Exponentials rarely maximize Fourier adjoint restriction estimates on cones

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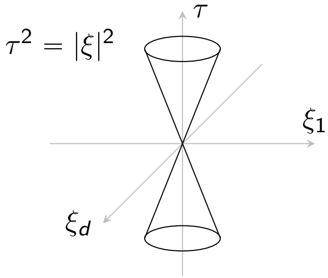
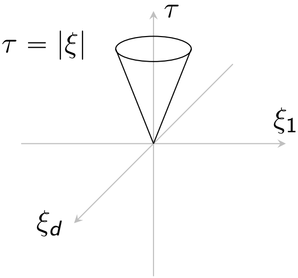
The adjoint restriction estimates on 2–Cone and 1–Cone



$\mu = \delta(\tau^2 - |\xi|^2)$



$\mu = \delta(\tau^2 - |\xi|^2)1_{\tau>0}$

Restriction Conjecture: for all $d \geqslant 2$,

$$\frac{\|\widehat{f\mu}\|_{L^q(\mathbb{R}^{d+1})}}{\|f\|_{L^p(\mu)}} \leqslant C_{p,d} < \infty, \quad \text{for} \begin{cases} q = \frac{d+1}{d-1} p', \\ \frac{1}{q} < \frac{d-1}{2d}. \end{cases}$$

We are interested in **maximizers** of this ratio (provided the estimate holds).

The Strichartz case: $p = 2, q = 2\frac{d+1}{d-1}$

Is $f_\star(\tau) = \exp(-|\tau|)$ a maximizer?

Spatial dim. d	2–Cone	1–Cone
2	NO	YES
3	YES	YES
4, 6, 8, ...	NO	Local
5, 7, 9, ...	Local	Local

Meaning that:

- $$\frac{\|\widehat{f\mu}\|_{L^q(\mathbb{R}^{d+1})}}{\|f\|_{L^2(\mu)}} \leqslant \frac{\|\widehat{f_\star\mu}\|_{L^q(\mathbb{R}^{d+1})}}{\|f_\star\|_{L^2(\mu)}} \text{ for } \begin{cases} \text{all } f \in L^2(\mu) & \text{(YES)} \\ \|f - f_\star\|_{L^2(\mu)} \leqslant C_d & \text{(Local)} \end{cases}$$
- $f_\star$ is not a critical point (see next slide). (NO)

Due to:

YES: Foschi 2007 | NO: G.N. 2018 | Local: G.N. & F.Gonçalves 2019

The general p case, on the 1–Cone

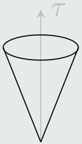
Recall: we want to maximize $\frac{\|\widehat{f\mu}\|_{L^q(\mathbb{R}^{d+1})}}{\|f\|_{L^p(\mu)}}$ around $f_\star(\tau) = \exp(-|\tau|)$.

Def.: $f_\star$ is a *critical point* iff

$$\boxed{\frac{\partial}{\partial\epsilon}\frac{\|(\widehat{f_\star+\epsilon f)\mu}\|_{L^q(\mathbb{R}^{d+1})}}{\|f_\star+\epsilon f\|_{L^p(\mu)}}\bigg|_{\epsilon=0} = 0, \qquad \forall f \in L^p(\mu).}$$

Theorem. (G.N.& D.Oliveira e Silva & B.Stovall & J.Tautges 2023)

(i) Maximizers exist for all p (for which the estimate holds).
(ii) $f_\star$ is a critical point $\iff$ $p = 2$.



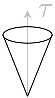
We say that $f_\star$ *rarely maximizes* the Fourier extension inequality.

Exactly the same happens on the paraboloid with $\mu = \delta(\tau - |\xi|^2)$

(i) Stovall 2020, (ii) Christ–Quilodrán 2012.

Digression: Foschi 2007. First appearance of $f_\star = e^{-|\tau|}$.

Let $\mu = \delta(\tau^2 - |\xi|^2)1_{\tau>0}$, $p = 2$, $d = 3$. Write $\zeta = (\tau, \xi)$.

