Lecture 4

Recall: Theorem: Suppose (Case 1) $R \neq \phi$. There exists a Λ-alg homomorphism $\varphi : \mathbb{T} \longrightarrow W$, surjective, where
$$W = E[\pi, \varepsilon_1, \dots, \varepsilon_r] \Big/ \left(\pi^{r_{an}+1}, \varepsilon_i^2, \varepsilon_i \pi, \varepsilon_1 \cdots \varepsilon_r + (-1)^r \mathcal{L}^*_{an}(\chi)\, \pi^{r_{an}}\right)$$

Sketch: $\Lambda_{(1)}$ = weight 1 localisation of Λ
$$= \Lambda_{(\ker \nu_1)} \qquad \nu_1 : \Lambda \longrightarrow E$$

e. $\mathcal{S}(n, \chi) = \mathcal{S}^\circ(n, \chi)$ - ordinary part of Λ-adic cusp forms.

$$\mathcal{S}^\circ(n, \chi)_{(1)} = \mathcal{S}^\circ(n, \chi) \otimes_{\Lambda} \Lambda_{(1)}$$

$$c : \mathcal{S}^\circ(n, \chi)_{(1)} \longrightarrow \prod_{0 \neq \mathfrak{a} \subseteq \mathcal{O}_F} \Lambda_{(1)}$$

$$\mathcal{F} \longmapsto (c(\mathfrak{a}, \mathcal{F}))_{\mathfrak{a} \subseteq \mathcal{O}_F}.$$

$\mathcal{H}$ = Image of Hecke orbit of $\mathcal{F}$ under c modulo $\pi^{r_{an}+1}$

Then we get $\varphi : \mathbb{T} \longrightarrow \mathrm{End}_{\mathcal{I}[\Gamma]/(\pi^{r_{an}+1})} \mathcal{H}$

The main point of the proof is to show that image of φ is isomorphic to W defined above.

Remark:

$$\varphi : \mathbb{T} \longrightarrow W$$

$T_\ell \mapsto 1 + \chi\varepsilon(\ell)$ $\quad \forall\ \ell \nmid np.$

$U_q \mapsto 1$ $\quad \forall\ \ell \mid n$ or $\ell \in R.$

$$U_{p_i} \longmapsto 1+\varepsilon_i \qquad 1 \le i \le r.$$

This "says" that all cuspidal eigenforms in $\mathcal{H}$ "look" eisenstein.

§ Construction of cohomology class.

$$W = \frac{E[\pi, \varepsilon_1, \dots, \varepsilon_r]}{\left(\pi^{r a_m+1},\ \varepsilon_i^2,\ \pi\varepsilon_i,\ \varepsilon_1 \cdots \varepsilon_r + (-1)^r \mathcal{L}_{a_m}(\chi)\pi^{r a_m}\right)}$$

$$\mathbb{T} \xrightarrow{\ \varphi\ } W \twoheadrightarrow W/\mathfrak{m}_W = E$$

$\mathfrak{m}$ be the kernel of this composition.

$\mathfrak{m}$ is generated by $\left(\underset{\ell \nmid Np}{T_\ell - 1 - \chi\varepsilon(\ell)},\ \underset{\ell \mid Np}{U_\ell - 1}\right)$.

i.e. $\mathfrak{m}$ is "eisenstein".

$\mathbb{T}_{(\mathfrak{m})}$ = localisation of $\mathbb{T}$ at $\mathfrak{m}$.

$$\mathbb{L} = \mathrm{Frac}(\mathbb{T}_{(\mathfrak{m})}) = \prod_{i=1}^{t} L_{\mathcal{X}_i}$$

product of fields

$\{\mathcal{H}_1, \dots, \mathcal{H}_t\}$ are Λ-adic cuspidal eigenform in $\mathcal{H}$.

$*$ $\quad \mathbb{T}_{(m)} \hookrightarrow \mathbb{L}$ $\quad$ (since $\mathbb{T}_{(m)}$ is reduced)

There are no old forms.

