Lecture 1 On the Gross-Stark conjecture.

Today: Introduction to L-functions, p-adic L-functions, modular forms.

F - number field $F/\mathbb{Q}$ finite ext^n $\mathcal{O}_F$ - ring of integers

Dedekind zeta functions

$$\zeta_F(s) = \sum_{0 \neq \mathfrak{a} \subseteq \mathcal{O}_F} N\mathfrak{a}^{-s} \qquad \mathrm{Re}(s) > 1$$

$$= \prod_{0 \neq \mathfrak{p} \text{ prime ideal}} \underbrace{(1 - N\mathfrak{p}^{-s})^{-1}}_{\text{Euler factor}}. \qquad \text{Euler product.}$$

More generally, for a character $\chi : G_F \to \bar{\mathbb{Q}}^\times$

$$L(\chi, s) = \sum_{\mathfrak{a} \subseteq \mathcal{O}_F} \chi(\mathfrak{a}) N\mathfrak{a}^{-s} = \prod_{\mathfrak{p}} (1 - \chi(\mathfrak{p}) N\mathfrak{p}^{-s})^{-1} \qquad \mathrm{Re}(s) > 1$$

Facts: ① $L(\chi, s)$ has meromorphic continuation to $\mathbb{C}$ with only a simple pole at $s=1$ when $\chi = 1$.

② $L(\chi, s)$ has a functional equation relating $L(\chi, s)$ to $L(\chi^{-1}, 1-s)$.

e.g. $[F:\mathbb{Q}] = r_1 + 2r_2$

degree

$\quad\;\; \| $

$\# \mathrm{Hom}(F, \mathbb{C})$

$r_1(F) = \#$ of real embedding of F in $\mathbb{C}$

$r_2(F) = \#$ of pairs of complex embeddings in $\mathbb{C}$.

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$$

$$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$$

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x}\, dx.$$

$\mathrm{Re}\, s > 0$

Δ_F = discriminant of F.

$$|\Delta_F|^{s-1} \Gamma_{\mathbb{R}}(s)^{r_1} \Gamma_{\mathbb{C}}(s)^{r_2} \zeta_F(s) = \Gamma_{\mathbb{R}}(1-s)^{r_1} \Gamma_{\mathbb{C}}(1-s)^{r_2} \zeta_F(1-s).$$

③ Dirichlet's class number formula.

$$\operatorname{Res}_{s=1} \zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} h_F R_F}{w_F}$$

$r_1+r_2-1 = \operatorname{rank}_{\mathbb{Z}} U$.

$$\zeta_F(s) = -\frac{h_F R_F}{w_F} s^{r_1+r_2-1} + O(s^{r_1+r_2})$$

Taylor expansion at $s=0$

$h_F = \# \mathcal{C}l(F)$ $\qquad$ $w_F = \# \mu(F)$

class number $\qquad$ ideal class group of F $\qquad$ $= \#$ of roots of 1 in F.

R_F = Dirichlet regulator.

$Y = Y_F =$ free abelian group generated by all archimedean places of F.

$X = X_F = \ker(Y \xrightarrow[\text{aug}]{} \mathbb{Z})$.

$\sum_v a_v v \longmapsto \sum_v a_v \in \mathbb{Z}$

$U = \mathcal{O}_F^\times$

Dirichlet's unit theorem.

$$\mathbb{R}U \xrightarrow{\varphi} \mathbb{R}X \subseteq \mathbb{R}Y$$

$$x \longmapsto \begin{cases} \log |x|_i & \text{if } i \text{ is Real} \\ 2\log|x|_i & \text{if } i \text{ is Complex.} \end{cases}$$

$R_F = |\det \varphi|$ wrt integral bases of U and X.

Digression ① H/F abelian extension. $G = \mathrm{Gal}(H/F)$.

$$\begin{array}{c} H \\ \big| \, G \\ F \end{array}$$

By Artin formalism.

$$L(\mathbb{1}_{H}, s) = \zeta_H(s) = \prod_{\chi \in \hat{G}} L(\chi, s) \stackrel{?}{=} \prod_{\chi \in \hat{G}} L(\chi_F, s)$$

Then
$$\prod_{\chi \in \hat{G}} L(\chi, s) = \frac{-h_H R_H}{w_H} s^{r_1(H)+r_2(H)-1} + O\left(s^{r_1(H)+r_2(H)}\right)$$

$h_H = \# \mathcal{Cl}(H)$ $\quad G \curvearrowright \mathcal{Cl}(H)$

① Is there a factorisation of RHS mirroring the one on LHS? (Stark).

