Module – 3.6

EXAMPLES OF STATIC INVERSE PROBLEMS

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THREE EXAMPLES

- Recovery of vertical temperature profile of the atmosphere from satellite radiance measurement – linear problem

- 1-D Spatial linear and 2-D bilinear interpolation – linear problem

- A nonlinear least squares problem

VERTICAL TEMPERATURE PROFILE

- Problem is to retrieve the vertical temperature profile of the atmosphere from satellite radiance measurements

PROBLEM 1: SATELLITE RADIANCE – A MODEL

- Energy R_f received by a satellite at a frequency, f is related to the vertical temperature profile, T(Þ) at the pressure level Þ of the atmosphere through a formula given by

$$R_f = \exp[-\gamma_f] + \int_0^1 T(Þ)W(Þ,\gamma_f)dÞ \qquad \text{-> (1)}$$

where $W(Þ,\gamma_f)$ is the weight function given by

$$W(Þ,\gamma_f) = Þ\gamma_f\exp(\gamma_f Þ) \qquad \text{-> (2)}$$

and γ_f is a constant that depends on f

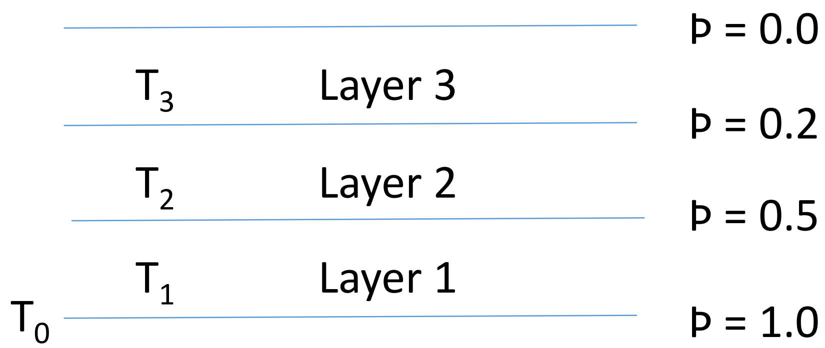
INPUT DATA

- The values of f and γ_{f} relevant to the problem are given by

i	1	2	3	4	5
f_i	0.9	1.0	1.1	1.2	1.3
γ_{f_i}	$\frac{1}{0.9}$	$\frac{1}{0.7}$	$\frac{1}{0.5}$	$\frac{1}{0.3}$	$\frac{1}{0.2}$

A DISCRETE MODEL

- The problem is to recover the function T(Þ) from a set of discrete measurements of R_{f_i}, $1 \le i \le 5$ – an underdetermined system
- We discretize the atmosphere by considering it as a 3-layered system



- T_0 is the temperature of the earth's surface
- T_i is the average temperature of the layer i, $1 \le i \le 3$
- Layers are bounded by isobaric surfaces at Þ = 1.0, 0.5, 0.2, and 0.0

A DISCRETE RELATION

- Discretizing (1) for the frequency f_i, $1 \le i \le 5$ using the 3-layer approximation:

$$Z_i = R_{f_i} - \exp[-\gamma_{f_i}]$$

$$= T_1 \int_{0.5}^{1.0} Þ\gamma_{f_i} \exp[-Þ\gamma_{f_i}] dÞ = T_1 a_{i1}$$

$$+ T_2 \int_{0.2}^{0.5} Þ\gamma_{f_i} \exp[-Þ\gamma_{f_i}] dÞ + T_2 a_{i2}$$

$$+ T_3 \int_{0.0}^{0.2} Þ\gamma_{f_i} \exp[-Þ\gamma_{f_i}] dÞ + T_3 a_{i3}$$

where the constant a_{i1}, a_{i2}, a_{i3} are the numerical values of the respective integrals obtained using the input data and by integration by parts

A LINEAR MODEL

- By collating the five linear relations between Z_i and T_1, T_2, T_3, for each frequency f_i, $1 \le i \le 5$, we get a linear model:

$$\begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \\ Z_5 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \\ a_{51} & a_{52} & a_{53} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \qquad \text{-> (3)}$$

