Module – 2.1

FINITE DIMENSIONAL VECTOR SPACES:

AN OVERVIEW

S. Lakshmivarahan

School of Computer Science

University of Oklahoma

Norman, Ok – 73069, USA

varahan@ou.edu

FINITE DIMENSIONAL VECTOR SPACE

- **R – Set of all real numbers – also called real scalars**
- **C – Set of all complex numbers – also called complex scalars**
- **R^n – Set of all real vectors of size n**
- **C^n – Set of all complex vectors of size n**

x ∈ R^n => x = $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ x_i ∈ R

0 = $\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ∈ R^n, null vector

X = $\begin{pmatrix} 3.2 \\ 1.5 \\ 9.9 \end{pmatrix}$ ∈ R^3

x ∈ C^n => x = $\begin{pmatrix} x_1 + iy_1 \\ \vdots \\ x_i + iy_i \\ \vdots \\ x_n + iy_n \end{pmatrix}$ x_i ,y_i ∈ R, i = $\sqrt{-1}$

X = $\begin{pmatrix} 1 + i \\ 1 - i \\ 1 \ - 2i \end{pmatrix}$ ∈ C^3

- We will largely deal with R^n

OPERATIONS VECTORS

- $x, y, z \in R^n$ $\quad$ $x = (x_1, x_2, \ldots, x_n)^T$

  $a, b, c \in R^n$ $\quad$ $y = (y_1, y_2, \ldots, y_n)^T$

  $z = (z_1, z_2, \ldots, z_n)^T$

- $z = x \pm y \Rightarrow z_i = x_i \pm y_i$ $\quad$ $1 \le i \le n$ – Vector addition /subtraction
- $y = ax \Rightarrow y_i = ax_i$ $\quad$ $1 \le i \le n$ – Scalar multiplication of a vector
- $z = ax + y \Rightarrow z_i = ax_i + y_i$ $\quad$ $1 \le i \le n$ – Scalar times a vector + a vector - Saxpy

LINEAR VECTOR SPACE

- Let V – denote a set or collection of real vectors of size n
- **V is called a (linear) <u>vector space</u> if it satisfies the following three conditions**:

  <u>C1). V is a group under addition</u>

  1) x + y ∈ V if x, y ∈ V – Closed under addition
  2) x + y + z = x + (y + z) = (x + y) + z – Associative property of addition
  3) V contains a zero vector 0: x + 0 = 0 + x = x ∀ x ∈ V
  4) For every x ∈ V, there is a unique y ∈ V: x + y = y + x = 0. y is called the additive inverse of x and y = -x

  <u>C2). Scalar multiplication</u>

  1) ax ∈ V if x ∈ V – Closed under scalar multiplication
  2) a(bx) = (ab)x – for all x ∈ V and a, b ∈ R
  3) 1x = x – for all x ∈ , 1 is the real number one

  <u>C3). Distributivity</u>

  1) a(x + y) = ax + ay – for all x, y ∈ V and a ∈ R
  2) (a + b)x = ax + bx – for all x ∈ V and a, b ∈ R

EXAMPLE OF VECTOR SPACES

- 1) R is a real vector space of real scalars
- 2) R^n is a real vector space of n-vector, ($n \geq 1$)
- 3) C^n is a complex vector space of n – complex vector, ($n \geq 1$)
- 4) The set of all nxn real matrices is a vector space
- 5) The set of all polynomials of degree n is a vector space
- 6) Let $x = (x_1, x_2, \ldots, x_n, \ldots)$ be an infinite sequence, with $\sum_{i=1}^{\infty} x_i^2 < \infty$ is a vector space – square summable sequences
- 7) The set of all continuous function over the interval [a,b] is a vector space

OPERATION ON VECTORS IN R^n

- Let x, y, z ϵ R^n, a, b, c ϵ R, <·, ·>: R^n x R^n -> R
- <u>Inner/scalar product of x and y is a scalar</u>

<x, y> = $x^T y$ = $\sum x_i y_i$ = $\sum y_i x_i$ = $y^T x$ = <y, x> - symmetry

- <u>Properties of <·, ·></u>

  1) <x, y> > 0 ∀x ≠ 0 – Positive definite
     = 0 only if x = 0
  2) <x, y> = <y,x> - symmetry
  3) <x + y, z> = <x, z> + <y, z> - additive
  4) <ax, y> = a<x, y> = <x, ay> - homogeneity
  5) <x, z> = <y, z> for all z -> x = y

<u>Note:</u> For x,y ϵ C^n,

<x, y> = $\sum x_i \bar{y}_i$, $\bar{y}_i$ is the complex conjugate of y_i

OPERATION ON VECTORS IN R^n

- Outer product of two vectors is a matrix:

$$xy^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} (y_1, y_2, \ldots, y_n) = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & \cdots & x_2y_n \\ \vdots & \vdots & \ddots & \vdots \\ x_ny_1 & x_ny_2 & \cdots & x_ny_n \end{bmatrix}$$

outer product: $R^n \times R^n \to R^{n \times n}$ = $[xy_1\ xy_2\ \ldots\ xy_n]$ - multiples of column x

$$= \begin{bmatrix} x_1y^T \\ x_2y^T \\ \vdots \\ x_ny^T \end{bmatrix}$$ - multiples of row y

