Montgomery's conjecture on large values of Dirichlet polynomials

In the early 1970s, Montgomery studied large values of Dirichlet polynomials.

This problem is somewhat related to the restriction problem and other problems in Fourier analysis.

Montgomery raised an interesting conjecture, but there has been little progress since his work.

In this talk, we discuss what is known about the problem and the issues that make it hard to go further.

Montgomery's conjecture on Dirichlet polynomials

$$D(t) = \sum_{n=N+1}^{2N} b_n e^{it \log n}.$$

Conjecture. (Montgomery) If $p \geq 2$ and $T \geq N$, then

$$\|D\|_{L^p([0,T])} \lessapprox \left(N + N^{1/2} T^{1/p}\right) \|b_n\|_{\ell^\infty}.$$

Montgomery's conjecture on Dirichlet polynomials

$$D(t) = \sum_{n=N+1}^{2N} b_n e^{it \log n}.$$

Conjecture. (Montgomery) If $p \geq 2$ and $T \geq N$, then

$$\|D\|_{L^p([0,T])} \lessapprox \left(N + N^{1/2} T^{1/p}\right) \|b_n\|_{\ell^\infty}.$$

Example 1. $b_n = 1$. Then $|D(t)| \sim N$ for $|t| \leq 1$.

Example 2. $b_n = \pm 1$ random. Then $|D(t)| \sim N^{1/2}$ for most t.

Montgomery's conjecture on Dirichlet polynomials

$$D(t) = \sum_{n=N+1}^{2N} b_n e^{it \log n}.$$

Conjecture. (Montgomery) If $p \geq 2$ and $T \geq N$, then

$$\|D\|_{L^p([0,T])} \lessapprox \left(N + N^{1/2} T^{1/p}\right) \|b_n\|_{\ell^\infty}.$$

Example 1. $b_n = 1$. Then $|D(t)| \sim N$ for $|t| \leq 1$.

Example 2. $b_n = \pm 1$ random. Then $|D(t)| \sim N^{1/2}$ for most t.

Motivation: Estimates for the number of zeroes of Riemann zeta in different parts of the critical strips. Distribution of primes in short intervals.

Part 1: Related questions in harmonic analysis

Montgomery's question is related to several questions in restriction theory.

We will now recall them.

General extension operators

μ a measure on $\mathbb{R}^d$ or another abelian group.

$E_\mu f := \widehat{f\mu}$.

Problem: For a given μ, estimate the best constant C in

$$\|E_\mu f\|_{L^p} \leq C\|f\|_{L^q}.$$

Example (Stein): μ is surface measure on S^{d-1}.

General extension operators

μ a measure on $\mathbb{R}^d$ or another abelian group.

$E_\mu f := \widehat{f\mu}$.

Problem: For a given μ, estimate the best constant C in

$$\|E_\mu f\|_{L^p} \leq C\|f\|_{L^q}.$$

Example (Stein): μ is surface measure on S^{d-1}.

$D(t) = \sum_{n=N+1}^{2N} b_n e^{it \log n}$.

$\mu = \sum_{n=N+1}^{2N} \delta_{\log n}$.
$g(\log n) = b_n$.
Then $D = E_\mu g$.

Random extension operators

$\mathbb{Z}_T := \mathbb{Z}/T\mathbb{Z}$.
$A \subset \mathbb{Z}_T$ with $|A| = N$.
$g : A \to \mathbb{C}$.
$E_A g(x) := \sum_{a \in A} g(a) e(\frac{ax}{T})$.

Theorem (Bourgain, late 80s). If A is a random subset of $\mathbb{Z}_T$ with $|A| = N$, then with high probability

$$\|E_A g\|_{\ell^p(\mathbb{Z}_T)} \lesssim \left(N^{1/2} + T^{1/p}\right) \|g\|_{\ell^2} \leq \left(N + N^{1/2} T^{1/p}\right) \|g\|_{\ell^\infty}.$$

Random extension operators

$\mathbb{Z}_T := \mathbb{Z}/T\mathbb{Z}$.
$A \subset \mathbb{Z}_T$ with $|A| = N$.
$g : A \to \mathbb{C}$.
$E_A g(x) := \sum_{a \in A} g(a) e(\frac{ax}{T})$.