Or $Z = Hx$, $Z \in R^5$, $H \in R^{5x3}$, $T \in R^3$ -> (4)

A TWIN EXPERIMENT – COMPUTER PROJECT: GENERATE OBSERVATION

- Set $T_1 = 0.9$, $T_2 = 0.85$, $T_3 = 0.875$, set $\bar{x} = (T_1, T_2, T_3)^T$
- Evaluate a_{i1}, a_{i2}, a_{i3} for $1 \leq i \leq 5$ using the input data
- This gives the matrix H
- Compute $\bar{Z} \in R^5$ using (3) as $\bar{Z} = H\bar{x}$
- Now generate an observation noise vector $V \in R^5$ such that $V \sim N(0, \sigma^2 I_5)$ where I_5 is the identity matrix of order 5 and σ^2 is the common variance of the radiance measurement
- Let $Z = \bar{Z} + V$, be the noisy observation

A TWIN EXPERIMENT – RECOVER T FROM NOISY OBSERVATION

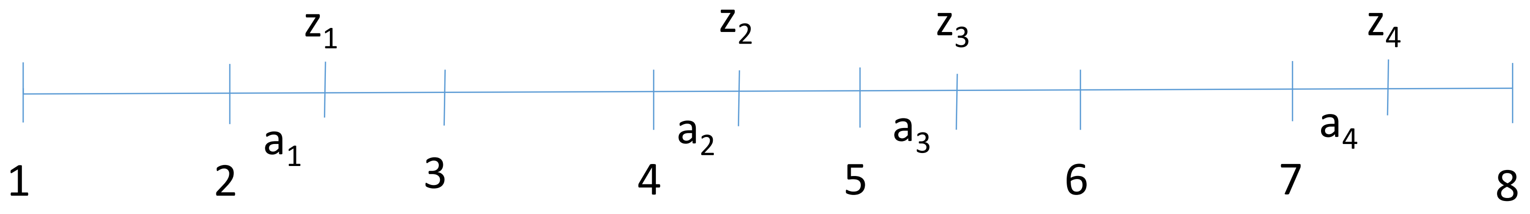
- Using this noisy observation vector Z, now solve the overdetermined linear least squares problem Z = Hx and recover x

- Compute the $||x - x_{LS}||_2$ and plot it as a function of σ^2 by repeatedly solving the problem for σ^2 = 0.0, 0.1, 0.4, 0.8, 1.0, 1.2

- Comment on your result

PROBLEM 2: SPATIAL INTERPOLATION – 1-D

- Consider a uniform spatial computational grid in 1-D with n points: n = 8



- The grid interval is assumed to be unity
- Let $x = (x_1, x_2, x_3, \ldots x_n)^T \in R^n$ be the unknown state vector
- Let $z_1, z_2, \ldots z_m$ be the m observations of a scalar field variable such as, say temperature, concentration of a pollutant, to name a few, where $m < n$

DISTRIBUTION OF THE OBSERVATIONS

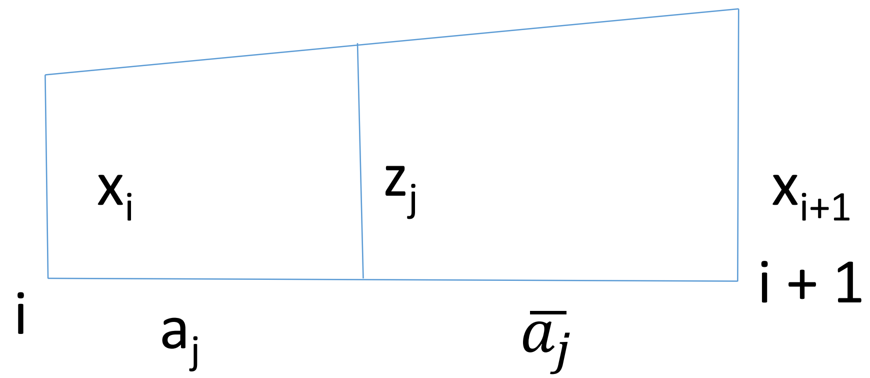
- Let the j^{th} observation z_j be contained in the interval [i, i + 1]

