$$f_\lambda(z) = a_\lambda(0) + \sum_{\substack{b \in t_\lambda \\ b \gg 0}} a_\lambda(b)\, e_F(bz)$$

$k = (k, \ldots, k)$ parallel weight

totally positive i.e. $\sigma(b) > 0$ for every $\sigma \in \mathrm{Hom}(F, \mathbb{R})$

$$e_F(x) = \exp\left(2\pi i \cdot \mathrm{Tr}_{F/\mathbb{Q}}(x)\right).$$

"Normalised Fourier co-efficients".

$\mathfrak{m} \subseteq \mathcal{O}_F$ in class λ

$$C(\mathfrak{m}, f) = a_\lambda(b)\, N(t_\lambda)^{-k/2}$$

$$c_\lambda(0, f) = a_\lambda(0)\, N(t_\lambda)^{-k/2}$$

$S_k(\mathfrak{n}, \psi)$ – cusp forms $a_\lambda(0) = 0 \ \forall \lambda$ and constant terms are 0 at other cusps

Lecture 2

Examples of HMFs : Eisenstein series.

The most basic and an important example are Eisenstein series.

Ref: · Duke paper of Shimura

$\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_F$ $\quad \eta$ modulo $\mathfrak{a}$, ψ modulo $\mathfrak{b}$.

• On constant terms of Eisenstein series (Dasgupta–K)

$$\mathrm{sgn}(\eta) = q \in (\mathbb{Z}/2\mathbb{Z})^d$$

$$\mathrm{sgn}(\psi) = r \in (\mathbb{Z}/2\mathbb{Z})^d$$

$$q + r \equiv (k, \ldots, k) \pmod 2 \qquad \text{(Parallel weight)}$$

Proposition: $\forall\ k \geq 1$, $\exists\ E_k(\eta, \psi) \in M_k(\mathfrak{ab}, \eta\psi)$ s.t.

s.t.

$$c(\mathfrak{m}, E_k(\eta,\psi)) = \sum_{\mathfrak{c} \mid \mathfrak{m}} \eta\left(\frac{\mathfrak{m}}{\mathfrak{c}}\right) \psi(\mathfrak{c})\, N\mathfrak{c}^{k-1} \qquad \text{(like division sums)},$$

$$c_\lambda(0, E_k(\eta,\psi)) = \begin{cases} 2^{-n}\eta^{-1}(t_\lambda) L(\psi\eta^{-1}, 1-k) & \text{if } \mathfrak{a} = 1,\ k \geq 2 \\ 2^{-n}\psi^{-1}(t_\lambda) L(\eta\psi^{-1}, 0) & \text{if } \mathfrak{b} = 1,\ \mathfrak{a} \neq 1,\ k = 1 \\ 2^{-n}\eta^{-1}(t_\lambda) L(\psi\eta^{-1}, 0) + 2^{-n}\psi^{-1}(t_\lambda) L(\eta\psi^{-1}, 0) & \text{if } \mathfrak{a} = \mathfrak{b} = 1,\ k = 1. \\ 0 & \text{otherwise.} \end{cases}$$

§ Λ-adic modular forms.

E - large enough extension of $\mathbb{Q}_p$.

$\Lambda = \mathcal{O}_E[[T]]$.

Over $\mathbb{Q}$

Power series in $\mathcal{O}[[T]][[q]]$.

weight k specialisation $\quad \nu_k : \Lambda \longrightarrow \mathcal{O}_E$

u - a fixed topological generator of $1+2p\mathbb{Z}_p$

$$\nu_k(f(T)) = f(u^{k-1}-1).$$

e.g. $\nu_1(f(T)) = f(0)$.