$L_{\mathcal{H}_i}$ is a finite extension of $\mathrm{Frac}(\Lambda)$

$\Lambda_{\mathcal{H}_i}$ integral closure of $\Lambda_{(1)}$ in $L_{\mathcal{H}_i}$.

$m_{\mathcal{H}_i}$ maximal ideal above (T).

$$c(\ell, \mathcal{H}_i) \equiv 1 + \chi\varepsilon(\ell) \mod m_{\mathcal{H}_i} \quad \forall\, \ell \nmid np$$

$$c(\ell, \mathcal{H}_i) \equiv 1 \mod m_{\mathcal{H}_i} \quad \forall\, \ell \mid np.$$

$\{\mathcal{H}_1, \dots, \mathcal{H}_t\}$ are all eigenforms with this property.

Theorem (Hida, Wiles) $\exists$ cont irr rep^n

$$\rho_i : G_F \longrightarrow GL_2(L_{\mathcal{H}_i}) \quad \text{s.t.}$$

① ρ_i is unramified outside $n p$.

② $\forall\ \ell \nmid np$

Hecke eigenvalue of T_ℓ.

$$\mathrm{char}(\rho(\mathrm{Frob}_\ell))(x) = x^2 - c(\ell, \mathcal{H}_i)x + \chi\varepsilon(\ell)$$

③ $\forall\ \mathfrak{p} \mid p$ ρ_i is ord at $\mathfrak{p}$

$$\rho_i|_{G_\mathfrak{p}} \sim \begin{pmatrix} \chi\varepsilon\eta_{\mathfrak{p},i}^{-1} & * \\ 0 & \eta_{\mathfrak{p},i} \end{pmatrix}$$

Hecke eigenvalue of $U_\mathfrak{p}$.

Here $\eta_{\mathfrak{p},i}$ is unram character

$$\eta_{\mathfrak{p},i}(\mathrm{rec}(\varpi^{-1})) = c(\mathfrak{p}, \mathcal{H}_i)$$

Remark: Locally at each $\mathfrak{p}$, there is a natural basis for the rep^n space i.e. the one that makes $\rho_i|_{G_\mathfrak{p}}$ upper triangular. Globally we need to choose a basis "incompatible" with all these local bases.

("Ribet's Wrench").

Notation: $v_{\mathfrak{p},i} \in L^2_{\mathfrak{H}_i}$ be the eigenvector for $\rho_i(G_\mathfrak{p})$

Lemma: $\exists$ $\tau \in G_F$ s.t. $\chi(\tau) \neq 1$ and s.t. $v_{\mathfrak{p},i}$ is not an eigenvector for $\rho_i(\tau)$ $\forall$ i & $\forall$ $\mathfrak{p} \in R_1$.

$\mathbb{T}_m$ = m-adic completion of $\mathbb{T}_{(m)}$.

$\mathbb{L}_m$ = Frac $(\mathbb{T}_m)$.

$\exists$ $\rho : G_F \longrightarrow GL_2(\mathbb{L}_m)$ s.t

Putting together all ρ_i's.

$\cong \prod_i \mathcal{L}_{\mathfrak{H}_i}$

① ρ unram outside $n\mathfrak{p}$

(2) $\mathrm{char}(\rho(\mathrm{Frob}_\ell))(x) = x^2 - \overline{T_\ell}\, x + \chi\varepsilon(\ell) \quad \forall\, \ell \nmid \mathfrak{n}p.$

(3) $\rho|_{G_\mathfrak{p}} \sim \begin{pmatrix} \chi\varepsilon\eta_\mathfrak{p}^{-1} & * \\ 0 & \eta_\mathfrak{p} \end{pmatrix} \quad \forall\, \mathfrak{p} | p.$