NORM AND DISTANCE

- **<u>Norm</u> of x denoted by $\|x\|$ is a scalar associated with x – denotes a measure of the size of x**

  1) Euclidean/ 2 – norm: $\|x\|_2 = (\sum_{i=1}^{n} x_i^2)^{½} = \langle x, x\rangle^{½}$

  2) Manhattan/ 1 – norm: $\|x\|_1 = \sum_{i=1}^{n} |x|$

  3) Chebyshev/ ∞ – norm: $\|x\|_\infty = \max_i\{|x_i|\}$

  4) Minkowski/ p – norm: $\|x\|_p = [\sum_{i=1}^{n} |x_i|^p]^{1/p}$

  5) Energy norm: $\|x\|_A = (x^T A x)^{½}$ - A symmetric positive definite matrix

- **<u>Distance:</u>** d(x, y) between x, y ∈ R^n

  $d(x,y) = \|x - y\|$ – depends on the choice of the norm

GENERAL PROPERTY OF A NORM

- Let N: R^n -> R then N(x) is a norm if it satisfies the following:

  1) $N(x) > 0$ when $x \neq 0$ – positive definite

     $= 0$ when $x = 0$

  2) $N(ax) = |a|N(x)$ – homogenous.

  3) $N(x + y) \leq N(x) + N(y)$ – Triangle inequality

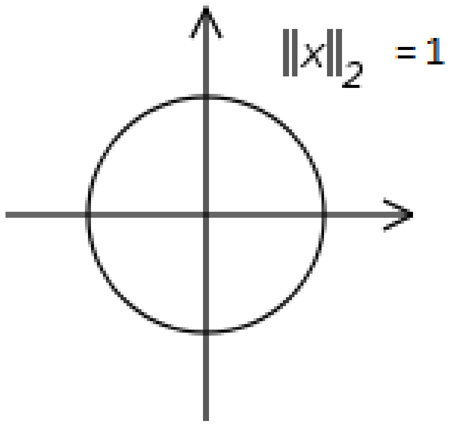
  Note:

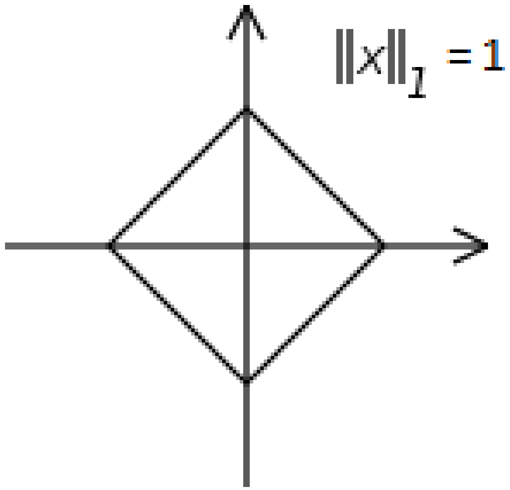
  - $||x||_2$ is derivable from inner product
  - Verify $||x + y||_2^2 + ||x - y||_2^2$ = 2($||x||_2^2 + ||x||_2^2$) called the parallelogram law

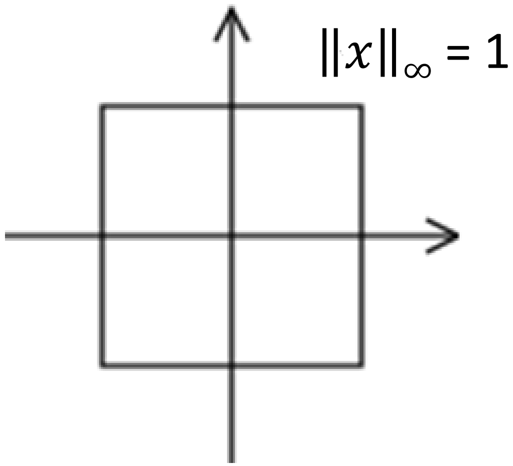
UNIT SPHERE

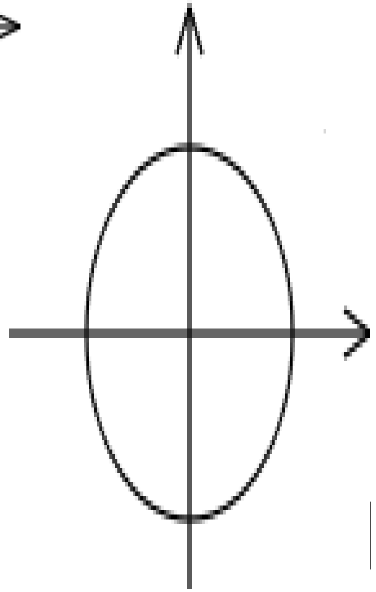
$$S = \{ x \in R^n \mid\mid \|x\| = 1\}$$

Variation of the shape of unit sphere under the norm.