- Referring to the Figure above, $m = 4$, $n = 8$, z_1 is in [2, 3], z_2 is in [4, 5], z_3 is in [5, 6] and z_4 is in [7, 8]

- <u>Problem:</u> Given $Z \in R^m$, find $x \in R^n$ where Z and x refer to the same quantities such as temperature, concentration, etc

A LINEAR INTERPOLATION

- Consider the interval [i, i + 1] containing z_j



$a_j + \overline{a_j} = 1$

- a_j is the distance of z_j from the left and i
- Relate x_i, x_{i+1} and z_j using a simple linear relation as:

$$\frac{z_j - x_i}{a_j} = \frac{x_{i+1} - z_j}{\overline{a_j}} \qquad \text{-> (4)}$$

- That is, $z_j = \overline{a_j}\,x_i + a_j x_{i+1}$ -> (5)

A LINEAR INVERSE PROBLEM: UNDERDETERMINED CASE

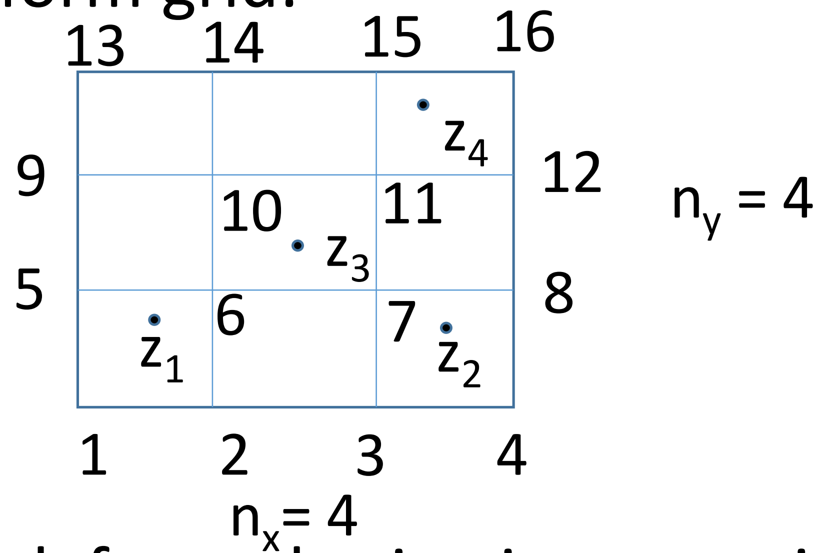
- Applying (5) to each of the m = 4 observations on the uniform grid with n = 8 points:

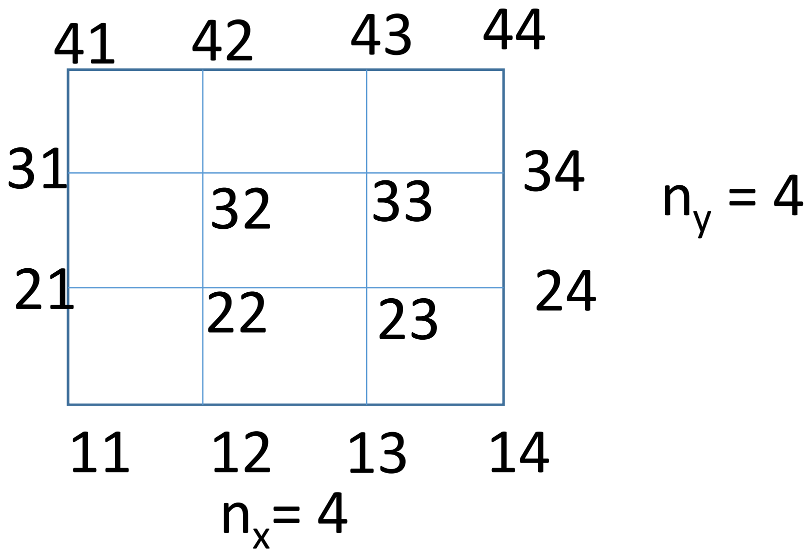
$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & \overline{a}_1 & a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{a}_2 & a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{a}_3 & a_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \overline{a}_4 & a_4 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} \quad \text{-> (7)}$$