Theorem (Bourgain, late 80s). If A is a random subset of $\mathbb{Z}_T$ with $|A| = N$, then with high probability

$$\|E_A g\|_{\ell^p(\mathbb{Z}_T)} \lesssim \left(N^{1/2} + T^{1/p}\right) \|g\|_{\ell^2} \leq \left(N + N^{1/2} T^{1/p}\right) \|g\|_{\ell^\infty}.$$

Compare with Montgomery

$$D(t) = \sum_{n=N+1}^{2N} b_n e^{it \log n}.$$

Conjecture. $\|D\|_{L^p([0,T])} \lessapprox \left(N + N^{1/2} T^{1/p}\right) \|b_n\|_{\ell^\infty}$.

Random extension operators

Theorem (Bourgain, late 80s). If A is a random subset of $\mathbb{Z}_T$ with $|A| = N$, then with high probability

$$\|E_A g\|_{\ell^p(\mathbb{Z}_T)} \lesssim \left(N^{1/2} + T^{1/p}\right) \|g\|_{\ell^2} \leq \left(N + N^{1/2} T^{1/p}\right) \|g\|_{\ell^\infty}.$$

Compare with Montgomery

$D(t) = \sum_{n=N+1}^{2N} b_n e^{it \log n}$.

Conjecture. $\|D\|_{L^p([0,T])} \lessapprox \left(N + N^{1/2} T^{1/p}\right) \|b_n\|_{\ell^\infty}$.

- Same inequality. Sharp because of same examples.
- But Bourgain's proof doesn't give any information about any particular set A. Open problem to give explicit examples of A.

Part 2: Methods

Next we describe some of the methods used to prove L^p estimates for the Montgomery problem and more generally for extension operators E_A.

We are looking for implications of the form:

Condition about A $\rightarrow$ L^p bounds for E_A.

Part 2: Methods

Next we describe some of the methods used to prove L^p estimates for the Montgomery problem and more generally for extension operators E_A.

We are looking for implications of the form:

Condition about A $\rightarrow$ L^p bounds for E_A.

Problem: Given a set $A \subset \mathbb{Z}_T$ which is secretly a random set of size N, what L^p bounds for E_A can we prove in polynomial time?

What we know about Montgomery's problem is very similar to what we know about this problem.

I'll describe the methods in the context of a set $A \subset \mathbb{Z}_T$. They apply equally well to the Montgomery problem.

Orthogonality

Recall
$A \subset \mathbb{Z}_T$ with $|A| = N$.
$g : A \to \mathbb{C}$.
$E_A g(x) := \sum_{a \in A} g(a) e(\frac{ax}{T})$.

Orthogonality: $\|E_A g\|^2_{\ell^2(\mathbb{Z}_T)} = T \|g\|^2_{\ell^2}$.

The T is a normalization because $\|e(\frac{ax}{T})\|^2_{\ell^2(\mathbb{Z}_T)} = T$.

The proof is just that the characters $e(\frac{ax}{T})$ are orthogonal on $\mathbb{Z}_T$.

Orthogonality

Recall
$A \subset \mathbb{Z}_T$ with $|A| = N$.
$g : A \to \mathbb{C}$.
$E_A g(x) := \sum_{a \in A} g(a) e(\frac{ax}{T})$.

Orthogonality: $\|E_A g\|^2_{\ell^2(\mathbb{Z}_T)} = T \|g\|^2_{\ell^2}$.

The T is a normalization because $\|e(\frac{ax}{T})\|^2_{\ell^2(\mathbb{Z}_T)} = T$.

The proof is just that the characters $e(\frac{ax}{T})$ are orthogonal on $\mathbb{Z}_T$.

For Dirichlet polynomials $D(t) = \sum_{n=N+1}^{2N} b_n e^{it \log n}$,

the functions $e^{it \log n}$ are approximately orthogonal on $[0, T]$ if $T \geq N$.

Even integer moments

Recall $E_A g(x) := \sum_{a \in A} g(a) e(\frac{ax}{T})$.