Defn: A Λ-adic modular form $\mathcal{F}$ of level n and character χ is a collection of elements in Λ

$$c(m, \mathcal{F}) \qquad \forall\ 0 \neq m \subseteq \mathcal{O}_F$$

$$c_\lambda(0, \mathcal{F}) \qquad \forall \quad \lambda \in Cl^+(F)$$

s.t for all but finitely many $k \geq 2$,

$\{ v_k(c(m,\mathfrak{J})),\ v_k(c_\lambda(0,\mathfrak{J})) : 0 \neq m \subseteq \mathcal{O}_F,\ \lambda \in Cl^+(F) \}$

are normalised Fourier coefficients of a HMF in $M_k(\mathfrak{n}p, \chi\omega^{1-k})$

$\boxed{\mathcal{M}(\mathfrak{n}, \chi) \text{ - the space of } \Lambda\text{-adic modular forms}}$ $\quad \underline{Q(\Lambda) = \mathrm{Frac}(\Lambda)}$

$\mathcal{M}(\mathfrak{n}, \chi) \otimes_\Lambda Q(\Lambda)$ - We also call this Λ-adic modular forms.

Example: (Eisenstein series) η, ψ characters of narrow ray class group modulo $\mathfrak{a}, \mathfrak{b}$, resp. $\exists$ a Λ-adic form $\mathcal{E}(\eta, \psi) \in \mathcal{M}(\mathfrak{a}\mathfrak{b}, \eta\psi) \otimes Q(\Lambda)$, s.t.

$$\forall\ k \geq 2 \quad v_k(\mathcal{E}(\eta, \psi)) = E_k(\eta, \psi\omega^{1-k}) \in M_k(\mathfrak{a}\mathfrak{b}p, \eta\psi\omega^{1-k})$$

Construction (Deligne - Ribet) For any ideal $\mathfrak{c}$, $(p, \mathfrak{c}) = 1$

$$\langle N_\mathfrak{c} \rangle = N_\mathfrak{c} / \omega(\mathfrak{c}) \in 1 + 2p\mathbb{Z}_p.$$

$N_\mathfrak{c}$ = Norm of ideal $\mathfrak{c}$ $\in \mathbb{Z}_p$

These terms can be p-adically interpolated as a function of k.

$$c\left(m, E_k(\eta, \psi\omega^{1-k})\right) = \sum_{\substack{c \mid m \\ (c,p)=1}} \eta\left(\frac{m}{c}\right) \psi(c) \langle Nc \rangle^{k-1}$$

$$k \equiv k' \mod \left(p^t(p-1)\right) \implies \langle Nc \rangle^{k-1} \equiv \langle Nc \rangle^{k'-1} \mod p^t.$$

$$\implies \exists \quad c(m, \mathcal{E}(\eta, \psi)) \in \Lambda \quad \text{s.t.}$$

Non-constant terms can be p-adically interpolated.

$$\nu_k\left(c(m, \mathcal{E}(\eta, \psi))\right) = c\left(m, E_k(\eta, \psi\omega^{1-k})\right).$$

<u>q-expansion Principle</u>: "q-expansion map is injective"

Ref: Deligne–Ribet "values of abelian L-functions..." Inventiones

"If non-constant co-efficients of a modular form are integral, then difference between constant terms at two cusps is also integral".

This gives p-adic L-function $L_{S,p}(\chi\omega, s)$.

§ Gross-Stark conjecture

Recall: S contains all primes above p.

$F^{\ker\chi} = H$

$$L_{S,p}(\chi\omega, n) = L_S(\chi\omega^n, n) = L(\chi\omega^n, n) \prod_{v \in S} (1 - \chi(v) N v^{-n})$$

F

$$L_{S,p}(\chi\omega, 0) = L_S(\chi, 0) = L(\chi, 0) \prod_{v \in S} (1 - \chi(v))$$

If $\chi(v) = 1$ for some $v \in S$, then $L_{S,p}(\chi\omega, 0) = 0$.