- That is,1-D interpolation matrix H is such that row sum is 1 and Z = Hx
- We can solve for $x_{LS} = H^T(HH^T)^{-1}Z$
- We can estimate the temperature, concentration at the computational grid from the observation

SPATIAL INTERPOLATION – 2D

- Consider 2-D version with $n = n_x n_y$ grid points arranged in an n_x by n_y uniform grid:





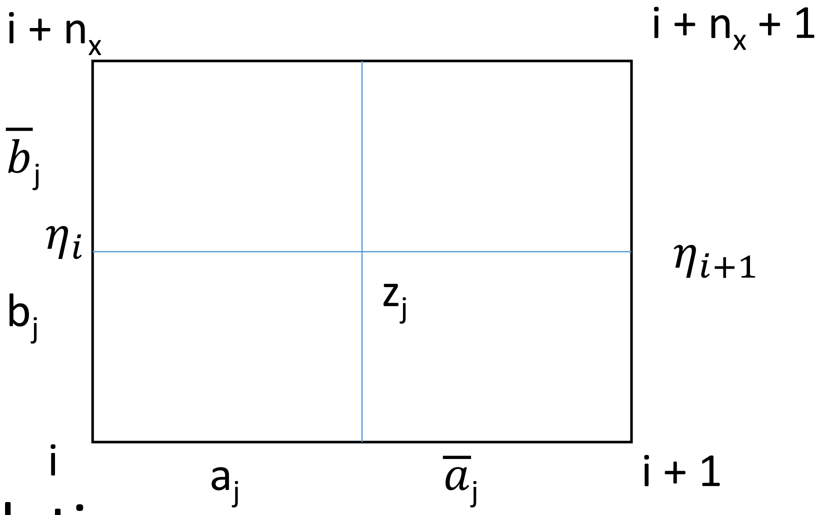
- The left numbering is row major order scheme and the right is the standard (i, j) notation
- The node label k in row major order related to the (i, j) scheme as

$$k = (i - 1)n_x + j$$

- With $n_x = 4$, the node label 7 correspond to (2,3) since 7 = (2-1)*4+3

A BILINEAR INTERPOLATION

- Let the j^{th} observation z_j be contained in the 2D-grid box whose south-east corner node has label i:



$a_j + \bar{a}_j = 1$

$b_j + \bar{b}_j = 1$

- By 1-D linear interpolation:

$$z_j = \bar{a}_j\eta_i + a_j\eta_{i+1} \qquad \text{-> (7)}$$

- Again, by 1-D linear interpolation

$$\eta_i = x_i\bar{b}_j + x_{i+nx}b_j \qquad \text{-> (8)}$$

$$\eta_{i+1} = x_{i+1}\bar{b}_j + x_{i+nx+1}b_j \qquad \text{-> (9)}$$

A LINEAR INVERSE PROBLEM

- Substituting (8) – (9) in (7) and simplifying

$$z_j = \bar{a}_j \bar{b}_j x_i + a_j \bar{b}_j x_{i+1} + \bar{a}_j b_j x_{i+nx} + a_j b_j x_{i+nx+1} \qquad \text{-> (10)}$$

- By collating the four relations for the four observation in the 2-D and:

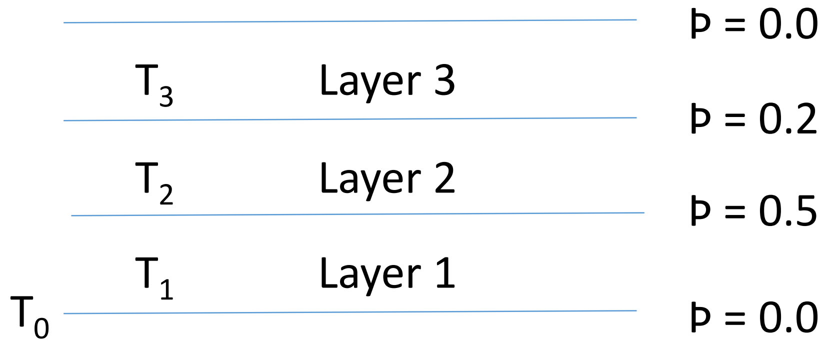
$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{array}{c} \begin{array}{cccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \end{array} \\ \begin{bmatrix} * & * & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ \vdots \\ x_8 \\ x_9 \\ \vdots \\ x_{12} \\ x_{13} \\ \vdots \\ x_{16} \end{bmatrix}$$