Note $\sum_{x \in \mathbb{Z}_T} |E_A g(x)|^{2s} = \sum_{x \in \mathbb{Z}_T} |E_A g(x)^s|^2$.

But $E_A g(x)^s = \sum_{b \in \mathbb{Z}_T} \left(\sum_{a_1 + \ldots + a_s = b} g(a_1) \ldots g(a_s) \right) e(\frac{bx}{T})$.

Even integer moments

Recall $E_A g(x) := \sum_{a \in A} g(a) e(\frac{ax}{T})$.

Note $\sum_{x \in \mathbb{Z}_T} |E_A g(x)|^{2s} = \sum_{x \in \mathbb{Z}_T} |E_A g(x)^s|^2$.

But $E_A g(x)^s = \sum_{b \in \mathbb{Z}_T} \left(\sum_{a_1 + ... + a_s = b} g(a_1)...g(a_s) \right) e(\frac{bx}{T})$.

Define $r_{s,A}(b) = \#\{(a_1, ..., a_s) \in A^s : a_1 + ... + a_s = b\}$.

Condition 1: $\|r_{s,A}\|_{\ell^\infty} \lessapprox N^s / T$.

- True for a random set A with high prob.
- Checkable in polynomial time.
- A version of this is true for $A = \{\log n\}_{n=N+1}^{2N}$.

Proposition. Condition 1 $\rightarrow \|E_A g\|_{L^{2s}(\mathbb{Z}_T)} \lessapprox \left(N^{\frac{1}{2}} + T^{\frac{1}{2s}} \right) \|g\|_{\ell^2}$.

Even integer moments

Because of the trick on the last slide, even integer moments are better understood than other moments in many problems.

Montgomery's conjecture is proven when p is an even integer and open in every other case.

Bourgain's theorem was proven for even integer p long before his work.

When p is an even integer, there are explicit sets $A \subset \mathbb{Z}_T$ with $|A| \sim N$ and

$$\|E_A g\|_{L^p(\mathbb{Z}_T)} \lesssim (N^{1/2} + T^{1/p})\|g\|_{\ell^2}.$$

Large value estimates

L^p estimates are closely related to estimates for superlevel sets.

$W_\lambda f := \{x : |f(x)| > \lambda\}.$

Montgomery's large value estimate

Recall $E_A g(x) := \sum_{a \in A} g(a) e(\frac{ax}{T})$.

$E_A 1(x) := \sum_{a \in A} e(\frac{ax}{T}) = \hat{A}(x)$.

Condition 2: $\hat{A}(0) = N$ and $|\hat{A}(x)| \lessapprox N^{1/2}$ for $x \neq 0$.

- True for random A with high prob.
- Checkable in polynomial time.
- A version of this is conjectured for Dirichlet polynomials. But the proven bounds are weaker:

Theorem. (Montgomery) If A obeys condition 2
and if $\lambda > N^{1/4+\epsilon} \|g\|_{\ell^2}$
then $|W_\lambda(E_A g)| \lessapprox \frac{N}{\lambda^2} \|g\|_{\ell^2}^2$.

Montgomery's large value estimate

Recall $E_A g(x) := \sum_{a \in A} g(a) e(\frac{ax}{T})$.

$E_A 1(x) := \sum_{a \in A} e(\frac{ax}{T}) = \hat{A}(x)$.

Condition 2: $\hat{A}(0) = N$ and $|\hat{A}(x)| \lessapprox N^{1/2}$ for $x \neq 0$.

Theorem. (Montgomery) If A obeys condition 2
and if $\lambda > N^{1/4+\epsilon} \|g\|_{\ell^2}$
then $|W_\lambda(E_A g)| \lessapprox \frac{N}{\lambda^2} \|g\|_{\ell^2}^2$.

Remark. The conclusion is very strong.
If we knew the conclusion for all $\lambda > N^\epsilon \|g\|_{\ell^2}$,
then it would be even stronger than Montgomery's conjecture from Slide 2.

BUT, if $\lambda < N^{1/4} \|g\|_{\ell^2}$, the theorem tells us nothing about W_λ.