F is called totally real if $r_1(F) = [F:\mathbb{Q}]$

② H/F abelian, H – CM extension. $2r_2(H) = [H:\mathbb{Q}]$

(totally complex quad ext^n of a totally real)

H^+ maximal totally real subfield of H.

$r_1(H^+) + r_2(H^+) - 1 = r_1(H) + r_2(H) - 1$

$r_2(H^+) = 0$, $r_1(H) = 0$

$$\frac{\zeta_H^*(0)}{\zeta_{H^+}^*(0)} = \prod_{\substack{\chi \in G \\ \chi \text{ totally odd}}} L(\chi, 0) = \frac{h_H R_H / w_H}{R_{H^+} h_{H^+} / w_{H^+}}$$

$w_H = \# \mu(H)$

$w_{H^+} = \# \mu(H^+)$

$$\frac{R_H}{R_{H^+}} = 2^{r_2(H)-1} Q$$

$$= \frac{h_H^- \, 2^{r_2(H)-1} Q}{(w_H / w_{H^+})} \in \mathbb{Q}^\times.$$

- $L(\chi, 0) \neq 0$ for χ totally odd

$$Q = [U_H : \mu(H) U_{H^+}].$$

Recall

$U_H = \mathcal{O}_H^\times$

$U_{H^+} = \mathcal{O}_{H^+}^\times$

§ p-adic L-function.

F - totally real. Fix a prime p.

χ totally odd character of F. $\Leftrightarrow$ $\chi : G_F \to \overline{\mathbb{Q}}^\times$

$H = \overline{F}^{\ker \chi}$ is a CM field.

ω Teichmüller character

$$\omega : \mathrm{Gal}(F(\mu_{2p})/F) \longrightarrow \mu_{\varphi(2p)} = \begin{cases} \mu_2 & \text{if } p=2 \\ \mu_{p-1} & \text{if } p \neq 2 \end{cases}$$

Fact: (Siegel, Klingen, Shintani)

$L(\chi\omega^n, n) \in \mathbb{Q}(\chi\omega)^\times$ for any $n \in \mathbb{Z}_{\leq 0}$.

Thus we have a function $\mathbb{Q}(\chi\omega) \subseteq \overline{\mathbb{Q}}_p$.

$$\mathbb{Z}_{\leq 0} \longrightarrow \overline{\mathbb{Q}}_p$$

$$n \longmapsto L(\chi\omega^n, n) \quad \text{(Coates-Lichtenbaum)}$$

(Technicality: There is a canonical way to do this).

Question: $\mathbb{Z}_{\leq 0}$ is dense in $\mathbb{Z}_p$.

Is there a continuation of the above function to $\mathbb{Z}_p$?

"Yes": need modification of the L-values. $L(\chi\omega^n, n)$

S – a finite set of places of F containing all places above p.

imprimitive version $\rightarrow$ $L_S(\chi, s) = L(\chi, s) \prod_{v \in S} (1 - \chi(v) N_v^{-s})$

Theorem (Cassou-Noguès, Deligne–Ribet)

$\exists$ $L_{S,p}(\chi\omega, s) : \mathbb{Z}_p \longrightarrow \overline{\mathbb{Q}}_p$,

$k \equiv k' \bmod p^t(p-1)$

Then $L_S(\chi\omega^k, k) \equiv L_S(\chi\omega^{k'}, k') \bmod (p^{t+1})$

meromorphic s.t.

$$L_{S,p}(\chi\omega, n) = L_S(\chi\omega^n, n) \quad \forall \; n \in \mathbb{Z}_{\leq 0}.$$

$L_{S,p}(\chi\omega, s)$ is analytic, except possibly a pole at $s=1$ if $\chi = \omega^{-1}$. i.e. $\chi\omega = \mathbb{1}$