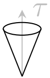


$$\begin{aligned}
\|\widehat{f\mu}(t,x)\|^4_{L^4(\mathbb{R}^4)} &= \int f(\zeta_1)f(\zeta_2)\overline{f(\zeta_3)f(\zeta_4)}e^{i(t,x)\cdot(\zeta_1+\zeta_2-\zeta_3-\zeta_4)}\, d\mu^{\otimes 4} dtdx \\
&= \int f(\zeta_1)f(\zeta_2)\overline{f(\zeta_3)f(\zeta_4)}\delta(\zeta_1+\zeta_2-\zeta_3-\zeta_4)d\mu^{\otimes 4} \\
\text{(Cauchy–Schwarz)} &\leqslant \int |f(\zeta_1)f(\zeta_2)|^2\delta(\zeta_1+\zeta_2-\zeta_3-\zeta_4)d\mu^{\otimes 4} \\
&= \int |f(\zeta_1)f(\zeta_2)|^2 \mu * \mu(\zeta_1+\zeta_2)d\mu^{\otimes 2} \\
\text{(Miracle)} &= (2\pi)^{\frac{1}{2}}\|f\|^4_{L^2(\mu)}.
\end{aligned}$$

Miracle: $\mu * \mu = (2\pi)^{\frac{1}{2}}$ on its support.

Digression: Foschi 2007. First appearance of $f_\star = e^{-|\tau|}$.

Let $\mu = \delta(\tau^2 - |\xi|^2) 1_{\tau > 0}$, $p = 2$, $d = 3$. Write $\zeta = (\tau, \xi)$.



$$\|\widehat{f\mu}(t,x)\|^4_{L^4(\mathbb{R}^4)} = \int f(\zeta_1) f(\zeta_2) \overline{f(\zeta_3) f(\zeta_4)} \delta(\zeta_1 + \zeta_2 - \zeta_3 - \zeta_4) d\mu^{\otimes 4}$$

$$(\text{C–S}) \leqslant \int |f(\zeta_1) f(\zeta_2)|^2 \delta(\zeta_1 + \zeta_2 - \zeta_3 - \zeta_4) d\mu^{\otimes 4}.$$

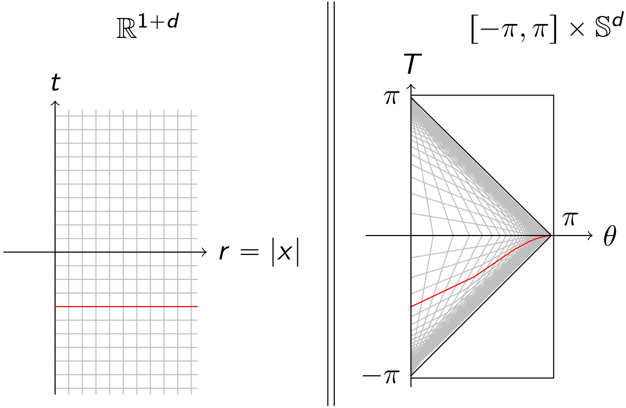
Why $f_\star$? Because (C–S) is an identity when $f = f_\star$.

Proof:

$$e^{-\tau_1 - \tau_2 - \tau_3 - \tau_4} \delta(\zeta_1 + \zeta_2 - \zeta_3 - \zeta_4) = e^{-2\tau_1 - 2\tau_2} \delta(\zeta_1 + \zeta_2 - \zeta_3 - \zeta_4).$$

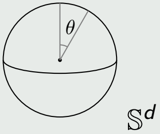
There must be a less technical reason for the appearance of $f_\star$!

The Penrose map $\mathbb{R}^{1+d} \to [-\pi, \pi] \times \mathbb{S}^d$



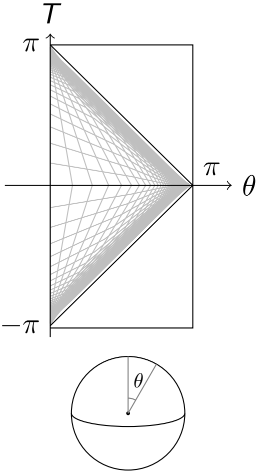
Equations of Penrose map (and definition of θ)

$$T = \arctan(t + r) + \arctan(t - r),$$
$$\theta = \arctan(t + r) - \arctan(t - r).$$



The Penrose map $\mathbb{R}^{1+d} \to [-\pi, \pi] \times \mathbb{S}^d$

We call $\mathbb{D}^{1+d}$ the range of the Penrose map - the *Penrose diamond*.



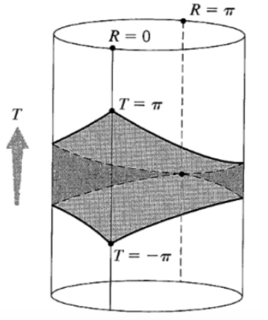


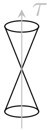
Figure: The Penrose diamond (left) is a submanifold of the cylinder $\mathbb{R} \times \mathbb{S}^d$.

