 symmetry breaking Wilson-Dirac mass term.



Topologial invariant: bi-orthogolalized nested polarization

How to characterize the MZMs?



Wannier center, polarization and Wilson loop PRB **47**, 1651(R) (1993), PRB **26**, 4269 (2016)

- Projection of the position operator in the occupied subspace $\Rightarrow$ Polarization $P_x = e\sum_{n=1}^{N}\langle W_n(j)|x|W_n(j)\rangle$ corresponds to sum of the Wannier centers of the occupied bands where $W_n(r-R_j)$ denotes the Wannier functions $\Rightarrow$ $= -\mathrm{Im}[\ln\Pi_{j=0}^{J-1}\langle u_n(k_j)|u_n(k_{j+1})\rangle]/(2\pi) + j$

- $W_\alpha = F_{\mathbf{k}+N\Delta k_\alpha}\ldots F_{\alpha,\mathbf{k}}$ with $[F_{\alpha,\mathbf{k}}]_{mn} = \left\langle u^m_{\mathbf{k}+\Delta k_\alpha}\middle| u^n_{\mathbf{k}}\right\rangle$, Wilson loop is unitary, eigenstates depend on the base point k, eigenvalues do not

- **Projected position operator using the Bloch functions (eigenvalues) $\Rightarrow$ Polarization (Wannier center)**, Polarization (Berry phase in the occupied sub-space) $p_x = -\frac{i}{2\pi}\mathrm{Log\ Det}\,[W_{k+2\pi\leftarrow k}]$

- Polarization $\leftrightarrow$ Wannier center $\leftrightarrow$ Wilson loop $\leftrightarrow$ Berry phase

- Example of FOT phase: $p_x = 1/2$ (mod 1) for 1D SSH model

Construction of static nth-order/ (nested)^n Wilson loops

PRB **96**, 245115 (2017)

Equlibrium or Floquet stroboscopic and insensitive to gap

The position operator $\hat{x}$ is **projected onto occupied subspace** $\rightarrow$ **first-order Wilson loop** (**Wannier bands and values**) and first-order polarization

$\Downarrow$

The position operator $\hat{y}$ is **projected onto subspace associated with first-order Wannier bands** $\rightarrow$ **second-order Wilson loop** and second-order polarization

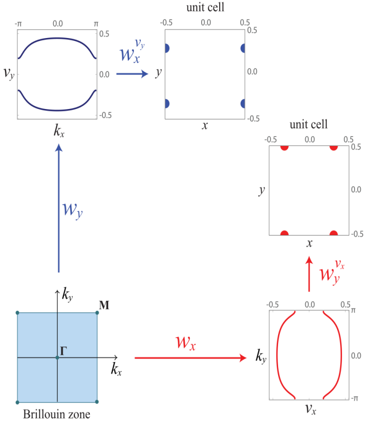
$\Downarrow$

The position operator $\hat{z}$ is **projected onto subspace associated with second-order Wannier bands** $\rightarrow$ **third-order Wilson loop** and third-order polarization

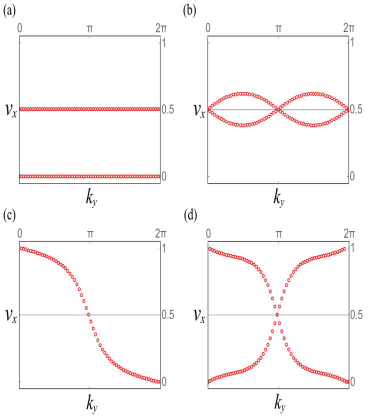
- FSOTSC in 2D: $p_y^{\pm\nu_x} = \frac{1}{N_x} \sum_{k_x} \nu_y^{\pm\nu_x}(k_x) = 1/2(\text{mod } 1)$ while first-order is gapped
- FTOTSC in 3D: $p_z^{\pm\nu_y^{\pm\nu_x}} = \frac{1}{N_x N_y} \sum_{k_x,k_y} \nu_z^{\pm\nu_y^{\pm\nu_x}} = 1/2(\text{mod } 1)$ while first-order and second-order are gapped out

Comparison with first-order PRB **96**, 245115 (2017)