We will see a construction of $L_{S,p}(\chi\omega, s)$ due to Deligne - Ribet.

$$L_{S,p}(\chi\omega, 0) = L_S(\chi, 0) = L(\chi, 0) \prod_{v \in S} (1 - \chi(v))$$

Fact: $L(\chi, 0) \neq 0$ (as χ is totally odd)

Functional equation & $L(\chi, 1) \neq 0$

But $L_{S,p}(\chi\omega, 0) = 0$ if $\chi(v) = 1$ for some $v \in S$

(Trivial zeroes / Exceptional zeroes)

Gross - Stark conjecture is about the leading term of $L_{S,p}(\chi\omega, s)$ at $s=0$.

§ Hilbert Modular Forms.

F totally real. $d = [F:\mathbb{Q}]$

- $\underline{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$. $\mathcal{H} = \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$

  $\underline{z} \in \mathcal{H}^d$, $\underline{a}, \underline{b} \in \mathbb{C}^d$ $\quad (\underline{a}\underline{z} + \underline{b})^{\underline{k}} = \prod_{i=1}^{d} (a_i z_i + b_i)^{k_i}$

- $\mathfrak{b} \subseteq \mathcal{O}_F$ $\quad \psi_f : (\mathcal{O}_F/\mathfrak{b})^\times \longrightarrow \overline{\mathbb{Q}}^\times$

- $Cl^+(F)$ narrow class group of F. $\lambda \in Cl^+(F)$

$$\Gamma_\lambda := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(F) \;\middle|\; \begin{array}{c} a, d \in \mathcal{O}_F \\ b \in t_\lambda^{-1}\partial^{-1} \quad c \in \mathfrak{b} t_\lambda \partial \\ ad - bc \in \mathcal{O}_F^\times \end{array} \right\}$$

Here ∂ = different of F $\quad t_\lambda$ is a fixed ideal in λ.

$M_{\underline{k}}(\mathfrak{n}, \psi)$ = Hilbert modular forms of level $\mathfrak{n}$, weight $\underline{k}$ are $h = \# Cl^+(F)$

Nebentype ψ, tuples of functions $f_\lambda : \mathfrak{H}^d \longrightarrow \mathbb{C}$,

$(f_\lambda)_{\lambda \in Cl^+(F)}$ holomorphic, s.t. for $\gamma \in \Gamma_\lambda$

$$f_\lambda \big|_{\underline{\gamma}}(\underline{z}) := \det(\gamma)^{\underline{k}/2} (\underline{c}\,\underline{z} + \underline{d})^{-\underline{k}} f_\lambda(\underline{\gamma}\underline{z}) = \psi_f(a) f_\lambda(\underline{z})$$

Here $GL_2^+(F) \ni \underline{\gamma} = \left(\sigma(\gamma)\right)_{\sigma \in \mathrm{Hom}(F, \mathbb{C})} \in GL_2^+(\mathbb{R})^d$.

Remark: Holomorphicity at cusps is automatic if $F \neq \mathbb{Q}$.

The modularity gives Fourier expansion for each f_λ ("q-expansion at infinity")

$k=(k,\dots,k)$ parallel weight

$$\cancel{f_\lambda(z) = a_\lambda(0) + \sum_{\substack{b \in t_\lambda \\ b \gg 0}} a_\lambda(b)\, e_F(bz)}$$

totally positive

i.e. $\sigma(b) > 0$

for every $\sigma \in \mathrm{Hom}(F,\mathbb{R})$

$$e_F(x) = \exp(2\pi i \cdot \mathrm{Tr}_{F/\mathbb{Q}}(x)).$$

"Normalised Fourier co-efficients".

$m \subseteq \mathcal{O}_F$ in class λ

$$C(m, f) = a_\lambda(b)\, N(t_\lambda)^{-k/2}$$

$$c_\lambda(0, f) = a_\lambda(0)\, N(t_\lambda)^{-k/2}.$$

Examples of HMFs : Eisenstein series.

The most basic and an important example are Eisenstein series.

$\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}_F$ $\quad$ η modulo $\mathfrak{a}$

ψ modulo $\mathfrak{b}$.