TT^* arguments

Montgomery proved his large value estimate using a TT^*-type argument.

This technique is somewhat similar to the proof of Tomas-Stein theorem in restriction theory, which was done a little bit later.

Recall $E_A g(x) := \sum_{a \in A} g(a) e(\frac{ax}{T})$.

For $W \subset \mathbb{Z}_T$, define

$E_{A,W} g(x) = 1_W(x) E_A(x)$.

Key ingredient: estimate $\|E_{A,W}\|_{2\to 2}^2 = \|E_{A,W} E_{A,W}^*\|_{2\to 2}$.

TT^* arguments

Recall $E_A g(x) := \sum_{a \in A} g(a) e(\frac{ax}{T})$.
For $W \subset \mathbb{Z}_T$, define

$E_{A,W} g(x) = 1_W(x) E_A(x)$.

Key ingredient: estimate $\|E_{A,W}\|_{2\to 2}^2 = \|E_{A,W} E_{A,W}^*\|_{2\to 2}$.

$E_{A,W} E_{A,W}^*$ is a matrix with rows and columns indexed by W.
The (x_1, x_2) entry is $\hat{A}(x_1 - x_2)$. Under condition 2:

- Diagonal entries are $N = |A|$.
- Off diagonal entries have norm $|\hat{A}(x_1 - x_2)| \leq C_\epsilon N^{1/2+\epsilon}$.

Therefore, $\|E_{A,W} E_{A,W}^*\|_{2\to 2} \leq N + C_\epsilon |W| N^{1/2+\epsilon}$.

TT^* arguments

Recall $E_A g(x) := \sum_{a \in A} g(a) e(\frac{ax}{T})$.
For $W \subset \mathbb{Z}_T$, define $E_{A,W} g(x) = 1_W(x) E_A(x)$.
Key ingredient: $\|E_{A,W} E_{A,W}^*\|_{2\to 2} \leq N + C_\epsilon |W| N^{1/2+\epsilon}$.

Set $W = W_\lambda(E_A g)$.

$\lambda^2 |W| \leq \|E_A g\|_{\ell^2(W)}^2 \leq \|E_{A,W}\|_{2\to 2}^2 \|g\|_{\ell^2}^2$.

TT^* arguments

Recall $E_A g(x) := \sum_{a \in A} g(a) e(\frac{ax}{T})$.
For $W \subset \mathbb{Z}_T$, define $E_{A,W} g(x) = 1_W(x) E_A(x)$.
Key ingredient: $\|E_{A,W} E_{A,W}^*\|_{2\to 2} \leq N + C_\epsilon |W| N^{1/2+\epsilon}$.

Set $W = W_\lambda(E_A g)$.

$$\lambda^2 |W| \leq \|E_A g\|_{\ell^2(W)}^2 \leq \|E_{A,W}\|_{2\to 2}^2 \|g\|_{\ell^2}^2.$$

$$\lambda^2 |W| \leq N \|g\|_{\ell^2}^2 + C_\epsilon N^{\frac{1/2}{+}\epsilon} \|g\|_{\ell^2}^2 \cdot |W|.$$

TT^* arguments

Recall $E_A g(x) := \sum_{a \in A} g(a) e(\frac{ax}{T})$.
For $W \subset \mathbb{Z}_T$, define $E_{A,W} g(x) = 1_W(x) E_A(x)$.
Key ingredient: $\|E_{A,W} E^*_{A,W}\|_{2 \to 2} \leq N + C_\epsilon |W| N^{1/2+\epsilon}$.

Set $W = W_\lambda(E_A g)$.

$$\lambda^2 |W| \leq \|E_A g\|^2_{\ell^2(W)} \leq \|E_{A,W}\|^2_{2 \to 2} \|g\|^2_{\ell^2}.$$

$$\lambda^2 |W| \leq N \|g\|^2_{\ell^2} + C_\epsilon N^{\frac{1/2}{+}\epsilon} \|g\|^2_{\ell^2} \cdot |W|.$$

If blue factor is at most $(1/2)\lambda^2$, we can rearrange, leaving

$$(1/2)\lambda^2 |W| \leq N \|g\|^2_{\ell^2} \text{ and so } |W| \lesssim \frac{N}{\lambda^2} \|g\|^2_{\ell^2}.$$

Limits of our knowledge

PART 3

Recall $D(t) = \sum_{n=N+1}^{2N} b_n e^{it \log n}$. Suppose $|b_n| \leq 1$ for all n.
Set $T = N^{3/2}$.