Second-order Wilson loop: SOT



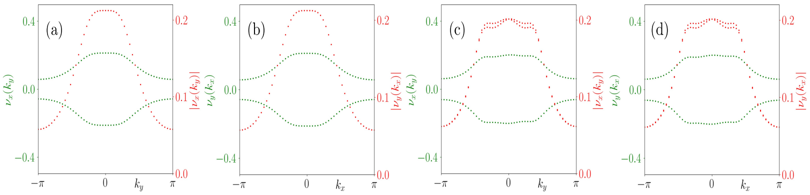
First-order Wilson loop: FOT



Bi-orthogonalized version for NH system

- The bi-orthogonalization guarantees $\sum_n |\Psi_n^{\rm R}(\mathbf{k'})\rangle\langle\Psi_n^{\rm L}(\mathbf{k'})| = \mathbb{I}$ and $\langle\Psi_n^{\rm L}(\mathbf{k'})|\Psi_n^{\rm R}(\mathbf{k'})\rangle = \delta_{mn}$; where, n runs over all the energy levels irrespective of their occupations.
- The non-Bloch form of the momentum $\mathbf{k'} \to \mathbf{k'}$ in $\mathcal{H}(\mathbf{k})$ i.e., $\mathcal{H}(\mathbf{k}) \to \mathcal{H}'(\mathbf{k'})$.
- Polarization along x $\to$ first-order Wilson loop $\to$ Wannier Hamiltonian $\log W_{x,\mathbf{k'}}$ $\to$ Wannier spectrum $\pm\nu_x$ and Wannier functions $|\nu_{x,\mu}^{\rm R}(\mathbf{k'})\rangle$ and $\langle\nu_{x,\mu}^{\rm L}(\mathbf{k'})|$
- Polarization along the perpendicular y-direction by projecting onto each $\pm\nu_x$ branch $\to$ nested Wilson loop $W_{y,\mathbf{k'}}^{\pm\nu_x} = F_{y,\mathbf{k'}+(L_y-1)\Delta_y\mathbf{e}_y}^{\pm\nu_x} \cdots F_{y,\mathbf{k'}+\Delta_y\mathbf{e}_y}^{\pm\nu_x} F_{y,\mathbf{k'}}^{\pm\nu_x}$ where $\left[F_{y,\mathbf{k'}}^{\pm\nu_x}\right]_{\mu_1\mu_2} = \sum_{mn}\left[\nu_{x,\mu_1}^{\rm L}(\mathbf{k'}+\Delta_y\mathbf{e}_y)\right]_m^* \left[F_{y,\mathbf{k'}}\right]_{mn} \left[\nu_{x,\mu_2}^{\rm R}(\mathbf{k'})\right]_n$ with $\left[F_{y,\mathbf{k'}}\right]_{mn} = \langle\Psi_m^{\rm L}(\mathbf{k'}+\Delta_y\mathbf{e}_y)|\Psi_n^{R}(\mathbf{k'})\rangle$
- Nested Wannier Hamiltonian $\log W_{y,\mathbf{k'}}^{\pm\nu_x}$ $\to$ Wannier spectrum $\nu_{y,\mu'}^{\pm\nu_x}(k'_x)$ $\to$ nested bi-orthogonalized polarization $\boxed{\langle\nu_{y,\mu'}^{\pm\nu_x}\rangle = \frac{1}{L_x}\sum_{k'_x} {\rm Re}\left[\nu_{y,\mu'}^{\pm\nu_x}(k'_x)\right]}$