$T_\ell \sim 1+\chi\varepsilon(\ell)$

$\lambda_1+\lambda_2 \sim 1+\chi\varepsilon(\ell)$

$\lambda_1\lambda_2 \sim \chi\varepsilon(\ell)$

$\eta_\mathfrak{p}$ unram $\qquad \eta_\mathfrak{p}(\mathrm{rec}(\varpi^{-1})) = \overline{U_\mathfrak{p}}.$

Fix a basis so that $\rho(\tau) = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$\lambda_1 \equiv 1 \pmod{\mathfrak{m}} \qquad \lambda_2 \equiv \chi(\tau) \pmod{\mathfrak{m}}.$

<u>Notation</u>: $\rho(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \in GL_2(\mathbb{T}_\mathfrak{m})$

$\forall\, \sigma \in G_F.$

Chebotarev density & (2) above gives $\boxed{\overline{\pi} = \text{image of } \pi \text{ in } \mathbb{T}_\mathfrak{m}}$

$\mathrm{tr}(\rho(\sigma)) = \underset{\substack{\uparrow \\ \mathbb{T}_\mathfrak{m}}}{a(\sigma) + d(\sigma)} \equiv 1 + \chi(\sigma) \pmod{\mathfrak{m}\overline{\pi}}$

$$\mathrm{tr}\,(\rho(\sigma\tau)) = a(\sigma\tau) + d(\sigma\tau) \equiv 1 + \chi(\sigma)\chi(\tau)$$

$$\rho(\sigma\tau) = \rho(\sigma)\rho(\tau)$$

$$= \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 a(\sigma) & \\ & \lambda_2 d(\sigma) \end{pmatrix}$$

$$\Rightarrow \quad \lambda_1 a(\sigma) + \lambda_2 d(\sigma) \equiv 1 + \chi(\sigma)\lambda_2$$

$$\Rightarrow \quad a(\sigma) \equiv 1 \pmod{m}$$

$$d(\sigma) \equiv \chi(\sigma) \pmod{m}$$

$\mathbb{T}/m$.

If $\sigma, \sigma' \in G_F$, $\quad \rho(\sigma\sigma') = \rho(\sigma)\rho(\sigma')$

$$\begin{pmatrix} a(\sigma\sigma') & b(\sigma\sigma') \\ c(\sigma\sigma') & d(\sigma\sigma') \end{pmatrix} = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}\begin{pmatrix} a(\sigma') & b(\sigma') \\ c(\sigma') & d(\sigma') \end{pmatrix}$$

$$\Rightarrow \quad b(\sigma\sigma') = a(\sigma)b(\sigma') + d(\sigma')b(\sigma)$$

$$b(\sigma\sigma') \equiv b(\sigma') + \chi(\sigma')b(\sigma) \pmod{m}$$

$\kappa(\sigma) = \chi(\sigma)^{-1} b(\sigma)$. Then κ is a cocycle.

$[\kappa] \in H^1(G_F, \overline{B}(\chi^{-1}))$

$B =$ $\mathbb{T}_m$-module generated by $b(\sigma)$, $\forall\ \sigma \in G_F$

$\overline{B} = B / \hat{m} B.$

§ Local behaviour of κ

$\begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix}$ be change of basis matrix to get upper triangular $\rho|_{G_{\mathfrak{p}_i}}$.

$$\begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix} \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \begin{pmatrix} \chi \varepsilon \eta_i^{-1} & * \\ 0 & \eta_i \end{pmatrix}$$

$\forall\ \sigma \in G_{\mathfrak{p}_i}$

$$\Rightarrow \quad b(\sigma) = \frac{A_i}{C_i} \left(\chi \varepsilon \eta_i^{-1}(\sigma) - a(\sigma) \right) \quad \forall\ \sigma \in G_{\mathfrak{p}_i}$$

Fact: Our choice of τ $\Rightarrow$ $A_i, C_i \in \mathbb{T}_m$ are nonzero divisors.