Proposition. $|W_{N^{3/4}}(D) \cap [0, T]| \lesssim N$.
Can prove with L^2 estimate or L^4 estimate.
If $\lambda = N^{3/4+\epsilon}$, the large value estimate gives much stronger bound fpr $|W_\lambda(D) \cap [0, T]|$.

Challenge: Prove $|W_{N^{3/4}}(D) \cap [0, T]| \lesssim N^{1-\delta}$.

Limits of our knowledge

Recall $D(t) = \sum_{n=N+1}^{2N} b_n e^{it \log n}$. Suppose $|b_n| \leq 1$ for all n.
Set $T = N^{3/2}$.

Proposition. $|W_{N^{3/4}}(D) \cap [0, T]| \lesssim N$.
Can prove with L^2 estimate or L^4 estimate.
If $\lambda = N^{3/4+\epsilon}$, the large value estimate gives much stronger bound fpr $|W_\lambda(D) \cap [0, T]|$.

Challenge: Prove $|W_{N^{3/4}}(D) \cap [0, T]| \lesssim N^{1-\delta}$.

The proposition has an easy proof (just compute the L^2 norm).
So I was surprised that it is difficult to improve it by a tiny bit.

Then I learned about a cousin problem where the Proposition is sharp.

Enemy scenario (Fu - G -Maldague-Cohen)

Define $a_n = \log n$. Recall $D(t) = \sum_{n=N+1}^{2N} b_n e^{ita_n}$.

Define $\tilde{a}_n = \sqrt{\frac{n}{N}}$. Define $\tilde{D}(t) = \sum_{n=N+1}^{2N} b_n e^{it\tilde{a}_n}$.

The sequences a_n and $\tilde{a}_n$ look similar on the real line.

Conjecturally, all the previous discussion applies to both a_n and $\tilde{a}_n$. In particular, conjecturally, a_n and $\tilde{a}_n$ both satisfy versions of Condition 1 and Condition 2 as long as $T \leq N^{3/2}$.

Enemy scenario (Fu - G -Maldague-Cohen)

Define $a_n = \log n$. Recall $D(t) = \sum_{n=N+1}^{2N} b_n e^{ita_n}$.

Define $\tilde{a}_n = \sqrt{\frac{n}{N}}$. Define $\tilde{D}(t) = \sum_{n=N+1}^{2N} b_n e^{it\tilde{a}_n}$.

Notice that the sequence $\{\tilde{a}_n\}_{n=N+1}^{2N}$ contains the integers from $\sqrt{N}$ to $\sqrt{2N}$.

We can use this to build an ‘enemy example’.

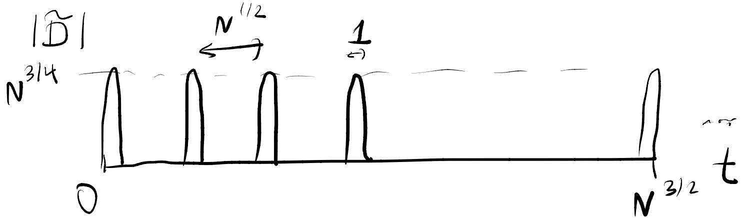
Enemy scenario (Fu - G -Maldague, Cohen)

Define $a_n = \log n$. Recall $D(t) = \sum_{n=N+1}^{2N} b_n e^{ita_n}$.

Define $\tilde{a}_n = \sqrt{\frac{n}{N}}$. Define $\tilde{D}(t) = \sum_{n=N+1}^{2N} b_n e^{it\tilde{a}_n}$.

Set $b_n = \begin{cases} N^{1/4} & n = m^2 \\ 0 & \text{else} \end{cases}$

Note $\tilde{D}(t) = N^{1/4} \sum_{\sqrt{N} < m \le \sqrt{2N}} e^{i \frac{m}{\sqrt{N}} t}$.