* - represents non-zero element based on (10)

- The 2-D interpolation matrix is such that the row sum is 1 and Z = Hx
- Hence $x_{LS} = H^T(HH^T)^{-1}Z$ is the optimal estimate

PROBLEM 3: A NON LINEAR PROBLEM

- Con sider a three layered atmosphere



- Let $T(Þ) = x_1(Þ - x_2)^2 + x_3$, $0 \leq Þ \leq 1$ -> (9)

- Let $x = (x_1, x_2, x_3)^T \in R^3$ be the unknown

RELATION BETWEEN TEMPERATURE AND RADIANCE

- The observations are measures of overlapping fractions of the area under the curve:

$$\overline{Z_{ij}} = \int_{Þ_i}^{Þ_j} T(Þ)\,dÞ \qquad \text{-> (10)}$$

- The observations are given by

$Þ_i$	$Þ_j$	$\overline{Z_{ij}}$
0.00	0.25	0.21
0.20	0.50	0.15
0.30	0.70	0.51
0.6	0.80	0.11

THE MODEL EQUATIONS

- After integration:

$$\overline{Z_{ij}} = \int_{Þ_i}^{Þ_j} [x_1(Þ - x_2)^2 + x_3]dÞ$$

$$= \frac{x_1}{3}[(Þ_j - x_2)^3 - (Þ_i - x_2)^3] + x_3(Þ_j - Þ_i))$$

- Referring to the Table in slide 19:

$$z_1 = 0.21 = \frac{x_1}{3}[(0.25 - x_2)^3 - {x_2}^3] + 0.25x_3 = h_1(x)$$

$$z_2 = 0.15 = \frac{x_1}{3}[(0.5 - x_2)^3 - (0.2 - x_2)^3] + 0.3x_3 = h_2(x)$$

$$z_3 = 0.51 = \frac{x_1}{3}[(0.7 - x_2)^3 - (0.3 - x_2)^3] + 0.4x_3 = h_3(x)$$

$$z_2 = 0.11 = \frac{x_1}{3}[(0.8 - x_2)^3 - (0.6 - x_2)^3] + 0.2x_3 = h_4(x)$$

NONLINEAR INVERSE PROBLEM

- Let $Z = h(x)$ with $Z = (z_1, z_2, z_3, z_4)^T$

$$h(x) = (h_1(x), h_2(x), h_3(x), h_4(x))^T$$

- Compute $r(x) = Z - h(x)$
- compute $f(x) = (Z - h(x))^T(Z - h(x))$
- Set $\nabla_x f(x) = 0$ and solve for x
- Check if $\nabla_x^2 f(x)$ is PD

APPROXIMATIONS

- Compute the Jacobian $D_x(h)$ and $D_x^2(h, y)$
- Build first and second order approximation to h(x)
- Solve the minimization arising from the first and second-order approximation

EXERCISES

11.1) (a) Compute the solution of ∇_xf(x) = 0 for the non linear problem described in slides 18-21, by using the nonlinear solvers in MATLAB

(b) Evaluate the Hessian ∇_x^2f(x) at each of the solution obtained in (a) and find the maxima and minima of f(x)

11.2) (a) Compute the Jacobian and the Hessian of h(x) described in slide 21

(b) Using these develop a first order and second order approximation to f(x)

(c) Starting from $x_c = (1, 1, 1)^T$, iterate twice and comment on the progress of your algorithm

REFERENCES

- This module follows closely chapters 5 through 7 of LLD (2006)