Consistency check: Bi-orthogonalized first-order polarization

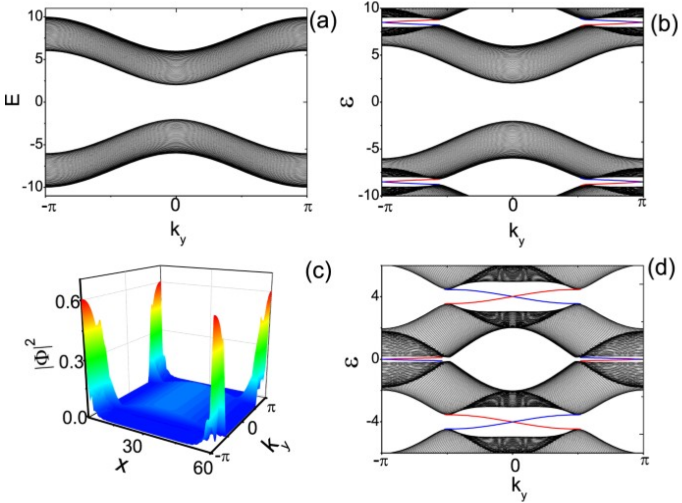


- M_x: $\nu_x(k_y) \to -\nu_x(k_y)$, and $\nu_y^{\nu_x}(k_x) \to \nu_y^{-\nu_x}(-k_x)$; M_x causes the first-order branches to appear in pairs
- M_y: $\nu_x(k_y) \to \nu_x(-k_y)$, and $\nu_y^{\nu_x}(k_x) \to -\nu_y^{\nu_x}(k_x)$; M_y defines the shape of the first-order branches
- The four-fold rotation C_4 and mirror rotations $\mathcal{M}_{xy}$, $\mathcal{M}_{x\bar{y}}$ interchange the branches, C_4: $\nu_x(k_y) \to -\nu_y(k_x)$, and $\nu_y^{\nu_x}(k_x) \to \nu_x^{-\nu_y}(-k_y)$, $\mathcal{M}_{xy}$: $\nu_x(k_y) \to \nu_y(k_x)$, and $\nu_y^{\nu_x}(k_x) \to \nu_x^{\nu_y}(k_y)$ $\mathcal{M}_{x\bar{y}}$: $\nu_x(k_y) \to -\nu_y(-k_x)$, and $\nu_y^{\nu_x}(k_x) \to -\nu_x^{-\nu_y}(-k_y)$.

Floquet exceptional HOTSC

How can one engineer the anomalous Floquet HOTSC phase for the NH case?

Anomalous phases in Floquet physics Scientific reports 8, 2243 (2018)



How do we treat? Floquet theory PR **138**, B979 (1965)

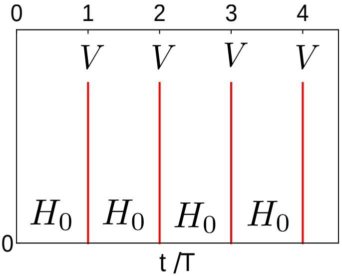
- Temporal analog of Bloch theorem for a time periodic Hamiltonian: $H(t) = H(t+T)$
- Wave function can be written in the Floquet basis: $|\Psi_j(t)\rangle = e^{-i\mu_j t}|\Phi_j(t)\rangle$, with $|\Phi_j(t+T)\rangle = |\Phi_j(t)\rangle$
- Wave-function of Schrödinger equation at the stroboscopic instant: $|\Psi(T)\rangle = \Sigma_j r_j e^{-i\mu_j T}|\Phi_j(0)\rangle$ with $r_j = \langle\Phi_j(0)|\Psi_j(0)\rangle$
- Time evolution operator: Floquet operator
  $U(T) = \mathcal{T} e^{-i\int_0^T H(t)dt} = \Sigma_j e^{-i\mu_j T}|\Phi_j(0)\rangle\langle\Phi_j(0)| = exp(-iH_F T)$
  where H_F is the Floquet Hamiltonian with eigenstates $|\Phi_j(0)\rangle$ and eigenvalue μ_j

Dynamic generation of second-order TI (SOTI): Floquet SOTI PRB **103**, 115308 (2021), PRB **106**, L140303 (2022)

Periodically kick in the FOT mass term

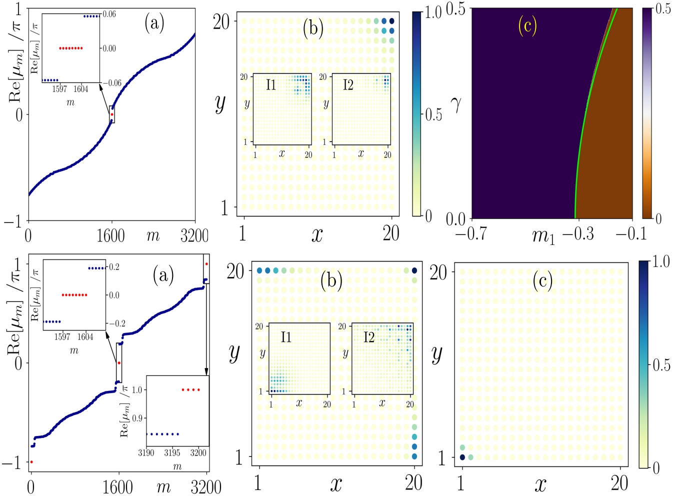
$$V(t) = m_1 \Gamma_3 \sum_{r=1}^{\infty} \delta\left(t - r\ T\right) \quad \text{with} \quad \Gamma_3 = \tau_z \sigma_z s_0$$

$$U(\mathbf{k}, T) = \exp\left[-i\mathcal{H}_0(\mathbf{k}) T\right] \exp\left[-i m_1 \Gamma_3\right]$$