So $|W_{N^{3/4}} \tilde{D} \cap [0, N^{3/2}]| \sim N$. This matches upper bound from L^2 norm.

Challenge problem vs. enemy scenario

Challenge problem.
Define $a_n = \log n$. Recall $D(t) = \sum_{n=N+1}^{2N} b_n e^{ita_n}$.
Suppose $|b_n| \leq 1$. (And so $\sum_n |b_n|^2 \leq N$.)
Try to prove $|W_{N^{3/4}}(D) \cap [0, N^{3/2}] < N^{1-\delta}$.

Enemy scenario.
Define $\tilde{a}_n = \sqrt{\frac{n}{N}}$. Define $\tilde{D}(t) = \sum_{n=N+1}^{2N} b_n e^{it\tilde{a}_n}$.

Set $b_n = \begin{cases} N^{1/4} & n = m^2 \\ 0 & \text{else} \end{cases}$. Note $\sum_{n=N+1}^{2N} |b_n|^2 \sim N$.

Then $|W_{N^{3/4}}(\tilde{D}) \cap [0, N^{3/2}] \sim N$.

Challenge problem vs. enemy scenario

Challenge problem.
Define $a_n = \log n$. Recall $D(t) = \sum_{n=N+1}^{2N} b_n e^{ita_n}$.
Suppose $|b_n| \leq 1$. (And so $\sum_n |b_n|^2 \leq N$.)
Try to prove $|W_{N^{3/4}}(D) \cap [0, N^{3/2}] < N^{1-\delta}$.

Enemy scenario.
Define $\tilde{a}_n = \sqrt{\frac{n}{N}}$. Define $\tilde{D}(t) = \sum_{n=N+1}^{2N} b_n e^{it\tilde{a}_n}$.
Set $b_n = \begin{cases} N^{1/4} & n = m^2 \\ 0 & \text{else} \end{cases}$. Note $\sum_{n=N+1}^{2N} |b_n|^2 \sim N$.
Then $|W_{N^{3/4}}(\tilde{D}) \cap [0, N^{3/2}] \sim N$.

To beat enemy scenario bound, we must either:

1. Distinguish a_n from $\tilde{a}_n$.
2. Distinguish $\|b_n\|_{\ell^\infty} \leq 1$ from $\|b_n\|_{\ell^2}^2 \leq N$.

Wave packets

PART 4

Wave packets have been a crucial tool for studying the extension operator for submanifolds of $\mathbb{R}^n$.

There is also a version of wave packets for Dirichlet polynomials, which goes back to Bourgain's work on the Montgomery conjecture in the late 80s.

We will recall this work.

But then we will argue that wave packets by themselves don't give enough information to make progress on the challenge problem.

Recall extension operator for circle vs. Dirichlet polynomials

Suppose μ is arc length measure on the unit circle S^1.
For $g : S^1 \to \mathbb{C}$, $E_{S^1} g(x) := \int_{S^1} g(\omega) e^{i\omega \cdot x} d\mu(\omega)$.

Let $A = \{\log n\}_{n=N+1}^{2N}$.
For $g : A \to \mathbb{C}$, define $E_A g(t) = \sum_{a \in A} g(a) e^{ita}$.
If we identify $g(\log n) = b_n$, then
$E_A g(t) = D(t) = \sum_{n=N+1}^{2N} b_n e^{it \log n}$.

Recall extension operator for circle vs. Dirichlet polynomials

Suppose μ is arc length measure on the unit circle S^1.
For $g : S^1 \to \mathbb{C}$, $E_{S^1} g(x) := \int_{S^1} g(\omega) e^{i\omega \cdot x} d\mu(\omega)$.

Let $A = \{\log n\}_{n=N+1}^{2N}$.
For $g : A \to \mathbb{C}$, define $E_A g(t) = \sum_{a \in A} g(a) e^{ita}$.
If we identify $g(\log n) = b_n$, then
$E_A g(t) = D(t) = \sum_{n=N+1}^{2N} b_n e^{it \log n}$.