(Right: Taken from S. Carroll, *Notes on general relativity*, arXiv:gr-qc/9712019.)

Where does $f_\star(\tau) = \exp(-|\tau|)$ come from?

For exposition purposes, let us focus on the 2–Cone only.

$u(t,x) = \widehat{f\mu}$ solves $\partial_t^2 u = \Delta u$. *Recall*: f is defined on



The Penrose map...

...yields a functional transform $(U_0, \dot{U}_0) \longleftrightarrow f$ such that

$$\int_{\mathbb{R}^{d+1}} |\widehat{f\mu}|^q = \int_{\mathbb{R}^{d+1}} |u|^q = \int_{\mathbb{D}^{d+1}} |U|^q \cdot (\text{some weight})^{\frac{d+1}{2(p-1)}(2-p)}, \qquad (*)$$

where $\left\{ \partial_T^2 U = \Delta_{\mathbb{S}^d} U - \frac{(d-1)^2}{4} U, \quad (U, \partial_T U)|_{T=0} = (U_0, \dot{U}_0) \right\}$.

Constant initial data $(1,0)$ correspond to $f_\star$.

$p = 2$: RHS of (*) appears to be rotationally invariant: we *can expect* a maximizer to be rotational invariant, i.e. constant.
$p \neq 2$: the weight breaks the rotational symmetry.

Back to the case $p = 2$, $q = 2\frac{d+1}{d-1}$

Recall: the estimate reads $\|\widehat{f\mu}\|_{L^q(\mathbb{R}^{d+1})} \leqslant C_d \|f\|_{L^2(\mu)}$.

Is $f_\star(\tau) = \exp(-|\tau|)$ a maximizer?

Spatial dim. d	2–Cone	1–Cone
2	NO	YES
3	YES	YES
4, 6, 8, ...	NO	Local
5, 7, 9, ...	Local	Local

Let $d = 2, 3$. We need to maximize $\int_{\mathbb{D}^{d+1}} |U|^q$, where U solves

$$\begin{cases} \partial_T^2 U = \Delta_{\mathbb{S}^2} U - \frac{1}{4} U, & d = 2, \\ \partial_T^2 U = \Delta_{\mathbb{S}^3} U - U, & d = 3, \end{cases} \quad \text{on } \mathbb{R} \times \mathbb{S}^d.$$

Theorem

$f_\star$ is a critical point for $d = 3$, but not for $d = 2$.

Proof. The nice case $d = 3$, $p = 2$: $f_\star$ is critical

Let X denote the generic point on $\mathbb{S}^3$.

Key Symmetry Lemma

Suppose that

$$\partial_T^2 U = \Delta_{\mathbb{S}^3} U - U, \quad \text{on } \mathbb{R} \times \mathbb{S}^3.$$

Then $U(T+\pi, -X) = -U(T, X)$.

Proof. We have $U = \cos(T\sqrt{1-\Delta_{\mathbb{S}^3}})U_0 + \frac{\sin(T\sqrt{1-\Delta_{\mathbb{S}^3}})}{\sqrt{1-\Delta_{\mathbb{S}^3}}}\dot{U}_0$. Expand

$$U_0 = \sum_{\ell=0}^{\infty} Y_\ell, \quad \text{with } -\Delta_{\mathbb{S}^3} Y_\ell = \ell(\ell+2)Y_\ell; \quad \text{so } Y_\ell(-X) = (-1)^\ell Y_\ell.$$

Then $U(T, X) = \sum_{\ell=0}^{\infty} \cos(T(\ell+1))Y_\ell + \ldots \quad \square$

We used: $\cos((T+\pi)(\ell+1)) = (-1)^{\ell+1}\cos(T(\ell+1))$.

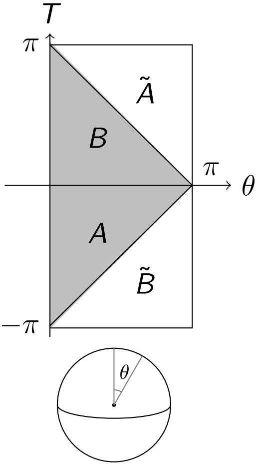
Proof. The nice case $d = 3$, $p = 2$: $f_\star$ is critical (cont.)