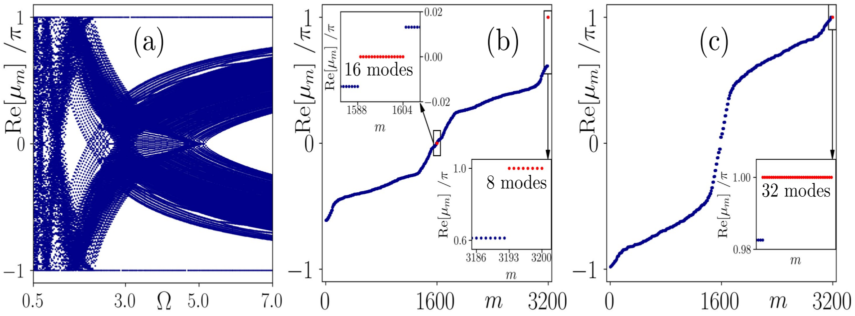


- $H^{\rm stat}_{\rm HOTSC} = H^{\rm stat}_{\rm HOTSC} + m_1 \Gamma_3$ exhibit trivial gapped phase as $m_0 > |t_x + t_y + \sqrt{\gamma_x^2 + \gamma_y^2}|$
- $H_{\rm Flq}(\mathbf{k}) \approx \mathcal{H}_0(\mathbf{k}) + \frac{m_1}{T}\Gamma_3 + m_1 \sum_{j=1,4}^{\neq 3} N_j \Gamma_{j1}$ and renormalized mass term $m_0' = m_0 - t_x - t_y - \frac{\gamma_x^2}{2\lambda_x^2} - \frac{\gamma_y^2}{2\lambda_y^2} + \frac{m_1}{T}$

Floquet NH HOTSC



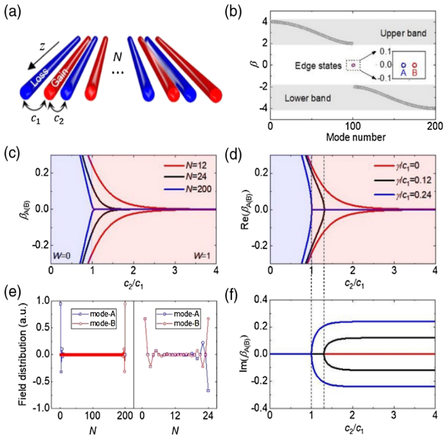
Tuning the MCMs dynamically

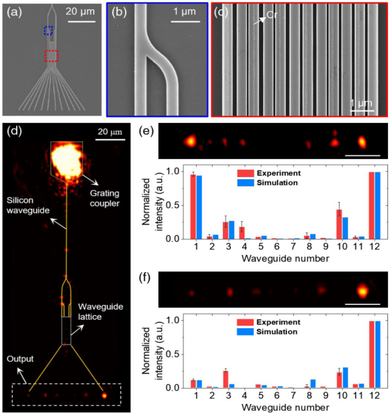


Conclusion

- **Considering 2D NH TI, proximized with d-wave superconductivity, we show the emergence of NH SOTSC phase**
- Breakdown of bulk-boundary correspondence for Bloch momenta $\rightarrow$ **recovery of bulk-boundary correspondence with non-Bloch momenta**
- MZMs are topologically characterized by the **bi-orthogonal nested polarization**
- **Floquet anomalous π-mode** following the mass kick

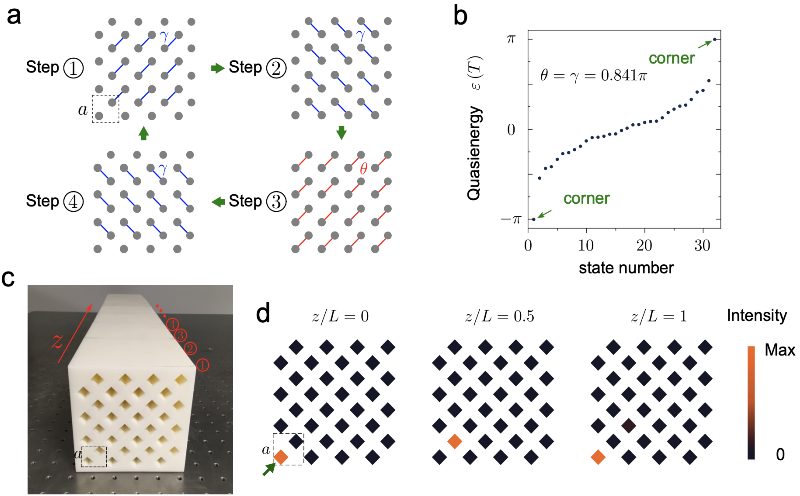
Experimental connections: NH topology PRL **123**, 165701 (2019)





- Non-Hermitian SSH model using a finite silicon waveguide lattice leading to topological phase $\rightarrow$ finite size effect of Hermitian system is overcome by the PT symmetric non-Hermitian terms such that topologial edge modes sustain

Experimental connections: Floquet HOT in acoustic system arXiv:2012.08847



☺ ☺ ☺ ☺ Any questions ...?