$\|x\|_A = 1,\qquad A = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

$$\|x\|_A^2 = x^T A x = 5x_1^2 + x_2^2 = \frac{x_1^2}{(\frac{1}{\sqrt{5}})^2} + \frac{x_2^2}{1} = 1$$

UNIT VECTOR

$$\hat{x} = \frac{X}{\|x\|_2} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)^T$$

- Cauchy Schwarz inequality (CS)
  - $\langle x, y\rangle = x^T y = \|x\|_2 \|y\|_2 \cos\theta \leq \|x\|_2 \|y\|_2$
  - Verify that x and y are parallel if $x^T y = \|x\|_2 \|y\|_2$
- Minkowski inequality: let p, q be integers: $\frac{1}{p} + \frac{1}{q} = 1$

$$\langle x, y\rangle = x^T y \leq \|x\|_p \|y\|_q$$

NORMS ARE EQUIVALENT

$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\ \|x\|_2$$

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n}\ \|x\|_\infty$$

$$\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$$

- Hence, in analysis one can pick any norm

FUNCTIONALS

- Let V be a vector space
- Any function that maps V into R: f: V -> R is called a functional
- f is called linear functional if

$$f(x_1 + x_2) = f(x_1) + f(x_2) \text{ – additive}$$

$$f(ax) = af(x) \text{ – homogenous}$$

- <u>Example:</u>

  1) $||x||$ is a nonlinear functional

  2) Let a be a fixed vector. f_a: R^n -> R, $f_a(x)$ = <a, x> is a linear functional

  3) $f_A(x) = \frac{1}{2} x^T A x$ is a nonlinear functional

ORTHOGONALITY AND CONJUGACY

- x, y are orthogonal denoted by $x \perp y$ if $\langle x, y \rangle = 0$
- x, y are A-conjugate if $x^T A y = 0$
- Let $S = \{x_1, x_2, \ldots, x_k\}$ $\quad x_i \in R^n$
  - S is said to be mutually orthogonal if

    $\langle x_i, x_j \rangle = 0$ for $i \neq j$

    $\neq 0$ for $i = j$
  - S is said to be orthonormal if

    $\langle x_i, x_j \rangle = 0$ for $i \neq j$

    $= 1$ for $i = j$
  - S is said to be A-conjugate if

    $x_i^T A x_j = 0$ for $i \neq j$

    $= \|x\|_A^2$ for $i = j$

LINEAR COMBINATION

- Let S = $\{x_1, x_2, \ldots, x_k\}$ be a set of k vector in R^n, where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T$
- Let $a_1, a_2, \ldots, a_k$ are real scalar
- Then y which is the sum of scalar multiples of vector in S is called linear combination

$$y = a_1 x_1 + a_2 x_2 + \ldots + a_k x_k$$

- When $a_i = \frac{1}{k}$, $\bar{x} = \frac{1}{k}\sum_{i=1}^{k} x_i$ is the centroid of S

LINEAR INDEPENDENCE

- Let $S = \{x_1, x_2, \ldots, x_k\}$ $x_i \in R^n$
- The vector in S are linear dependent if there exists a linear combination

$$y = a_1x_1 + a_2x_2 + \ldots + a_kx_k = 0$$

when not all the scalars a_i are zero

- The set S is linearly independent if it is not linearly dependent

SPAN OF A SET OF VECTORS

- Let S = $\{x_1, x_2, \ldots, x_k\} \subset R^n$ (a finite subset)

  SPAN(S) = $\{ y \mid y = \sum_{i=1}^{k} a_i x_i , a_i \in R, x_i \in R^n \}$

  = set of all linear combination of vectors in S

- Clearly SPAN(S) = vector space which is a subset of R^n

- SPAN(S) is called a subspace generated by S

BASIS AND DIMENSION

- Let B be a finite subset of a vector space V
- If every vector x in V can be obtained as a linear combination of those in B, then B is called the generator for V
- If the set of vector in B are linearly independent, then B is the basis for SPAN(S) = V
- Let e_i = i^{th} unit vector with 1 as the i^{th} element and zero else where
- Then $B_n = \{ e_i \mid 1 \leq I \leq n\}$ is the basis for R^n
- The number of elements in B is called the dimension of SPAN(B)

EXERCISES

2.1) Verify the parallelogram law:

$$\|x + y\|_2^2 + \|x - y\|_2^2 = 2(\|x\|_2^2 + \|x\|_2^2)$$

2.2) Verify the triangle inequality for 2 –Norm, 1-Norm and the ∞ - norm

2.3) Prove that if $x^T y = \|x\|_2 \|y\|_2$, then x and y are parallel vectors

2.4) Using MATLAB, plot the Contours of $f(x) = x^T A x$ when $x = (x_1, x_2)^T$ and $A = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$

2.5) Verify that $(x_1 + x_2)$, $(x_2 + x_3)$, $(x_3 + x_1)$ are also linearly independent when $\{x_1, x_2, x_3\}$ are linearly independent

2.6) Let $x = (1, 2, 3)^T$. Verify the relations between the 1, 2 and ∞ - norms given in slide 12, Module - 2

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