Key Symmetry Lemma

$\partial_T^2 U = \Delta_{\mathbb{S}^3} U - U$ on $\mathbb{R} \times \mathbb{S}^3$. Then $U(T + \pi, \pi - \theta) = -U(T, \theta)$.

Corollary. $\displaystyle\int_{\mathbb{D}^{1+3}} |U|^q = \frac{1}{2} \int_{-\pi}^{\pi} \int_{\mathbb{S}^3} |U|^q.$

Proof. $\mathbb{D}^{1+3} = A \cup B$. By the lemma,
$\displaystyle\int_A |U|^q = \int_{\tilde{A}} |U|^q$ and $\displaystyle\int_B |U|^q = \int_{\tilde{B}} |U|^q$. □



Remark

For $d = 2$ the symmetry lemma fails. For example,

$$\partial_T^2 U_\star = \Delta_{\mathbb{S}^2} U_\star - \tfrac{1}{4} U_\star, \quad (U_0, \dot{U}_0) = (1, 0)$$

yields $U_\star = \cos \frac{1}{2} T$, so $|U_\star(T + \pi)| \neq |U_\star(T)|$.

Proof. The nice case $d = 3$, $p = 2$: $f_\star$ is critical (end)

Recall: $q = 2\frac{d+1}{d-1} = 4$. We want: $\max\{\|\widehat{f\mu}\|_{L^4} \ : \ \|f\|_{L^2(\mu)} = 1\}$.

Claim

$$I = \int_{\mathbb{R}^{3+1}} \left|(f_\star \widehat{+ \epsilon f_\perp})\mu\right|^4 - \int_{\mathbb{R}^{1+3}} \left|\widehat{f_\star \mu}\right|^4 = o(\epsilon) \text{ for } \langle f_\star | f_\perp \rangle_{L^2(\mu)} = 0.$$

Proof. By Penrose, $f_\perp \equiv (U_0^\perp, \dot{U}_0^\perp)$ and $f_\star \equiv (1, 0)$.
Note $\partial_T^2 U_\star = \Delta_{\mathbb{S}^3} U_\star - U_\star$, $(U_0, \dot{U}_0) = (1, 0)$ yields $U_\star = \cos T$.

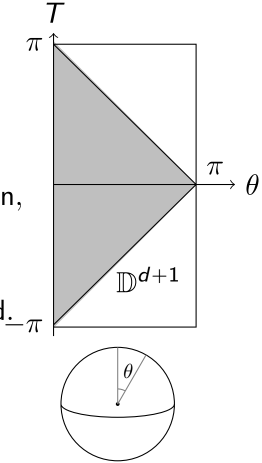
$$\begin{aligned} 2I &= \int_{-\pi}^{\pi} \int_{\mathbb{S}^3} |U_\star + \epsilon U^\perp|^4 - \int_{-\pi}^{\pi} \int_{\mathbb{S}^3} |U_\star|^4 \\ &= 4\epsilon \int_{-\pi}^{\pi} \int_{\mathbb{S}^3} U_\star^3 U^\perp + o(\epsilon) = 4\epsilon \int_{-\pi}^{\pi} (\cos T)^3 \left(\int_{\mathbb{S}^3} U^\perp(T, \cdot) \right) + o(\epsilon). \end{aligned}$$

Now $\langle f_\star | f_\perp \rangle = 0 \Rightarrow \langle U^\perp | 1 \rangle = 0$ at $T = 0$. Therefore $\langle U^\perp | 1 \rangle = 0$ at all times T. Hence $\int_{\mathbb{S}^3} U^\perp(T, \cdot) \equiv 0$ and we conclude $I = 0$.$\square$

Final summary

We want to maximize $\displaystyle\int_{\mathbb{R}^{1+d}} |\widehat{f\mu}|^q$, which equals:

$$\begin{cases} \displaystyle\int_{\mathbb{D}^{d+1}} |U|^q (\cos T + \cos\theta)^{\frac{d+1}{2(p-1)}(2-p)}, & p \neq 2, \\ \displaystyle\int_{\mathbb{D}^{d+1}} |U|^q, & p = 2, d \text{ even}, \\ \displaystyle\int_{-1}^{1}\int_{\mathbb{S}^d} |U|^q, & p = 2, d \text{ odd}. \end{cases}$$



Only in the latter case,
$f_\star = \exp(-|\tau|)$ is critical (and so has a chance of being a maximizer).