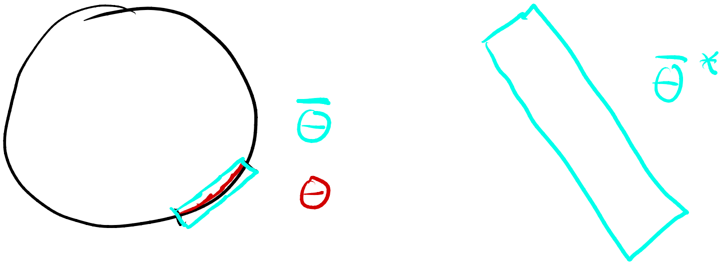
Circle is not a straight line, leading to non-trivial estimates.

Analogously A is not an arithmetic progression, leading to non-trivial estimates.

Wave packets

Suppose μ is arc length measure on the unit circle S^1.
For $g : S^1 \to \mathbb{C}$, $E_{S^1} g(x) := \int_{S^1} g(\omega) e^{i\omega \cdot x} d\mu(\omega)$.

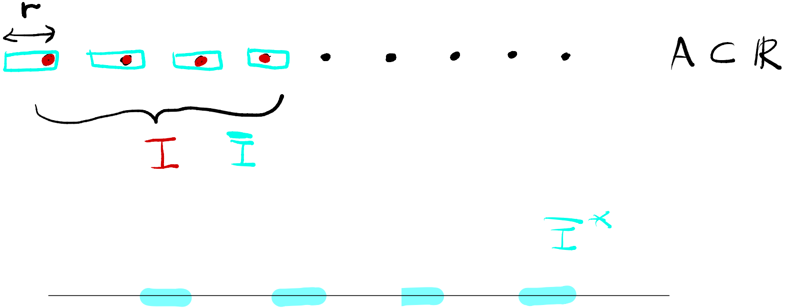
Suppose $\theta \subset S^1$ arc. $\bar{\theta}$ rectangular box around θ. $\bar{\theta}^*$ the dual box.



Lemma (vague) If g_θ supported in θ,
then $|E_{S^1} g_\theta| \approx$ constant on translates of $\bar{\theta}^*$.

Wave packets for Dirichlet polynomials

Let $A = \{\log n\}_{n=N+1}^{2N}$.
For $g : A \to \mathbb{C}$, define $E_A g(t) = \sum_{a \in A} g(a) e^{ita}$.
Suppose I is the intersection of A with an interval.
Then $I \subset N_r(\mathrm{Arith.\ Progr.}) =: \bar{I}$.



Lemma (vague) If g_I is supported on I,
then $|E_A g_I| \approx$ constant on translates of $\bar{I}^*$.

Montgomery and Kakeya

Theorem (Bourgain, late 80s): Montgomery conjecture implies Kakeya conjecture.

Also, in Montgomery conjecture, need a factor of $\log T$.

Proof idea. Can choose g_I so that Eg_I is concentrated on a single translate of $\bar{I}^*$.
Arrange these translates to overlap a lot.
Analogous to Fefferman's counterexample to ball multiplier.

Each $\bar{I}^*$ is a fat AP.
Different I have APs with different common difference.
This leads to an arithmetic variant of the Kakeya problem.
The arithmetic variant problem is probably harder than the original Kakeya problem.
Bourgain also found a clever way to relate original Kakeya to arithmetic Kakeya.

Montgomery and Kakeya 2

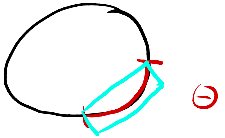
Wave packets were used to show that
restriction conjecture implies Kakeya conjecture.
But they have also been used to make a lot of progress on restriction conj.

Wave packets were used to show that
Montgomery conjecture implies Kakeya conjecture.
But can they lead to progess on Montgomery conjecture?

Fu, Maldague and I tried to adapt wave-packet based ideas from restriction/decoupling to Montgomery conjecture.