p-adic function

If v is not above p and $\chi(v) = 1$

$$L_{S,p}(\chi\omega, s) = L_{S \setminus \{v\}, p}(\chi\omega, s) \cdot (1 - \chi(v)\langle Nv\rangle^{-s})$$

$$= L_{S\setminus\{v\},p}(\chi\omega, s) \cdot (1 - \langle Nv\rangle^{-s})$$

$k \equiv k' \pmod{p^t(p-1)}$

$\Rightarrow \langle Nv\rangle^k \equiv \langle Nv\rangle^{k'} \pmod{p^{t+1}}$

$\because p \nmid Nv$

Leading term

$$\Longrightarrow \quad L^*_{S,p}(\chi\omega,0) = L^*_{S\setminus\{v\},p}(\chi\omega,0)\,\log_p\langle Nv\rangle$$

$\log_p : \mathbb{Q}_p^\times \longrightarrow \mathbb{Q}_p$ Iwasawa's p-adic log.

This does not work if $v \mid p$.

From now on: $S = \{v \mid p\}$.

$$R = \{v\in S \mid \chi(v)\neq 1\} \qquad R_1 = \{v\in S \mid \chi(v)=1\} = S\setminus R.$$

$$r = \# R_1.$$

Notation: $L_{S,p}(\chi\omega,s)$ by $L_p(\chi\omega,s)$.

It is easy to see that if $r\geq 1$, then

$$L_p(\chi\omega,0)=0$$

Conjecture 1 (Gross / Kuzmin)

$$\mathrm{ord}_{s=0} L_p(\chi\omega, s) = r.$$

Remark: • This conjecture is open in general.

• Okay for $r=0$

• Known if $r=1$. It uses Baker-Brumer theorem.

• Known that $\mathrm{ord}_{s=0} L_p(\chi\omega, s) \geq r$.

(Wiles, $\underbrace{\text{Charollois - Dasgupta - Greenberg}}_{\text{"Shintani's formula"}}$)

Theorem (Dasgupta - Darmon - Pollack) (Gross - Stark conjecture)
Dasgupta - K - Ventullo

$$L_p^{(r)}(\chi\omega, 0) = r!\, L_R(\chi, 0)\, R_p(\chi)$$

Here $R_p(\chi)$ is the Gross regulator defined below:

$$H = \bar{F}^{\ker \chi} \qquad U = \mathcal{O}_H\left[\tfrac{1}{p}\right]^\times$$

$$\begin{array}{c} H \\ \big|\, G \\ F \end{array}$$

X = Free abelian group generated by $S_p(H) = \{v \mid p,\ \text{prime of } H\}$.

$U^- = \{x \in U \mid c(x) = x^{-1}\}$.

$X^- = \{x \in X \mid c(x) = -x\}$.

$c \in G$. complex conjugation (χ totally odd).

$E/\mathbb{Q}_p$ sufficiently large finite ext^n.

There are two G-module maps.

$$l_p,\ o_p : E \underset{\mathbb{Z}}{\otimes} U^- \longrightarrow E \underset{\mathbb{Z}}{\otimes} X^-$$

$$o_p(x) = \left(\mathrm{ord}_{\mathfrak{p}}(x)\right)_{\mathfrak{p}\in S_p(H)} = \sum_{\mathfrak{p}\in S_p(H)} \mathrm{ord}_{\mathfrak{p}}(x)\cdot \mathfrak{p}$$

$$l_p(x) = \left(-\log_p \mathrm{Norm}_{H_{\mathfrak{p}}/\mathbb{Q}_p}(x)\right)_{\mathfrak{p}\in S_p(H)}$$

$\mathcal{O}_H^{+}$

$\mathcal{O}_H\left[\frac{1}{p}\right]^{+}$

- Dirichlet's unit theorem: o_p is an isomorphism.
- Conjecturally l_p is also an isomorphism

$$o_p^{\chi},\ l_p^{\chi} : \left(E\otimes U^-\right)^{\chi} \longrightarrow \left(E\otimes X^-\right)^{\chi}$$

$$\parallel$$

$$\{x\in E\otimes U^- \mid \chi(x) = u^{\chi^{-1}(\sigma)}\}$$

$$\boxed{R_p(\chi) = \det\left(l_p^{\chi}\circ\left(o_p^{\chi}\right)^{-1}\right) \in E}$$

Gross Regulator.