Lemma: $[\kappa] \in H^1(G_F, \overline{B}(\chi^{-1}))$ is unramified outside primes in R_1

E v.s.

el_{I_v} is trivial for all $v \notin R_1$

Lemma: $B \subseteq \frac{A_1}{c_1}\hat{m} + \frac{A_2}{c_2}\hat{m} + \cdots + \frac{A_r}{c_r}\hat{m}$

§ Regulator

Recall we have fixed a basis $u_1, \dots, u_r$ of $(E \otimes U)^\chi$.

Then (Poitou-Tate) $\sum_{i=1}^{r} \mathrm{res}_i(\kappa)(u_j) = 0$ in $\overline{B}$ $\forall j$

Technical point

$u_j \in (E \otimes U)^\chi$, so u is of the form $u = \sum_k y_{jk} \otimes e_{jk}$ for $y_{jk} \in E$ & $e_{jk} \in U$.

But we pretend, for simplicity, that $u_j \in U := \mathcal{O}_H[\frac{1}{p}]^\times$

$$\operatorname{res}_i \kappa(u_j) = \bar{b}(\sigma_{ij}) = \frac{A_i}{c_i}\left(\eta_i^{-1}\varepsilon(\sigma_{ij}) - a(\sigma_{ij})\right)$$

$\sigma_{ij} = \operatorname{rec}_i(u_j)$ Local Artin reciprocity map

$I = \ker \varphi$

- $\eta_i^{-1}(\sigma_{ij}) = U_{\beta_i}^{o_i(u_j)} \equiv 1 + o_i(u_j)(U_{\beta_i} - 1) \pmod{I}$
- $\varepsilon(\sigma_{ij}) \equiv 1 + l_i(u_j)\pi \pmod{\pi^2}$
- $a(\sigma_{ij}) \equiv 1 + a_i'(\sigma_{ij}) \pmod{(m^2, I)}$

$$\Rightarrow \quad b(\sigma_{ij}) = \frac{A_i}{c_i}\left(l_i(u_j)\pi + o_i(u_j)(U_{\beta_i} - 1) - a_i'(u_j) + m_{ij}\right)$$

in (m^2, I).

Lemma $B \subseteq \sum \frac{A_i}{c_i} m$ is used

$$0 = \sum_i b(\sigma_{ij}) = \sum_i \frac{A_i}{c_i}\left(l_i(u_j)\pi + o_i(u_j)(U_{\beta_i} - 1) - a_i'(u_j) + m_{ij}\right)$$

$$\Rightarrow 0 = \det\left(\frac{A_i}{c_i}\left(l_i(u_j)\pi + o_i(u_j)(U_{\beta_i} - 1) - a_i'(u_j) + m_{ij}\right)\right)_{1 \le i, j \le r}$$

Choice of $\tau \Rightarrow A_i, C_i \in \mathbb{L}_m^{\times}$

$$\Rightarrow \quad 0 = \det\left(l_i(u_j)\pi + O_i(u_j)\left(U_{f_i} - 1\right) - a_i'(u_j) + m_{ij} \right)$$

Apply φ.

$$0 = \det\left(l_i(u_j)\pi + O_i(u_j)\varepsilon_i + n_{ij} \right) \quad \leftarrow m_W^2 .$$

$$\rotatebox{90}{$\equiv$}$$

$\pi\varepsilon_i = 0$ in W

$$\det\left(l_i(u_j)\right)_{i,j}\pi^r + \det\left(O_i(u_j)\right)_{i,j}\,\pi\varepsilon_i \pmod{m_W^{r+1}}$$

$$\equiv \det\left(l_i(u_j)\right)\pi^r + \det\left(O_i(u_j)\right)(-1)^{r+1}\mathcal{L}_{an}^*(x)\,\pi^{r_{an}} \pmod{m_W^{r+1}}$$

- If $r_{an} > r$, then $\pi^r \det\left(l_i(u_j)\right) = 0 \Rightarrow R_p(x) = 0$
- If $r_{an} = r$, then $\mathcal{L}_{an}(x) = (-1)^r \dfrac{\det\left(l_i(u_j)\right)}{\det\left(O_i(u_j)\right)} = R_p(x)$ $\square$