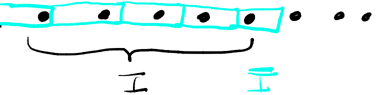
Issue with wave packets for Montgomery

Wavepackets for circle
Suppose $\theta \subset S^1$ arc. $\bar{\theta}$ rectangular box around θ.
Key fact: $|E_{S^1} g_\theta| \approx$ constant on translates of $\bar{\theta}^*$.
If $\mathrm{diam}(\theta) \ll 1$, then $\bar{\theta}$ is "non-trivial".



Wave packets for Dirichlet polynomials. Recall $A = \{\log n\}_{n=N+1}^{2N}$.
Suppose $I \subset A$. $\bar{I}$ smallest fat AP containing I
Key fact: $|E_A g_I| \approx$ constant on translates of $\bar{I}^*$.
If $\mathrm{diam}(I) < N^{-1/2}$, $\bar{I}$ is "non-trivial".

If $\mathrm{diam}(I) > N^{-1/2}$, $\bar{I}$ is just an interval.



Issue with wave packets for Montgomery

Wavepackets for circle
Suppose $\theta \subset S^1$ arc. $\bar{\theta}$ rectangular box around θ.
Key fact: $|E_{S^1} g_\theta| \approx$ constant on translates of $\bar{\theta}^*$.
If $\mathrm{diam}(\theta) \ll 1$, then $\bar{\theta}$ is "non-trivial".

Wave packets for Dirichlet polynomials. Recall $A = \{\log n\}_{n=N+1}^{2N}$.
Suppose $I \subset A$. $\bar{I}$ smallest fat AP containing I
Key fact: $|E_A g_I| \approx$ constant on translates of $\bar{I}^*$.
If $\mathrm{diam}(I) < N^{-1/2}$, $\bar{I}$ is "non-trivial".

If $\mathrm{diam}(I) > N^{-1/2}$, $\bar{I}$ is just an interval.

Wave packets are not enough to rule out enemy scenario

Issues:

1. $A = \{\log n\}_{n=N+1}^{2N}$ and $\tilde{A} = \sqrt{\frac{n}{N}}$ have the same wave packets.

2. In enemy scenario, $|E_{\tilde{A}} g_I(t)|$ is contant for I with $\mathrm{diam}(I) < N^{-1/2}$. There is no interesting behavior to try to bound.

But if $\mathrm{diam}(I) > N^{-1/2}$, then there are no wave packets to try to use anyway.

Short Dirichlet polynomials

$D_{short}(t) = \sum_{n=N+1}^{N+N^{1/2}} b_n e^{ita_n}$, with $a_n = \log n$ OR $a_n = \sqrt{\frac{n}{N}}$.

In this range, there is a very close analogy between Dirichlet polynomials and the extension operator for the circle or parabola.

Fu-G-Maldague proved analogues of restriction estimates, decoupling, small cap decoupling, ...

Theorem (FGM): If $N \leq T \leq N^2$ and $p \geq 2$, then

$$\|D_{short}\|_{L^p([0,T])} \lessapprox \left(N^{\frac{1}{2}} N^{\frac{1}{2p}} + T^{\frac{1}{p}} N^{\frac{1}{4}} \right) \|b_n\|_{\ell^\infty}.$$

This bound is sharp for all T from $N^{3/2}$ to N^2, for every $p \geq 2$.

Short Dirichlet polynomials

$D_{short}(t) = \sum_{n=N+1}^{N+N^{1/2}} b_n e^{ita_n}$, with $a_n = \log n$ OR $a_n = \sqrt{\frac{n}{N}}$.

In this range, there is a very close analogy between Dirichlet polynomials and the extension operator for the circle or parabola.

Fu-G-Maldague proved analogues of restriction estimates, decoupling, small cap decoupling, ...

Theorem (FGM): If $N \le T \le N^2$ and $p \ge 2$, then

$$\|D_{short}\|_{L^p([0,T])} \lessapprox \left(N^{\frac{1}{2}} N^{\frac{1}{2p}} + T^{\frac{1}{p}} N^{\frac{1}{4}} \right) \|b_n\|_{\ell^\infty}.$$

This bound is sharp for all T from $N^{3/2}$ to N^2, for every $p \ge 2$.

BUT for original Montgomery, wave packets don't distinguish D from $\tilde{D}$ and don't rule out enemy scenario.